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The use of Reduce Differential Transform Method for solving Partial differential Equation

A Research

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بسم الله الرحمن الرحيم

"لا يكلف الله نفسا الا وسعها لها ما كسبت وعليها ما اكتسبت ربنا لا تؤاخذنا ان نسينا أو اخطأنا ربنا لا تحمل علينا اصرا كما حملته على الذين من قبلنا ربنا ولا تحملنا ما لا طاقة لنا به واعف عنا واغفر لنا وارحمنا انت مولانا فانصرنا على القوم الكافرين "

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Abstract

In this paper, we apply Reduced Differential Transform Method (RDTM) for solving partial differential equations (Heat equation with external force), many numerical application are shown for implementing this method. The results show that this method is very effective and simple.

Introduction

M any physical problems can be described by mathematical models that involve partial differential equation. A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Several numerical methods were developed for solving partial differential equation with variable coefficients such us He's polynomials [1], the homology perturbation method [2], homology analysis method [3], and the modified variation iteration method [4].

The main aim of this paper is to apply the reduced differential transform method (RDTM) [5-6] to obtain the exact solution for heat equation with external force of the form

$$u_t = u_{xx} + F(x,t)$$

With the initial condition

$$u(x,0) = f(x)$$

The proving that the (RDTM), is very efficient. Suitable, quite accurate and simple to such types of equations.

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Chapter one

Basic Definition and Rules

1.1-Introduction

In this chapter the basic definition and rules accorder to partial differential equation and how this can be classified accorder to their solutions.

1.2-Definition

Differential Equation

Definition 1.2.1:

A differential equation is an equation that relates the derivatives of a function depending on one or more variables,

For example:

$$\frac{d^2u}{dx^2}+\frac{du}{dx}=\cos x \qquad \dots (1, 1)$$

Is a differential equation involving an unknown function u(x) depending on one variables and

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=\frac{\partial u}{\partial t} \qquad \dots (1, 2)$$

Is a differential equation involving an unknown function u (t, x, y) depending on three variables.

Partial Differential Equation (PDE)

Definition 1.2.2:

A partial differential equation (PDF) is an equation that contains, in addition to the dependent and independent variables, one or more partial derivatives of the dependent variable.

Suppose that our unknown function is u and it depends on the tow independent

Variables than the general from of the (PDE) is

$$F\left(x, y, \dots, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots\right) = 0$$

Here subscripts denotes the partial derivatives, for example

$$u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial y}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

Order of partial differential equation

Definition 1.2.3:

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation,

For example:

In the following example, our unknown function is u and it depends on three variables t, x and y.

■ $u_{xx} + 2xu_{xy} + u_{yy} = e^y;\ Order\ is\ Two$

■$u_{xxy} + xu_{yy} + 8u = 7y\,; Order\ is\ Three$

■$u_t - 6uu_x + u_{xxx} = 0; Order\ is\ Three$

■$u_t + uu_x = u_{xx}; Order\ is\ Tow$

■$u_{xxx} + xu_{xy} + yu^2 = x + y; Oeder\ isThree$

■$u_x + u_y = 0\,; Order\ is\ One$

Degree of partial differential equation

Definition 1.2.4:

The degree of a partial differential equation is the degree of the highest order partial derivative occurring in the equation,

For example:

In the following examples, our unknown function is u and it depends on two t, x and y

■$u_{xx} + 2xu_{xy} + u_{yy} = e^y\,; Degree\ is\ One$

■$u_{xxy} + 2xu^2{}_{yy} + 8u = 7y\,; Degree\ is\ One$

■$u_t - 6uu_xu^3{}_{xxx} = 0\,; Degree\ is\ Three$

■$u_t + uu^3{}_x = u_{xx}\,; Degree\ is\ One$

■$u^2{}_{xxx} + xu^3{}_{xy} + yu^2 = x + y\,; Degree\ is\ Two$

■$u_x + y_y = 0\,; Degree\ is\ One$

Definition 1.2.5:

The Laplace transform of a function $f(x)$ is denoted by L $(f(x))$ and is defined as the integral of $e^{-sx}f(x)$ between the limits $x = 0$ and $x = \infty$,

i.e.

$$L(f(x)) = \int_0^\infty e^{-sx} f(x)dx$$

The constant parameter (s) is assumed to be positive and large enough to ensure that the product $e^{-sx}f(x)$ converges to zero as $x \to \infty$, for most common function $f(x)$.

In determining the transform of any function, you will appreciate the limits are substituted for (x), so that the result will be a function of s

$$\therefore \; L(f(x)) = \int_0^\infty e^{-sx} f(x)dx = F(s)$$

1.3-Properties of Laplace transformations

At this point, we will state two important properties of Laplace transformation then precede how to use it to solve differential equation.

PROPERTY 1:-

If $f(x) = Ag(x) + Bh(x)$, where A and B are constants and the function $g(x)$ and $h(x)$ have Laplace transform $G(x) and\ H(x)$, respectively, then

$$L(f(x)) = F(s) = AG(s) + BH(s)$$

In general

If $f_1(x), f_2(x), \dots, f_n(x)\ are\ (n)$ function such that $x > 0$ and if $G_1, G_2, \dots, G_n\ are\ (n)$ numbers then:

$$L[G_1 f_1(x) + G_2 f_2(x) + \cdots + G_n f_n(x)] = G_1 L[f_1(x)] + G_2 L[f_2(x)] + \cdots + G_n L[f_n(x)]$$

This property is called property of linearity.

PROPERTY 2:- (property of uniqueness)

Let $f(x), and\ g(x)$, be two function piecewise continuous with an exponential order at infinity.

Assume that $L(f) = L(g)$,

Then $f(x) = g(x) for\ x \in [0, B], for\ every\ B > 0$, except may be for a finite set of points.

1.4-Classification of partial differential equation

1-The general from of the first order P.D.E is given by:

$f(x, y, u, u_x, u_y) = 0$, where x and y are independent variables and u is the dependent variable and this equation may be linear and may be non-linear equation.

2-The general from of the second order P.D.E is given by:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Whereas A, B, C, D, E and F are function of x and y or constants and G is function of x and y.

3-The general from of the P.D.E of higher order is given by:

$$\sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}(x, y) \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = f(x, y)$$

1.5-Laplace transform for partial differential equation

There are many applications for Laplace transformations such as solutions of linear partial equation with constant coefficients which we show it in this paragraph. And before starting in the method of solution we need to know Laplace transformation for partial derivatives.

Definition 1.2.6:-

Laplace transformation for function $u(x, t)$ is define by following from:

$$L\big(u(x,t)\big) = \int_0^\infty e^{-st} u(x,t)\,dt = v(x,s), s > o$$

From the above definition, we get the following laws:-

(1) $L(u_t) = sv(x,s) - u(x,0)$

(2) $L(u_{tt}) = s^2 v(x,s) - su(x,0) - u_t(x,0)$

(3) $L(u_x) = \frac{d}{dx} v(x,s)$

Chapter tow

Solving partial differential equation

By differential transform

2.1-Introduction

In this chapter we will deliberates the method of differential transformation to find a complete solution for partial differential equation .

2.2-Analysis of the method

The basic definitions of reduced differential transform method are introduced as follows:

2.3-Definitions

Definition 2.3.1

If the function is analytic and differentiated continuously with respect to time and space in the domain of interest then

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \quad \dots \ (2.1)$$

Where the t-dimensional spectrum function $U_k(x)$ is the transformed function in this paper, the lower case represent the original function while the uppercase stand for the transformed function .

Definition 2.3.2

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \quad \dots \ (2.2)$$

Then combining equation (2.1) and (2.2) we write

$$u(x,y) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,y)\right]_{t=0} t^k \qquad \dots (2.3)$$

From the above definitions , it can be found that the concept of the reduced differential transform is derived from the power series expansion . The function mathematical operations performed by RDTM can be readily obtained and are listed in table 1 .

2.4-Table1: Reduced differential transform

Function Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$w_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,y)$	$w_k(x) = \alpha U_k(x)$ α *is a constant*
$w(x,t) = x^m t^n$	$w_k(x) = x^m \delta(k-n), \delta(k) = \begin{cases} 1 & ,k=0 \\ 0 & ,k\neq 0 \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$w_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$w_k(x) = \sum_{r=0}^{k} V_r U_{k-r} = \sum_{r=0}^{k} U_r V_{k=r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$w_k(x) = (k+1)\ldots(k+r)U_{k+r}(x)$

$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$w_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$	$w_k(x) = \frac{\partial^2}{\partial x^2} U_k(x)$

2.5-Numerical Applications

In this section, we use reduced differential transform method (RDTM) for solving heat equations.

2.6-Exampls

Example 2.6.1

Consider the initial problem which describes the heat equations

$$u_t(x,t) = u_{xx}(x,t) - u_x(x,t) \qquad \text{... (2.4)}$$

Which the initial

$$u(x,0) = e^{\frac{1}{2}x} \qquad \text{... (2.5)}$$

Where $u = u(x,t)$ is a function of the variables x and t.

Taking the reduced differential transform of (4), we obtain the

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2} U_k(x) - \frac{\partial}{\partial x} U_k(x) \qquad \text{.... (2.6)}$$

Where the t-dimensional spectrum function is the transform function . From the initial condition (5), we write

$$U_0(x) = e^{\frac{1}{2}x} \qquad \text{.... (2.7)}$$

Substituting (2.7) into (2.6), we obtain the following $U_k(x)$ **values successively**

$$U_1(x) = \frac{-1}{4} e^{\frac{1}{2}x}$$

$$U_2(x) = \frac{1}{2!}\frac{1}{4^2} e^{\frac{1}{2}x}$$

$$U_3(x) = \frac{1}{3!}\frac{1}{4^3} e^{\frac{1}{2}x}$$

$$U_4(x) = \frac{1}{4!}\frac{1}{4^4} e^{\frac{1}{2}x}$$

.

.

.

Finally the differential inverse transform (2, 2) of $U_k(x)$ **gives**

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k = \left(1 - \frac{1}{4}t + \frac{1}{2!}\frac{1}{4^2}t^2 + \frac{1}{3!}\frac{1}{4^3}t^3 + \cdots\right) e^{\frac{1}{2}x}$$
$$= e^{\frac{-1}{4}t} e^{\frac{1}{2}x}$$

This is the exact solution.

Example 2.6.2

To solve the equation:

$$u_t(x,t) = u_{xx}(x,t) + \frac{x}{2} \qquad \text{... (2.8)}$$

With the initial condition

$$u(x,t) = x \qquad \text{... (2,9)}$$

Applying the reduced differential transform to (2.8) we obtain the recurrence equation

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) - \frac{x}{2}\delta(k) \qquad \text{... (2.10)}$$

Where the t-dimensional spectrum function is the transform function . From the initial condition (2.5) we write

$$U_0(x) = x \qquad \text{... (2.11)}$$

Substituting the initial condition (2.11) into (2.10), and using the recurrence relation (2.10) we can

Obtain the following $U_k(x)$ values;

$U_1(x) = \frac{x}{2}$

$U_2(x) = 0$

$U_3(x) = 0$

$U_4(x) = 0$

.

.

.

i.e $U_k(x) = 0 \quad , \forall k \geq 1$

Finally the differential inverse transform (2.2) of $U_k(x)$ gives

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k=\left(1+\frac{1}{2}t\right)x$$

With is the exact solution

Example 2.6.3

Consider the initial value problem which describes the heat equation

$u_t(x,t)=u_{xx}(x,t)+\sin(x)$ ... (2, 12)

With the initial condition

$$u(x,0)=0 \quad \text{... (2, 13)}$$

Similarly , by using the reduced differential transform to (2, 12) , we obtain the recurrence equation

$(k+1)U_{k+1}(x)=\frac{\partial^2}{\partial x^2}U_k(x)+(\sin(x))\delta(k)$ ... (2, 14)

From the initial condition by given of (2, 13)

$$U_0(x)=0 \quad \text{... (2, 15)}$$

Substituting (2, 15) into (2, 14), we have

$U_1(x)=\sin(x)$

$U_2(x)=\frac{-1}{2}\sin(x)$

$U_3(x)=\frac{1}{6}\sin(x)$

$U_4(x)=\frac{-1}{24}\sin(x)$

$U_5(x)=\frac{1}{120}\sin(x)$

.

.

.

The series solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k$$

$$= \left(t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \cdots\right)\sin(x) = (1 + e^{-t})\sin x$$

This is the exact solution.

Example 2.6.4

Consider the initial value problem which describes the heat equation

$$u_t(x,t) = u_{xx}(x,t) - u(x,t) + x \qquad \dots (2.16)$$

With the initial condition

$$u(x,t) = 0 \qquad \dots (2.17)$$

Apply the reduced differential transform method to (2. 16)

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) - U_k(x) + x\delta(k) \qquad \dots (2.18)$$

From the initial condition (2. 17), we can write

$U_0(x) = 0$

$U_1(x) = x$

$U_2(x) = \frac{-1}{2}x$

$U_3(x) = \frac{1}{6}x$

$U_4(x) = \frac{-1}{24}x$

.

.

.

By using definition (2.3.2), we get the exact solution

$$u(x,t) = x\left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \cdots\right) = -xe^{-t} + x \ .$$

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