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Differential transform method for solving partial Differential equations with variable coefficients

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Supervised By

Noor Ali Hussein



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> إعداد الطالب سجاد علي كنان

بإشراف الأستاذة نور على حسين

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Abstract

In this search, we consider the differential transform method (DTM) for finding approximate and exact solutions of some partial differential equations with variable coefficients. The efficiency of the considered method is illustrated by some examples, the results reveal that the proposed method is very effective and simple and can be applied for other linear and nonlinear problems in mathematical physics.

Chapter one Basic Definition and Rules

Introduction

Many physical problems can be described by mathematical models that involve partial differential equations . A mathematical model is a simplified description of physical reality expressed in mathematical terms . Thus , the investigation of the exact or approximation solution helps us to understand the means of these mathematical models . Several numerical methods were developed for solving partial differential equations with variable coefficients such uh He's polynomials , the homotopy perturbation method , homotopy analysis method and the modified variational iteration method . In this search, we consider the differential transform method (DTM) for finding approximate and exact solutions of some partial differential equations with variable coefficients . The efficiency of the considered method is illustrated by some examples , the results reveal that the proposed method is very effective and simple and can be applied for other linear and nonlinear problems in mathematical physics .

1.1-Introduction

In this chapter the basic definition and rules accorder to partial differential equation and how this can be classified accorder to their solutions.

1.2-Definition

Differential Equation

Definition 1.2.1:

A differential equation is an equation that relates the derivatives of a function depending on one or more variables,

For example:

$$\frac{d^2u}{dx^2} + \frac{du}{dx} = \cos x \qquad ... (1, 1)$$

Is a differential equation involving an unknown function u(x) depending on one variables and

 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \qquad \dots (1, 2)$

Is a differential equation involving an unknown function u (t, x, y) depending on three variables.

Partial Differential Equation (PDE)

Definition 1.2.2:

A partial differential equation (PDF) is an equation that contains, in addition to the dependent and independent variables, one or more partial derivatives of the dependent variable.

Suppose that our unknown function is u and it depends on the tow independent

Variables than the general from of the (PDE) is

$$F(x, y, \ldots, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = \mathbf{0}$$

Here subscripts denotes the partial derivatives, for example

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y},$$

 $u_{yy} = \frac{\partial^2 u}{\partial y^2}$

Order of partial differential equation

Definition 1.2.3:

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation,

For example:

In the following example, our unknown function is u and it depends on three variables t, x and y.

 $u_{xx} + 2xu_{xy} + u_{yy} = e^{y}; \text{ Order is Two}$ $u_{xxy} + xu_{yy} + 8u = 7y; \text{ Order is Three}$ $u_t - 6uu_x + u_{xxx} = 0; \text{ Order is Three}$ $u_t + uu_x = u_{xx}; \text{ Order is Tow}$ $u_{xxx} + xu_{xy} + yu^2 = x + y; \text{ Oeder is Three}$ $u_x + u_y = 0; \text{ Order is One}$

Degree of partial differential equation

Definition 1.2.4:

The degree of a partial differential equation is the degree of the highest order partial derivative occurring in the equation,

For example:

In the following examples, our unknown function is u and it depends on two t, x and y

$$u_{xx} + 2xu_{xy} + u_{yy} = e^{y}; Degree is One$$

$$u_{xxy} + 2xu^{2}_{yy} + 8u = 7y; Degree is One$$

$$u_{t} - 6uu_{x}u^{3}_{xxx} = 0; Degree is Three$$

$$u_{t} + uu^{3}_{x} = u_{xx}; Degree is One$$

$$u^{2}_{xxx} + xu^{3}_{xy} + yu^{2} = x + y; Degree is Two$$

$$u_{x} + y_{y} = 0; Degree is One$$

Definition 1.2.5:

The Laplace transform of a function f(x) is denoted by L (f(x)) and is defined as the integral of $e^{-sx}f(x)$ between the limits x = 0 and $x = \infty$,

i.e.

$$L(f(x)) = \int_0^\infty e^{-sx} f(x) dx$$

The constant parameter (s) is assumed to be positive and large enough to ensure that the product $e^{-sx}f(x)$ converges to zero as $x \to \infty$, for most common function f(x).

In determining the transform of any function, you will appreciate the limits are substituted for (x), so that the result will be a function of s

$$\therefore L(f(x)) = \int_0^\infty e^{-sx} f(x) dx = F(s)$$

1.3-Properties of Laplace transformations

At this point, we will state two important properties of Laplace transformation then precede how to use it to solve differential equation.

PROPERTY 1:-

If f(x) = Ag(x) + Bh(x), where A and B are constants and the function g(x) and h(x) have Laplace transform G(x) and H(x), respectively, then

$$L(f(x)) = F(s) = AG(s) + BH(s)$$

In general

If $f_1(x), f_2(x), \dots, f_n(x)$ are (n) function such that x > 0 and if G_1, G_2, \dots, G_n are (n) numbers then:

$$L[G_1f_1(x) + G_2f_2(x) + \dots + G_nf_n(x)] = G_1L[f_1(x)] + G_2L[f_2(x)] + \dots + G_nL[f_n(x)]$$

This property is called property of linearity.

PROPERTY 2:- (property of uniqueness)

Let f(x), and g(x), be two function piecewise continuous with an exponential order at infinity.

Assume that L(f) = L(g),

Then f(x) = g(x) for $x \in [0, B]$, for every B > 0, except may be for a finite set of points.

1.4-Classification of partial differential equation

1-The general from of the first order P.D.E is given by:

 $f(x, y, u, u_x, u_y) = 0$, where x and y are independent variables and u is the dependent variable and this equation may be linear and may be non-linear equation.

2-The general from of the second order P.D.E is given by:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Whereas A, B, C, D, E and F are function of x and y or constants and G is function of x and y.

3-The general from of the P.D.E of higher order is given by:

$$\sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}(x, y) \frac{\partial^{i+j} u}{\partial x^{i} \partial y^{j}} = f(x, y)$$

1.5-Laplace transform for partial differential equation

There are many applications for Laplace transformations such as solutions of linear partial equation with constant coefficients which we show it in this paragraph. And before starting in the method of solution we need to know Laplace transformation for partial derivatives.

Definition 1.2.6:-

Laplace transformation for function u(x, t) is define by following from:

$$L(u(x,t)) = \int_0^\infty e^{-st} u(x,t) dt = v(x,s), s > o$$

From the above definition, we get the following laws:-

(1)
$$L(u_t) = sv(x, s) - u(x, 0)$$

(2)
$$L(u_{tt}) = s^2 v(x,s) - su(x,0) - u_t(x,0)$$

(3)
$$L(u_x) = \frac{d}{dx}v(x,s)$$

Chapter Two

Solving partial differential Equations by using Differential transformation

2-1 INTRODUCTION

In This Chapter we will deliberates The method differential transformation to find a complete solution for partial differential equation with variable coefficients

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2-2- Methodolog

To illustrate the basic of the DTM, we considered u(x, t) is analytic and differenuously in the domain of interest then let

Where the spectrum uk(x) is the transformed function, which is called T-function in brief the differential inverse transform of

 $U_k(x)$ is defined as follows.

Combining (2-1) and (2-2), it can be obtained that

When (t) are taken as (t = 0) then equation (2-3) is expressed

And equation (2-2) is shown as

In real application, the function u(x, t) by a finite series of

Usually, the values of n is decided by convergence of the series coefficients. The following theorems that can be deduced from equation (2-3) and equation (2-4) are given as:

2-3-Theorems:

Theorem 2-3-1: if the original function is $u(x, t) = w(x, t) \pm v(x, t)$ then the transformed function is:

 $U_k(x) = w_k(x) \pm V_k(x).$

Theorem 2-3-2: if the original function as U(x, t) = a v(x, t) then the transformed function is

 $U_{k}(x)=aV_{k}\left(x\right) .$

Theorem 2-3-3: if the original function is $U(x, t) = \frac{\partial^m w(x,t)}{\partial t^m}$, then

The transformed function is :

 $U_k(x) = \frac{(k+m)!}{k!} w_k(x) .$

<u>Theorem 2-3-4</u>: if the original function is $U(x, t) = \frac{\partial w(x,t)}{\partial x}$, then the transformed function is $U_k(x) = \frac{\partial}{\partial x} w_k(x)$.

Theorem 2-3-5: if the original function is $U(x, y, t) = \frac{\partial w(x, y, t)}{\partial y}$, then the transformed function is

$$U_k(x,y) = \frac{\partial}{\partial y} w_k(x,y).$$

Theorem 2-3-6: if the original function is $U(x, y, z, t) = \frac{\partial w(x,y,z,t)}{\partial z}$, then the function transformed function is $U_k(x, y, z) = \frac{\partial}{\partial z} w_k(x, y, z)$.

Theorem 2-3-7: if the original function is $U(x, t) = x^m t^n$ then the transformed function i:s

 $U_k(x) = x^m \delta(k - n).$

Theorem 2-3-8: if the original function is $U(x, t) = x^m t^n w(x, t)$, Then the transformed function is

 $u_k(x) = x^m w_{k-n}(x).$

Theorem 2-3-9: if the original function is U(x,t) = w(x,t)v(x,t)then the transformed function is:

$$U_k(x) = \sum_{r=0}^k w_r(x) v_{k-r}(x).$$

To illustrate the aforementioned theory, some examples of partial differential equations with variable coefficients are discussed in details and the obtained results are exactly the same which is found by varitional iteration method.

2-4-application

Here, extended differential transformation method(DTM) is used to fine solutions of the PDEs in one, two and three dimensions with variable coefficients, and compared with that obtained by other methods

Example 2-4-1

Consider the one-dimensional heat equation with variable coefficients in the form

And the initial condition

$$U(x, 0) = x^2$$
(2-8)

Where U = U(x, t) is a function of the variables x and t the exact solution of this problem is $U(x, t) = x^2 e^t$ then by using the basic properties of the reduced differential transformation, we can find transformed form of equation (7) as:

$$(k+1)v_k(x) = \frac{x^2\partial^2 v_k(x)}{2\partial x^2},$$
(2-9)

Using the initial condition, equation (2-8) we have

Now, substituting equation (2-10) into (2-9), we obtain the following $U_k(x)$ values successively

Finally the differential inverse transform of $U_k(x)$ gives

$$U(x,t) = \sum_{k=0}^{\infty} U_k(x) t^2 = X^2 \sum_{k=0}^{\infty} \frac{t^k}{k!} . \qquad (2-12)$$

Then, the exact solution is given as:

.....

Example 2-4-2

Consider the tow dimensional heat equation with variable coefficients a:

$$U_1(x, y, t) - \frac{y^2}{2} U_{xx}(x, y, t) - \frac{x^2}{2} U_{uu}(xx, y, t) = 0$$
(2-14)

Where the initial condition is

$$U(x, y, 0) = y^2$$
.(2-15)

Taking differential transform of equation(2-14) and the initial condition Equation (2-15) respectively ,

$$(k+1)V_k(x,y) = Y^2 \frac{\partial^2}{\partial x^2} U_k(x,y) + x^2 \frac{\partial^2}{\partial y^2} U_k(x,y), \dots (2-16)$$

Using the initial condition, equation (2-15) we have:

$$U_0(x, y) = y^2$$
(2-17)

Now, substituting equation (2-17) into(2-16), we obtain the following

 $U_k(x, y)$ values successively

$$U_{1}(x,y) = X^{2}, U_{2}(x,y) = \frac{x^{2}}{2}, U_{3}(x,y) = \frac{x^{2}}{6}, U_{4}(x,y) = \frac{Y^{2}}{24}, U_{5}(x,y) = \frac{x^{2}}{120}, U_{6}(x,y) = \frac{y^{2}}{720}, U_{7} = \frac{y^{2}}{5040}, \dots, U_{k}(x,y) = \begin{cases} \frac{x^{2}}{k!}k \text{ is even} \\ \frac{y^{2}}{k!}k \text{ is even} \end{cases}$$
(2-18)

Finally the differential transform $U_k(x, y)$

$$U(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y) t^k = x^2 \sum_{k=0}^{\infty} \frac{t^k}{k!} + Y^2 \sum_{k=0}^{\infty} \frac{t^k}{k!} \dots (2-19)$$

Then the exact solution is given by:

$$U(x, y, t) = x^{2} \left(t + \frac{t^{3}}{3!} + \frac{t^{5}}{5!} \dots \frac{t^{2n+1}}{(2n+1)} \right) + Y^{2} \left(1 + \frac{t^{2}}{2!} + \frac{t^{4}}{5!} + \dots \frac{t^{2n+1}}{(2n)!} \right)^{sin ht}$$

= $X^{2} sinht + Y^{2} cosht \dots (2-20)$

Which is the exact solution of equation (2-14) initial condition equation

(2-22)respective

Example 2-4-3

Considering the one-dimensional wave equation with variable coefficients as:

$$U_{tt} - \frac{x^2}{2} U_{xx}(x, t) = 0$$
,(2-21)

With the initial condition

$$U(x, 0) = x$$
, $U_t(x, 0) = X^2$,(2-22)

Taking deferential transform of equation

$$(k+1)(k+2)U_k(x) - \frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x) = 0$$
,(2-23)

Using the initial condition, equation (2-22) we have

$$U_0(x) = X_U_1(x) = x^2$$
.(2-24)

Now, substituting equation (2-24) into (2-23), we have obtain the following $U_k(x)$ values successively

.....(2-21)

 $U_x(x) = 0, k = 2, 4, 6, \dots$ $U_3(x) = \frac{1}{6}x^2, U_5(x) = \frac{1}{120}x^2, U_{73}(x) = \frac{1}{5040}x^2$ (2-25) finally the differential invers transform of $U_k(x)$ gives

Thus, the exact solution is given in the closed form as:

Example 2-4-4

Considering the tow-dimension wave equation with variable coefficients as:

$$U_{tt}(x, y, t) - \frac{x^2}{12} u_{xx}(x, y, t) - \frac{x^2}{12} u_{yy}(x, y, t) = 0$$
(2-28)

With the initial condition

$$U(x, y, 0) = x^2, u_t(x, y, o) = y^2,$$
(2-29)

Taking differential transform of equation (2-28) and the initial condition equation (2-29) respectively

$$(k = 1)u_{k+1}(x, y) - \frac{x^2}{12}\frac{\partial^2}{\partial x^2}U_k(x, y) - \frac{y^2}{12}\frac{\partial^2}{\partial y^2}U_k(x, y) = 0$$
 (2-30)

Using the initial condition equation (2-28) we have,

$$u_0(x, y) = x^2 u_1(x, y) = y^4$$
(2-31)

Now, substituting equation (2-31) into (2-30), we obtain the following $U_k(x)$ Values successively

Finally the differential inverse transform of $U_k(x, y)$

$$U(x, y, t) = x^{2} \left(1 + \frac{t^{2}}{2!} + \frac{t^{4}}{4!} + \cdots \right) + y^{2} \left(t + \frac{t^{3}}{3!} + \frac{t^{45}}{5!} + \cdots \right)$$

Hence, the exact solution is:

$$U(x, y, t) = x^2 sosh(t) + Y^2 sinh(t)$$
(2-34)

.....

Example 2-4-5

Considering the three-dimension wave equation with variable coefficients as:

$$U_{tt} - (x^2 + y^2 + z^2) - \frac{1}{2}(x^2 U_{tt} + y^2 U_{tt} + Z^2 U_{tt}) = 0, \quad \dots \dots \quad (2-35)$$

With the initial condition

$$U(x, y, z, 0)=0, u_1(x, y, z, 0) = x^2 + y^2 - z^2$$
,(2-36)

Taking differential transform of equation (2-35) and the initial condition equation (2-36) respectively ,

.

$$(k+1)(k+2)u_{k+2}(x,y,z) - \frac{x^2}{12}\frac{\partial^2}{\partial x^2}u_k(x,y,z)$$

$$-\frac{y^2}{12}\frac{\partial^2}{\partial y^2}u_k(x, y, z) = 0$$
,(2-37)

Using the initial condition, equation (2-36) we have

$$U_0(x, y, z) = 0, U_1(x, y, z) = x^2 + y^2 - z^2$$
,(2-38)

Now, substituting equation (2-38) into (2-38), we obtain the following $U_k(x, y, z)$ Values successively

$$U_{2}(x, y, z) = \frac{1}{2}(x^{2} + y^{2} + z^{2}), U_{3}(x, y, z) = \frac{1}{6}(x^{2} + y^{2} - z^{2}), U_{4}(x, y, z) = \frac{1}{24}(x^{2} + y^{2} + z^{2}), U_{5}(x, y, z) = \frac{1}{120}(x^{2} + y^{2} - z^{2}), U_{6}(x, y, z) = \frac{1}{720}(x^{2} + y^{2} - z^{2}), \dots (2-39)$$

Finally the differential inverse transform of $U_k(x, y, z)$ gives

$$U(x, y, z) = \sum_{k=0}^{\infty} U_k(x, y, z) t^k$$

= $(x^2 + y^2 + z^2) \sum_{k=0,2,3}^{\infty} U_k(x, y, z) t^k + (x^2 + y^2 - z^2) \sum_{k=1,3,5}^{\infty} U_k(x, y, z) t^k$ (2-40)
 $u(x, y, z, t) = (x^2 + y^2) (t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots) + z^2 (-t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots)$
Then, the exact solution is given in the closed form by

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