Ministry of Higher Education
and Scientific Research
University of Qadisiyah
College of Education
Department of Mathematics

Using ELzaki Transform to Solve the Ordinary Differential Equations

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By

Mortadha Rahim Rahi

Supervisor

Alaa Kamel Jaber

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بِسْمِ اللَّـهِ الرَّحْمَـٰنِ الرَّحِيمِ

(إِنَّمَا يَخْشَى اللَّهَ مِنْ عِبَادِهِ الْعُلَمَاءُ إِنَّ اللَّهَ عَزِيزٌ غَفُورٌ)

صدق اللهَّ العظيم

سُوَرة فاطر

أيه ٢٨

الإهداء

لرجل ذكراه دائماً تتردد بالخير على ذاكرتي .. الى من سعى وشقي لأنعم بالراحه والعناء الى من علمني أن أرتقي سلم الحياة بحكمة وصبرالى من حبه يجي بعروقي .. والدي الحبيب

لامرأة صنعت مني رجلاً طموح يعشق التحديات كما عشقتها هي .. لتلك التي منها تعرفت على ألقوه والثقة بالنفس .

لمن رضاها يخلق لي التوفيق ...لجنتي ..أمي

الى من عرفت كيف أجدهم وعلموني أن لا أضيعهم... أصدقائي

الشكر والتقدير

(من لا يشكر الناس . . . لا يشكر الله)

أتقدم بالشكر أولاً وأخيراً لله سبحانه وتعالى ..

لـ أساتذتي ...حتى وإن كانت هذه الكلمة قليلة في حقكم، ولكن

أنتم السبب المهم ليكون هذا البحث واقع ملموس وليس حلم من أحلامي البسيطة ..

شكرًا لكل إحساس صادق التمسته بعيونكم وقلوبكم طوال فترة دراستي،،

وبأشد الكلمات الطيبة اقدم شكري وامتناني لمن كان السبب في استكمال هذا البحث ولتحفيزي على المثابرة والاستمرار وعدم اليأس لأستاذي ..

مشرف البحث الدكتور علاء كامل جابر الموسوي ...

Abstract

In this paper a new integral transform namely Elzaki transform was applied to solve linear ordinary differential equations.

Introduction

The differential equations represent the most important phenomena occurring in the world. This phenomenon is importance in applied mathematics, physics, and issues related to engineering. The importance of obtaining the exact solution of differential equations is still a big problem that needs new methods to discover new exact or approximate solutions.

Several techniques such as A domain decomposition method [1], Variational iteration method [2, 3], Homotopy perturbation method [4], Laplace decomposition method [5, 6, 8], Sumudu decomposition method [7], have been used Such that is the original function and to solve linear and nonlinear partial differential equations. The main aim of this paper is to solve the differential equations by using of ELzaki transform.

Chapter one:

Basic Concepts

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Basic Concepts

1.1. Introduction

In this chapter we will introduce the definition of a differential equations DE. In section 1.2 and the properties such as order , linearity and homogeneous and discuss in the sections 1.3, 1.4 and 1.5 respectively , in section 1.6 we will display the definition of the solution of a DE. Boundary and initial conditions will introduce in 1.8 and 1.9.

1.2. Definition of DE

- **Ordinary differential equations** containing dependent variables with one independent variable and derivatives of this variable.
- **Partial differential equations** contain mathematical functions of more than one independent variable with their partial derivatives.

Examples of the DEs are given by

$$u_t = k\, u_{xx}, \qquad (1.1)$$

$$u_t = k\,(u_{xx}+u_{yy}), \qquad (1.2)$$

$$x=5y-4y' \qquad (1.3)$$

These equations describe the heat flow in one-dimensional space, two-dimensional space, and three-dimensional space respectively. In (1.1), the dependent variable $u = u(x, t)$ depends on the position x and on the time variable t. However, in (1.2), $u = u(x, y, t)$ depends on three independent variables, the space variables x, y and the time variable t. Other examples of DEs are given by

$$u_{tt} = c^2 u_{xx} \qquad (1.4)$$

$$u_{tt}= c2\,(u_{xx}+u_{yy}) \qquad (1.5)$$

$$u_{tt}= c^2\,(u_{xx}+u_{yy}+u_{yy}) \qquad (1.6)$$

These equations describe the wave propagation in one-dimensional space, two-dimensional space, and three-dimensional space respectively. Moreover, the unknown functions in (1.4), (1.5), and (1.6) are defined by u = u(x, t), u = u(x, y, t), and u = u(x, y, z, t) respectively.

The well-known Laplace equation is given by::

$$u_{xx}+ u_{yy}= 0, \tag{1.7}$$

$$u_{xx}+ u_{yy}+u_{zz} = 0, \tag{1.8}$$

where the function u does not depend on the time variable t .

Moreover, the Burgers equation and the KdV equation are given by

$$u_t+ uu_x-\nu u_{xx}= 0, \tag{1.9}$$

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.10}$$

Respectively, where the function u depends on x and t

1.3. Order of DE

The order of a DE is the order of the highest derivative that appears in the equation. For example, the following equations

$$\begin{aligned} u_x-u_y &= 0, \\ u_{xx}- u_t &= 0, \\ u_y - uu_{xxx} &= 0 \end{aligned} \tag{1.11}$$

<u>Example</u> 1. The order of the following DEs:

(a) $u_t= u_{xx}+ u_{yy}$ (b) $u_x+u_y = 0$

(c) $u\ 4u_{xx} + u_{xxx}= 2$ (d) $u_{xx}+ u_{yxxy}= 1$

(a)The highest derivative contained in this equation is u_{xx} or u_{yy}. The DE is therefore of order two.

(b) The highest derivative contained in this equation is u_x or u_y. The DE is therefore of order one.

(c) The highest derivative contained in this equation is u_{xxx} . The DE is therefore of order three.

(d) The highest derivative contained in this equation is u_{yxxy} . The DE is therefore of order four.

1.4. Linear and Nonlinear DEs

Differential equations are classified as linear or nonlinear. A differential equation is called linear if**:**

1. The power of the dependent variable and each derivative contained in the equation is one, and
2. The coefficients of the dependent variable and the coefficients of each derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear**.**

Example 2. To classify the following DEs as linear or nonlinear

(a) $xu_{xx} + yu_{yy} = 0$

(b) $uu_t + xu_x = 2$

(c) $u_x + \sqrt{u} = x$

(d) $u_{rr} + r\, u_r + r^2 u_{\theta\theta} = 0$

a) The power of each derivative u_{xx} and u_{yy} is one. In addition, the coefficients of the derivatives are the independent variables x and y respectively. Hence, the DE is linear**.**

b) Although the power of each derivative is one, but u_t has the dependent variable u as its coefficient. Therefore, the DE is nonlinear**.**

c) The equation is nonlinear because of the term $\sqrt{u}$**.**

d) The equation is linear because it satisfies the two necessary conditions.

1.4.1. Some Linear Differential Equations

As stated before, linear differential equations arise in many areas of scientific applications, such as the diffusion equation and the wave equation. In what follows, we list some of the well-known models that are of important concern**:**

1. The heat equation in one-dimensional space is given by

$$u_t = ku_{xx} \quad , \qquad (1.13)$$

where *k* is a constant.

2. The *wave equation* in one-dimensional space is given by

$$u_{tt} = c^2 u_{xx} \,, \qquad (1.14)$$

where c is a constant.

3. The *Laplace equation* is given by

$$u_{xx} + u_{yy} = 0. \qquad (1.15)$$

4. The Linear Schrodinger's equation is given by

$$iu_t + u_{xx} = 0, \quad i = \sqrt{-1}. \qquad (1.16)$$

5. The *Telegraph equation* is given by

$$u_{xx} = a u_{tt} + b u_t + cu, \qquad (1.17)$$

where a, b and c are constants. It is to be noted that these linear models and others will be studied in details in the forthcoming chapters.

1.4.2. Some Nonlinear Differential Equations

It was mentioned earlier that differential equations arise in different areas of mathematical physics and engineering, including fluid dynamics, plasma physics, quantum field theory, nonlinear wave propagation and nonlinear fiber optics [8]. In what follows we list some of the well-known nonlinear models that are of great interest:

1. The Advection equation is given by

$$u_t + uu_x = f(x, t). \qquad (1.18)$$

2. The Burgers equation is given by

$$u_t + uu_x = \alpha u_{xx} \qquad (1.19)$$

3. The Korteweg de-Vries (KdV) equation is given by

$$u_t + auu_x + bu_{xxx} = 0. \qquad (1.20)$$

4. The *modified KdV equation* (*mKdV*) is given by

$$u_t - 6u^2 u_x + u_{xxx} = 0. \qquad (1.21)$$

5. The *Boussinesq equation* is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0. \qquad (1.22)$$

6. The *sine-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sin u. \qquad (1.23)$$

7. The *sinh-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sinh u. \qquad (1.24)$$

8. The *Liouville equation* is given by

$$u_{tt} - u_{xx} = e^{\pm u}. \qquad (1.25)$$

9. The Fisher equation is

$$u_t = Du_{xx} + u(1-u). \qquad (1.26)$$

10. The *Kadomtsev-Petviashvili* (*KP*)*equation* is given by

$$(u_t + auu_x + bu_{xxx})_x + u_{yy} = 0. \qquad (1.27)$$

11. The *K*(*n*,*n*)*equation* is given by

$$u_t + a(u^n)_x + b(u^n)_{xx} = 0, \quad n > 1. \qquad (1.28)$$

12. The *Nonlinear Schrodinger* (*NLS*) *equation* is

$$iu_t + u_{xx} + \gamma|u|^2 u = 0. \qquad (1.29)$$

13. The *Camassa-Holm*(*CH*)*equation* is given by

$$u_t - u_{xxt} + au_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \qquad (1.30)$$

14. The *Degasperis-Procesi* (*DP*) *equation* is given by

$$u_t - u_{xxt} + au_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \qquad (1.31)$$

The above-mentioned nonlinear differential equations and many others will be examined in the forthcoming chapters. These equations are important and many give rise to solitary wave solutions.

1.5. Homogeneous and Inhomogeneous DEs

Differential equations are also classified as homogeneous or inhomogeneous. Adifferential equation of any order is called homogeneous if every term of the DE contains the dependent variable u or one of its derivatives, other- wise, it is called an inhomogeneous DE. This can be illustrated by the following example**.**

Example 3. To classify the following differential equations as homogeneous or inhomogeneous

(a) $u_t = 4u_{xx}$

(b) $u_t = u_{xx} + x$

(c) $u_{xx} + u_{yy} = 0$

(d) $u_x + u_y = u + 4$

We have:

a) The terms of the equation contain derivatives of u only, therefore it is a homogeneous DE.

b) The equation is an inhomogeneous DE, because one term contains the independent variable x.

c) The equation is a homogeneous DE.

d) The equation is an inhomogeneous DE.

1.6. Solution of a DE

A solution of a DE is a function u such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for u to satisfy the equation, the left hand side of the DE and the right hand side should be the same upon substituting the resulting solution. This concept will be illustrated by examining the following examples. Examples of differential equations subject to specific conditions will be examined in the coming chapters.

Example 4. The function $u(x,t) = \sin x\, e^{-4t}$ is a solution of the following DE

$$u_t = 4u_{xx}. \qquad (1.33)$$

Since:

Left Hand Side (LHS)= $u_t = -4\sin x\, e^{-4t}$

Right Hand Side (RHS)=$4u_{xx} = -4\sin x\, e^{-4t}$ =LHS

Example 5. The function $u(x,y) = \sin x \sin y + x^2$ is a solution of the following DE

$$u_{xx} = u_{yy} + 2. \qquad (1.34)$$

Since:

Left Hand Side (LHS)= $u_{xx} = -\sin x \sin y + 2$

Right Hand Side (RHS)= $u_{yy} + 2 = -\sin x \sin y + 2$=LHS

<u>Example</u> 6. Show that $u(x,t) = \cos x \cos t$ is a solution of the following DE

$$u_{tt} = u_{xx}. \tag{1.35}$$

Since:

Left Hand Side (LHS)= $u_{tt} = -\cos x \cos t$

Right Hand Side (RHS)= $u_{xx} = -\cos x \cos t$=LHS

1.7. Initial Conditions

It was indicated before that the DEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation phenomena and phenomena of quantum mechanics. Most of the DEs, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time t = 0 should be prescribed. It will be discussed later that for the heat case, the initial value u(t = 0), that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions u(t = 0) and u_t(t = 0) should also be prescribed.

Chapter tow:

Application of ELzaki Transform of Ordinary Differential Equations.

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2.1. Introduction

In this chapter, we will introduce the basic properties for the ELzaki Transform in sections 2.2, 2.3 and 2.4, then we will apply this transform to solve the ordinary differential equations in section 2.5.

2.2. ELzaki Transform

ELzaki Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Elzaki transform and its fundamental properties. Elzaki transform was introduced by Tarig ELzaki to facilitate the process of solving ordinary and partial differential equations in the time domain.

Typically, Fourier, Laplace and Sumudu transforms are the convenient mathematical tools for solving differential equations, also ELzaki transform and some of its fundamental properties are used to solve differential equations.

A new transform called the ELzaki transform defined for function of exponential order we consider functions in the set A defined by:

$$A = \{ f\ (t) : \exists M, k1, k2\ >\ 0, |\ f\ (t)|\ <\ Me^{\frac{|t|}{kj}}, if\ t \in \left(-1\right)^{j} X\ [0, \infty)\} \quad (2.1)$$

For a given function in the set A, the constant M must be finite number, k_1 ,k_2 may be finite or infinite. The ELzaki transform denoted by the operator E (.) defined by the integral equations

$$E[f\,(t)] = T\,(u) = u\int_0^{\infty} f\ (t) e^{\frac{-t}{u}}\, dt\ ,\ t \geq 0\ ,\ k_1 \leq u \leq k_2 \qquad (2.2)$$

The variable u in this transform is used to factor the variable t in the argument of the function f. This transform has deeper Connection with the Laplace transform.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

2.3. ELzaki Transform of the Some Functions

For any function $f\,(t)$, we assume that the integral equation (2.2) exist. The sufficient conditions for the existence of ELzaki transform are that $f\,(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, Otherwise ELzaki transform may or may not exist.

In this section we find ELzaki transform of simple functions.

1) Let $f\,(t)$=1 , then:

$$E(1) = u\int_0^\infty e^{\frac{-t}{u}}dt = u[-ue^{\frac{-t}{u}}]_0^\infty = u^2$$

2) Let $f\,(t)=t$, then:

$$E(t)=u\int_0^\infty t\, e^{\frac{-t}{u}}\; dt \qquad ,$$

We can integral by parts to find that: $E(t)=u^3$

In the general case if $n > 0$ is integer number, then.

$$E(t^n)=n!\, u^{n+2}$$

3) $E[e^{at}] = \int_0^\infty e^{\frac{-t}{u}}\; e^{at}\; dt = \frac{u^2}{1-au}$

This result will be useful, to find ELzaki transform of:

i $E[\sin at]= \frac{au^3}{1+a^2u^2}$

ii $E[\sinh at]= \frac{au^3}{1-a^2u^2}$

iii $E[\cos at]= \frac{u^2}{1+a^2u^2}$

iv $E[\cosh at]= \frac{u^2}{1+a^2u^2}$

Theorem:

Let $T\,(u)$ is the ELzaki transform of $[E\,(f\,(t))=T\,(u)]$. then:

i $E[f'(t)]=\frac{T(u)}{u}-u\,f\,(0)$

ii $E[f''(t)] =\frac{T(u)}{u^2} - f\,(0)-u\,f'(0)$

iii $E[f^{(n)}(t)]= \frac{T(u)}{u^2} - \sum_{k=0}^{n-1} u^{2-n+k}\; f^{(k)}(0)$

Proof:

(i) $E[f'(t)]=u\int_0^{\infty} f'(t)e_u^{-t}\,dt$

Integrating by parts to find that: $E[f'(t)]=\frac{T(U)}{u}-u\,f(0)$

(ii) Let $g(t)=f'(t)$, then: $E[g'(t)]=\frac{1}{u}E[g(t)]-u\,g(0)$, by (i) we find :

$$E[f''(t)]=\frac{T(u)}{u^2}-f(0)-uf'(0)$$

(iii) Can be proof by mathematical induction

2.4. The Inverse of Elzaki Transform

Definition: Let the functions $\bar{f}(u) = E\{f\}$ is the Elzaki transform of the function $f(t)$, then $f(t)$ called the inverse transform of the function $\bar{f}(u)$ and we will write it as :

$$f(t) = E^{-1}\{\bar{f}(u)\}$$

Remark: The inverse transform has the linear combination property, i.e.

$$E^{-1}\left\{\sum_{k=1}^{n} a_k \bar{f}_k(u)\right\} = \sum_{k=1}^{n} a_k E^{-1}\{\bar{f}_k(u)\}$$

2.5. Solve the Differential Equations

As stated in the introduction of this paper, the ELzaki transform can be used as an effective tool. For analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of the ELzaki transform in solving certain initial value problems described by ordinary differential equations.

Consider the first-order ordinary differential equation.

$$\frac{dx}{dt} + px = f(t) \quad , \quad t > 0 \qquad (2.3)$$

With the initial Condition

$$x(0) = a \qquad (2.4)$$

Where p and a are constants and $f(t)$ is an external input function so that its ELzaki transform exists.

Applying ELzaki transform of the equation (2.3) we have:

$$\frac{1}{u} E(x) - u\, x(0) + p\, E(x) = E(\mathrm{f})$$

$$E(x) = \frac{uE(f)}{1+pu} + \frac{au^2}{1+pu}$$

The inverse ELzaki transform leads to the solution.

The second order linear ordinary differential equation has the general form.

$$\frac{d^2y}{dx^2} + 2\,\mathrm{p}\,\frac{dx}{dy} + q\,y = f(x)\ , x > 0 \qquad (2.5)$$

The initial conditions are

$$y(0) = a\ , \quad \frac{dy}{dx}(0) = b \qquad (2.6)$$

Are constants. Application of the ELzaki transforms b and p,q,a where to this general initial value problem gives

$$\frac{1}{u^2} E(y) - y(0) - u\grave{y}(0) + 2p\left[\frac{1}{u} E(y) - uy(0)\right] + \mathrm{q}E(y) = E(f)$$

$$E(y) = \frac{u^2E(f)}{qu^2+2pu+1} + \frac{au^2}{qu^2+2pu+1} + \frac{(b+2pa)u^3}{qu^2+2pu+1}$$

The inverse transform gives the solution.

<u>Example</u> (1): Consider the first order differential equation

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} + \mathrm{y} = 0 \quad , \quad y(0) = 1$$

We have

$$\frac{1}{u}\mathrm{E(y)} - \mathrm{uy(a)} + \mathrm{E(y)} = 0 \quad \rightarrow \quad \left(\frac{1}{u} + 1\right)\mathrm{E(y)} - \mathrm{u} = 0 \quad \rightarrow \quad \mathrm{E(y)} = \frac{u}{\frac{1+u}{u}}$$

$$\rightarrow \quad \mathrm{E(y)} = \frac{u^2}{1+u}$$

By the inverse of Elzaki transform, we get the Solution: y= e^{-x}

<u>Example</u> (2): To solve the differential equation

$$y' + 2\,y = x \quad , \quad y(0) = 1$$

We get

$$\mathrm{E}(y') + 2\,\mathrm{E}(y) = \mathrm{E(x)} \quad \rightarrow \quad \frac{1}{u}\mathrm{E(y)} - \mathrm{u\,y(0)} + 2\,\mathrm{E(y)} = \mathrm{u}^3 \quad \rightarrow$$

$$\rightarrow \quad \mathrm{E(y)} = \frac{(u^3+u)u}{1+2u} \quad \rightarrow \quad \mathrm{E(y)} = \frac{u^3}{2} + \frac{5}{4}\left(\frac{u^2}{1+2u}\right) - \frac{1}{4}u^2$$

Then we can get the solution of this problem by the inverse

$$y(x) = \frac{x}{2} + \frac{5}{4}e^{-2x} - \frac{1}{4}$$

Example (3): Let we have

$$y'' + y = 0\ , \quad y(0) = y'(0) = 1$$

By take ELzaki transform

$$E(y'') + E(y) = 0 \ \rightarrow\ \frac{1}{u^2}E(y) - y(0) - uy'(0) + E(y) = 0 \ \rightarrow\ \frac{1}{u^2}E(y) - 1 - u + E(y) = 0$$

$$\rightarrow\quad E(y) = \frac{u^2+u^3}{1+u^2} \qquad \rightarrow \qquad E(y) = \frac{u^2}{1+u^2} + \frac{u^3}{1+u^2}$$

Then the solution is: $\quad y(x) = \sin x + \cos x$

Example (4): Consider the following equation

$$y'' - 3y' + 2y = 0 \quad , y(0) = 1\ , y'(0) = 4$$

Take ELzaki transform of this equation we find that:

$$E(y) = \frac{u^2(u+1)}{(2u-1)(u-1)} = \frac{-2u^2}{1-u} + \frac{3u^2}{1-2u}$$

Then the solution is $y(x) = -2e^{x} + 3e^{2x}$

Example (5)

$$y'' + 9y = \cos 2t \qquad y(0) = 1\ , y'(0) = 1\ .$$

Take Elzaki transform of this equation and using the conditions, we have

$$\frac{E(y)}{u^2} - u + 9\,E(y) = \frac{u^2}{4u^2+1}$$

Which we can write as:

$$E(y) = u^2\left[\frac{4}{5(1+9u^2)} + \frac{u}{3(1+9u^2)} + \frac{1}{5(1+4u^2)}\right]$$

Therefore, the solution is

$$y = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$$

Example (6):

To solve the differential equation**:**

$$y'' - 3y' + 2y = 4e^{3x}\ , y(0) = -3\ , y'(0) = 5$$

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Taking the Elzaki transforms both side of the differential equation and using the given conditions we have,

$$\frac{E(y)}{u^2} + 3\text{-}5u\text{-}3[\frac{E(y)}{u} + 3u] + 2\,E(y) = \frac{4u^2}{1-3u} \quad \rightarrow$$

$$T(u) = u^2\frac{4}{1-2u} + \frac{2}{1-3u} - \frac{9}{1-u}$$

The solution is:

$$y(x) = 4e^{2x} + 2e^{3x} - 9e^{x}$$

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