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الإهداء

الى من كان ولم يزال معلمي عند جهلي في حياتي وضيائي في الظلمات الى التي أمطرت من زمن الجفاف والظمأ...........

والدي العزيز

الى من لا يكل اللسان بالدعاء لها وفاء ....

الى من إلا تمل العين من رؤية وجهها ....

الى منبع التضحية وبحر الحنان وحضن الأمان ....

والدتي العزيزة

الى ابلغ المعاني واصدق المشاعر وأحلى الصور.....

أعمامي الأحبة

أولاد خالي الأوفياء

الى من عبروا معي محطات الزمن خطوة بعد خطوة......

أصدقائي الأوفياء

الى من بنو بنيات لبنة تلو الأخرى ....ينابيع العطاء.....

أساتذتي المخلصين

مع خالص احترامي وتقديري

الى كل من ساعدوني في انجاز هذا البحث

اهدي هذا الجهد المتواضع

الشكر و التقدير

الحمد الله رب العالمين والصلاة والسلام على سيد المرسلين محمد (صَلَّى اللَّهُ عَلَيْهِ وَآلِهِ وَسَلَّمَ).

يسرني وقد انتهيت من إعداد بحثي هذا ، أن اشكر أولا الخالق الباري عز وجل .

كما أتقدم بالشكر الجزيل الى من ساعدني في انجاز هذا البحث وخصوصا الدكتورة

(الاء حسين محمد)التي تّفضل مشكورة بالإشراف على هذا البحث ،

فقد كانت لي الأستاذة لما قدمت من توجيهات وأراء علميه وعلاقة إنسانية طيبة، اسال الله تعالى أن يمن على جميع بالصحة والعافية العمر المديد وان يسدد خطاهم.

والله ولي التوفيق

Introduction :

We take a closer look at linear continuous maps between Hilbert spaces these are often called bounded operators, and the branch of functional Analysis that studies these objects is called operator theory.

Abstract :

In this research , we introduce the notation of Operator between Hilbert spaces, and given some properties of them .

CHAPTER ONE:

Preliminariers .

Definition (1-1)

Let N be a vector space over a field F (F= R or C).

N is called a normal space over a field F , if there exists a map

$\|.\|: \mathbf{N} \to \mathbf{R}^{+}$ satisfies the folloming axioms :

1- $\|\mathbf{x}\| \geq \mathbf{0}$, $\forall\, \mathbf{x} \in \mathbf{N}$, $\|\mathbf{x}\| = 0$ iff **x**= 0 .
2- $\|\boldsymbol{\alpha}\mathbf{x}\| = |\boldsymbol{\alpha}|\,\|\mathbf{x}\|$, $\forall\, \mathbf{x} \in \mathbf{N}$, $\forall\, \boldsymbol{\alpha} \in \mathbf{F}$.
3- $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, $\forall\, \mathbf{x},\mathbf{y} \in \mathbf{N}$, (triangle inquality) .

$\|.\|$ is called a normal on **N** .

(**N**, $\|.\|$) is called a normal space .

Remark (1-2)

Any norm space is a vector space but the converse is not true in general .

Definition(1-3)

Let **N** be a normed spacs over afield **F** and let $\langle \mathbf{x_n} \rangle$ be sequence in **N** , $\langle \mathbf{x_n} \rangle$ is said to be convergent sequence , if there exists **a** ∈ **N** such that $\forall\, \boldsymbol{\epsilon} > 0\ \exists\ \mathbf{K} \in \mathbb{N}$ such that $\|\mathbf{x_n} - a\| < \epsilon$, $\forall\, \mathbf{n} > \mathbf{K}$.

Definition (1-4)

Let **N** be a normed space over a field **F** and let $\langle \mathbf{x_n} \rangle$ sequence in **N** , $\langle \mathbf{x_n} \rangle$ is said to be Cauchy sequence , if $\forall\, \boldsymbol{\epsilon} > 0$, $\exists\ \mathbf{K} \in \mathbb{N}$ such that $|\mathbf{x_n} - \mathbf{y_m}| < \boldsymbol{\epsilon}$, $\forall\, \mathbf{n},\mathbf{m} > \mathbf{K}$.

Remark(1-5)

Every convergent sequence is a Cauchy but the converse is not true .

Definition(1-6)

Let **X** be a normed space . **X** is said to be complete if every Cauchy sequence in **X** is convergent.

Definition (1-7)

Every complete normed space is called Banach space.

Definition(1-8)

Let **X**,**Y** are normed space and **f**: **X** → **Y** be a function , **f** is said to be continuous on $\mathbf{x_0} \in$ **X** if such that $\mathbf{x_n} \to \mathbf{x_0}$ then $\mathbf{f(x_n)} \to \mathbf{f(x_0)}$.

Proposition (1-9)

A norm $\|.\| : \mathbf{N} \to \mathbf{R}^+$ is continuous function on **N** . i.e if $\mathbf{x_n} \to \mathbf{x_0}$ in **N**, Then $\|\mathbf{x_n}\| \to \|\mathbf{x_0}\|$ in $\mathbf{R}^+$.

Proof:

Since $\mathbf{x_n} \to \mathbf{x_0}$ then $\|\mathbf{x_n}-\mathbf{x_0}\| \to 0$, as **n**→ ∞

Claim : $\|\mathbf{x_n}\| \to \|\mathbf{x_0}\|$,

i.e $|\|\mathbf{x_n}\| - \|\mathbf{x_0}\|| \to 0$

$|\|\mathbf{x_n}\| - \|\mathbf{x_0}\|| \leq \|\mathbf{x_n}-\mathbf{x_0}\| \to 0$, as **n**→ ∞.

Thus $|\|\mathbf{x_n}\| - \|\mathbf{x_0}\|| \to 0$, as **n**→ ∞.

Thus the claim hold .

i.e $\|.\|$ is a continuous function .

Definition (1-10)

L(**N** , **N***)={ **T**: **N** → **N*** } ,**L** (**N**,**N***) is vector space over **F** , **T** is linear .

Proof;

1- Let $\mathbf{T_{1\ x}}$, $\mathbf{T_{2\ x}}$ ∈ **L**(**N**,**N***) $\mathbf{T_1}$,$\mathbf{T_2}$ are linear transformation .
$\Longrightarrow$ $(\mathbf{T_1}\ ,\ \mathbf{T_2}\ )_{\mathbf{x}}$ is linear .

Thus $\mathbf{T_1} + \mathbf{T_2}$ = **L**(**N** , **N***) .

2- **c**$(\mathbf{T_1} + \mathbf{T_2}\ )_{\mathbf{x}}=(\mathbf{T_1}\ ,\ \mathbf{T_2}\ )_{\mathbf{cx}}$

$= \mathbf{T_{1\,(\,cx)}}+\mathbf{T_{2\,(\,cx)}}=c\mathbf{T_{1\ x}}+\mathbf{c}\ \mathbf{T_{2\ x}}$.

3 – $(\mathbf{c_1}\ \mathbf{c_2})\ \mathbf{T_x} = \mathbf{T_{(c_1\,c_2\,x)}} = c_1(\mathbf{T_{c_2\,x}}) = \mathbf{c_1}(\mathbf{c_2}\mathbf{T_x})$.

4 - **1**. $\mathbf{T_x} = \mathbf{T_{(1.x)}} = \mathbf{T_x}$.

5 - $(\mathbf{T_1} + \mathbf{T_2}\ )_{\mathbf{x}}=\mathbf{T_{1\ x}}+\mathbf{T_{2\ x}}=\mathbf{T_{2\ x}}+\mathbf{T_{1\ x}}=(\mathbf{T_2} + \mathbf{T_1}\ )_{\mathbf{x}}$.

Thus **L**(**N** , **N***) is vector space .

Definition (1-11)

Let **X** be a vector space over a field **F** , **X** is said to be inner product space over **F** if there exists a function $\langle , \rangle$: **X** × **X**⟶ **F** , (**F**=**R** or **C**) satisfies the following axioms :

1- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle$ =0 iff **x**=0 , ∀ **x** ∈ **X** .
2- $\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$, ∀ **x**,**y** ∈ **X** , λ ∈**F**.
3- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$, ∀ **x**,**y** ∈ **X** .
4- $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.

Theorem (1-12) **(Cauchy Shwartz inquality)**

let X be an inner product space then

$|\langle \mathbf{x},\mathbf{y}\rangle| \leq \|\mathbf{x}\|\ \|\mathbf{y}\|$, $\forall\ \mathbf{x},\mathbf{y} \in \mathbf{X}$.

Theorem (1-13)

Every inner product space is normed space .

Proof:

Let **X** be inner product space .

There exists function $\langle , \rangle : \mathbf{X} \times \mathbf{X} \longrightarrow \mathbf{F}$ satisfies the previous (**4**) axioms above .

We must prove that **X** is normed space .

Then we define the function $\|.\| : \mathbf{X} \longrightarrow \mathbf{R}$ is follows

$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$, $\mathbf{x} \in \mathbf{X}$

1- $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x},\mathbf{x}\rangle} \geq 0$, $\forall\ \mathbf{x}$.

$\|\mathbf{x}\|=0$ iff $\sqrt{\langle \mathbf{x},\mathbf{x}\rangle}=0$ iff $\langle \mathbf{x},\mathbf{x}\rangle=0$ iff $\mathbf{x}=0$.

2- $\|\lambda\mathbf{x}\| = \sqrt{\langle \lambda\mathbf{x},\lambda\mathbf{x}\rangle} = \sqrt{\lambda\,\langle \mathbf{x},\lambda\mathbf{x}\rangle}$

$= \sqrt{\lambda\,\overline{\langle \lambda\mathbf{x},\mathbf{x}\rangle}} = \sqrt{\lambda\,\overline{\lambda}\overline{\langle \mathbf{x},\mathbf{x}\rangle}} = \sqrt{\lambda\,\overline{\lambda}\,\langle \mathbf{x},\mathbf{x}\rangle}$

$= \sqrt{|\lambda|^2\ \|\mathbf{x}\|^2} = |\lambda|\ \|\mathbf{x}\|$.

3- $\|\mathbf{x}+\mathbf{y}\| = \sqrt{\langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle} = \sqrt{\langle \mathbf{x}+\mathbf{y},\mathbf{x}\rangle,\langle \mathbf{x}+\mathbf{y},\mathbf{y}\rangle}$

$= \sqrt{\overline{\langle \mathbf{x},\mathbf{x}+\mathbf{y}\rangle} + \overline{\langle \mathbf{y},\mathbf{x}+\mathbf{y}\rangle}}$

$= \sqrt{\overline{\langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{x},\mathbf{y}\rangle} + \overline{\langle \mathbf{y},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle}}$

$$= \sqrt{\overline{\langle \mathbf{x},\mathbf{x}\rangle} + \overline{\langle \mathbf{x},\mathbf{y}\rangle} + \overline{\langle \mathbf{y},\mathbf{x}\rangle} + \overline{\langle \mathbf{y},\mathbf{y}\rangle}}$$

$$= \sqrt{\|\mathbf{x}\|^2 + \overline{\langle \mathbf{x},\mathbf{y}\rangle} + \overline{\langle \mathbf{y},\mathbf{x}\rangle} + \|\mathbf{y}\|^2}$$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\,\langle x,y\rangle$.

By Theorem (1-12)

Re $\langle \mathbf{x},\mathbf{y}\rangle \le |$ **Re** $\langle \mathbf{x},\mathbf{y}\rangle| \le |\langle \mathbf{x},\mathbf{y}\rangle| \le \|\mathbf{x}\|\ \|\mathbf{y}\|$.

Thus $\|x+y\|^2 \le \|x\|^2 + \|y\|^2 + 2\|x\|\ \|y\|$

$= (\|x\| + \|y\|)^2$

Thus $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$

From (**1**), (**2**) and (**3**) we have ($\mathbf{X}$, $\|.\|$) is normed space .

Definition (1-14)

A Hilbert space over **F** is a complete inner product space .

Remark (1-15)

Every Hilbert space is a Banach space but the converse is not true in general.

Definition (1-16)

Given operator **T** ∈**B**(**H1**, **H2**), the unique operator

S ∈**B**(**H2**, **H1**) that satisfies[(**Te**1|**e**2)H2 =(**e**1|**Se**2)H1,

∀ (**e**1,**e**2) ∈ **H1xH2**] is called the adjoint of **T**, and is denoted by $\mathbf{T}^*$. Bythe above Remark, for any two vectors **e1** ∈**H1**, **e2** ∈**H2**, we have the identities :

(**T e1**|**e2**)H2 = (**e1**| $\mathbf{T}^*$**e2**)H1.

(**e2**|**T e1**)H2 = ($\mathbf{T}^*$**e2**|**e1**)H1.

Example:

$\mathbf{B} : \mathbf{L}1(\mathbf{C}) \to \mathbf{L}2(\mathbf{C})$ $\quad \mathbf{B}(\mathbf{x}1, \mathbf{x}2, \ldots) = (\mathbf{x}2, \mathbf{x}3, \ldots)$

$\mathbf{U}(\mathbf{x}1, \mathbf{x}2, \ldots) = (0, \mathbf{x}1, \mathbf{x}2, \ldots)$

Claim $\mathbf{U}^* = \mathbf{B}$, $\mathbf{B}^* = \mathbf{U}$

Proof:

Let $\mathbf{x}, \mathbf{y} \in \mathbf{H}$

To prove $\langle \mathbf{Ux}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{U}^* \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{By} \rangle$.

$\langle \mathbf{Ux}, \mathbf{y} \rangle = \langle (0, \mathbf{x}1\ \mathbf{x}2, \ldots), (\mathbf{y}1, \mathbf{y}2, \ldots) \rangle$

$= 0.\mathbf{y}1 + \mathbf{x}1\mathbf{y}2 + \mathbf{x}2\mathbf{y}3 + \ldots$

$= \mathbf{x}1\mathbf{y}2 + \mathbf{x}2\mathbf{y}3 + \ldots$

$= \langle (\mathbf{x}1\ \mathbf{x}2, \ldots), (\mathbf{y}2, \mathbf{y}3, \ldots) \rangle$

Thus $\langle \mathbf{Ux}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{By} \rangle$.

Thus $\langle \mathbf{Ux}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{By} \rangle = \langle \mathbf{x}, \mathbf{U}^* \mathbf{y} \rangle$.

Thus $\mathbf{By} = \mathbf{U}^* \mathbf{y}$, $\forall\ \mathbf{y} \in \mathbf{H}$, Thus $\mathbf{B} = \mathbf{U}^*$

Now $\mathbf{U}^* = \mathbf{B} \implies \mathbf{U}^{**} = \mathbf{B}^* \implies \mathbf{U} = \mathbf{B}^*$.

Definition (1-17)

T : v(f) $\to$ u(f) , then Range of T,

R(T) = { y $\in$ u(f) ,such that y = T(x) , x $\in$ V } ,

And Kernal space of T ,N(T)

N(T) = {x $\in$ V(f), such that T(x) = 0 } .

R(T) is subspace of u(f) .

N(T) is subspace of v(f) .

CHAPTER TOW:

Main Results .

Lemma(2.1)

Let **X** and **Y** be normed vector spaces. For a sesquilinear map $\boldsymbol{\varphi} : \mathbf{X} \times \mathbf{Y} \longrightarrow \mathbf{C}$, the following are equivalent:

(**i**) $\boldsymbol{\varphi}$ is→ continuous;

(**ii**) $\boldsymbol{\varphi}$ is continuous at (0, 0);

(**iii**) $\sup\{|\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| : (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathrm{Y}, \|\mathbf{x}\| . \|\mathbf{y}\| \leq 1\} < \infty$;

(**iv**) there exists some constant $\mathbf{C} \geq 0$, such that

$$|\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| \leq \mathbf{C} \cdot \|\mathbf{x}\| . \|\mathbf{y}\|, \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y} .$$

Moreover, the number in (**iii**) is equal to

$\min\{ \mathbf{C} \geq 0 : |\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| \leq \mathbf{C} \cdot \|\mathbf{x}\| . \|\mathbf{y}\| ,\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}\}$ …….(1).

Proof:

The implication (**i**) ⇒ (**ii**) is trivial.

(**ii**) ⇒ (**iii**)

. Assume φ is continuous at (0, 0).

We prove (**iii**) by contradiction.

Assume, for each integer $\mathbf{n} \geq 1$ there are vectors $\mathbf{x_n} \in \mathbf{X}$ and $\mathbf{y_n} \in \mathbf{Y}$ with $\|\mathbf{x}\|$, $\|\mathbf{y}\| \leq 1$, but such that $|\boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y_n})| \geq \mathbf{n^2}$.

If we take $\mathbf{v_n} = \frac{1}{n}\mathbf{x_n}$ and $\mathbf{w_{n=}y_n}\frac{1}{n}$

then on the one hand we have $\|\mathbf{x}\| . \|\mathbf{y}\| \leq \frac{1}{n}, \forall\, \mathbf{n} \geq 1$, which forces $\lim_{n\to\infty} (\mathbf{v_n}, \mathbf{w_n}) = (0, 0)$ in $\mathbf{X} \times \mathbf{Y}$, so by (**iii**) we have $\lim_{n\to\infty} \boldsymbol{\varphi}(\mathbf{v_n}, \mathbf{w_n}) = 0$.

the other hand, we also haveo

$$|\boldsymbol{\varphi}(\mathbf{v_n}, \mathbf{w_n})| = \frac{|\varphi(\mathbf{x_n}),(\mathbf{w_n})|}{\mathbf{n^2}} \geq 1, \forall\, \mathbf{n} \geq 1,$$

which is impossible.

(**iii**) ⇒ (**iv**).

Assume $\boldsymbol{\varphi}$ has property (**iii**), and denote the number

$\sup\{ |\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| :(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}, \|\mathbf{x}\| . \|\mathbf{y}\| \leq 1\}$

simply by **M**.

In order to prove (**iv**) we are going to prove the inequality

$|\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| \leq \mathbf{M}. \|\mathbf{x}\| . \|\mathbf{y}\|, \ \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$ ....(2)

Fix $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$.

If either $\mathbf{x} = 0$ or $\mathbf{y} = 0$, the above inequality is trivial, so we can assume both **x** and **y** are non-zero.

Consider the vectors $\mathbf{v} = \frac{1}{\|\mathbf{x}\|} \mathbf{x}$ and $w = \frac{1}{\|\mathbf{y}\|} \mathbf{y}$.

We clearly have

$|\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| = | \boldsymbol{\varphi}(\|\mathbf{x}\| \mathbf{v}, \|\mathbf{y}\| \mathbf{w}) = \|\mathbf{x}\| \, \|\mathbf{y}\| . |\boldsymbol{\varphi}(\mathbf{v}, \mathbf{w})|$.

Since $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$, we have $|\boldsymbol{\varphi}(\mathbf{v}, \mathbf{w})| \leq \mathbf{M}$, so the above inequality gives (**2**).

(**iv**) $\Rightarrow$ (**i**).

Assume φ has property (**iv**) and let us show that $\boldsymbol{\varphi}$ is continuous.

Let $\mathbf{C} \geq 0$ is as in (**iv**). Let $(\mathbf{x_n})_{n\to\infty} \subset \mathbf{X}$ and $(\mathbf{y_n})_{n\to\infty} \subset \mathbf{Y}$ be convergent sequences with $\lim\limits_{n\to\infty} \mathbf{x_n} = \mathbf{x}$ and $\lim\limits_{n\to\infty} \mathbf{y_n} = \mathbf{y}$, and let us prove that $\lim\limits_{n\to\infty} \boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y_n}) = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})$.

Using (**iv**) we have

$|\boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y_n}) - \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| \leq |\boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y_n}) - \boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y})| + |\boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y}) - \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})|$

$= |\boldsymbol{\varphi}(\mathbf{x_n}. \mathbf{y_n} - \mathbf{y})| + |\boldsymbol{\varphi}(\mathbf{x_n} - \mathbf{x}, \mathbf{y})|$

$\leq \mathbf{C} . \|\mathbf{x_n}\| . \|\mathbf{y_n} - \mathbf{y}\| + \mathbf{C} . \|\mathbf{x_n} - \mathbf{x}\| . \|\mathbf{y}\|, \ \forall \mathbf{n} \geq 1$,

which clearly forces $\lim\limits_{n\to\infty} |\boldsymbol{\varphi}(\mathbf{x_n}, \mathbf{y_n}) - \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| = 0$, and we are done.

To prove the last assertion we observe first that every **C** ≥ 0 with
$|\boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})| \leq \mathbf{C} \,.\, \|\mathbf{x}\| . \|\mathbf{y}\|$, $\forall$ $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{Y}$,
automatically satisfies the inequality $\mathbf{C} \geq \mathbf{M}$.

This is a consequence of the above inequality, restricted to those

$(\mathbf{x}, \mathbf{y}) \in \mathbf{X}\times\mathbf{Y}$, with $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$.

To finish the proof ,all we have to prove is the fact that $\mathbf{C} = \mathbf{M}$ satisfies (**iv**).

But this has already been obtained when we proved the implication

(**iii**) $\Rightarrow$ (**iv**).

Notation (2-2)

With the notations above, the number defined in (**iii**), which is also equal to the quantity (1), is denoted by $\|\boldsymbol{\varphi}\|$. This is justified by the following.

Lemma(2-3)

let **M** subspace actally is closed in normal space **X** and let **Y** a number real so that $\mathbf{0<Y<1}$ the exists $\mathbf{X_Y} \in \mathbf{X}$, $\|\mathbf{x} - \mathbf{X_Y}\| \geq \mathbf{Y}$, $\forall$ $\mathbf{x} \in \mathbf{M}$

Notation (2-4)

Let **X** and **Y** be normed vector spaces over **C**.

Prove that the space

S(**X**, **Y**) ={ **φ** : **X** × **Y** → **C** : **φ** sesquilinear continuous}

is a vector space, when equipped with pointwise addition and scalar multiplication.

Prove that the map

S(**X**, **Y**) ∋ **φ** -→ ‖**φ** ‖∈ [0, ∞)

defines a norm.

With this terminology, we have the following technical result.

Theorem(2-5).

Let **H1** and **H2** be Hilbert spaces, and let **φ** : **H1** × **H2** → **C** be a sesquilinear map. The following are equivalent .

(**i**) **φ** is continuous;

(**ii**) there exists **T** ∈**B** (**H1**, **H2**), such that

φ(**e1**, **e2**) = (**T e1**/**e2**)H2, ∀**e1**, **e2**) ∈**H1** × **H2**,

where (. | .)H2 denotes the inner product on **H2**.

Moreover, the operator **T** ∈ **B**(**H1**, **H2**) is unique, and has norm

‖**T**‖=‖**φ**‖.

Proof. (**i**) ⇒ (**ii**).

Assume φ is continuous, so by Lemma(2-1) we have

|**φ**(**e**,**z**)| ≤ ‖**φ** ‖.‖e ‖ . ‖z ‖, ∀ e ∈ **H1**, **z** ∈**H2**……….(3).

Fix for the moment e ∈**H1**, and consider the map

φe: **H2** ∋ **z** -→ **φ**(**e**, **z**)∈ C.

Using (**3**), it is clear that **φe** : **H2** → **C** is linear continuous, and has norm $\|\boldsymbol{\varphi}e\| \leq \|\boldsymbol{\varphi}\| \,.\, \|e\|$.

Using Riesz' Theorem, it follows that there exists a unique

vector $\tilde{\mathbf{e}} \in \mathbf{H2}$, such that

$\boldsymbol{\varphi}e(z) = (\tilde{\mathbf{e}}|\mathbf{z})H2, \ \forall \mathbf{z} \in \mathbf{H2}.$

Moreover, one has the equality

$\|\tilde{\mathbf{e}}\|H2 = \|\boldsymbol{\varphi}_{\mathbf{e}}\| \leq \|\boldsymbol{\varphi}\| \,.\, \|\mathbf{e}\|H1$……………(4).

Remark that, if we start with two vectors **e**, **q** ×∈H1, then we have

$(\tilde{\mathbf{e}}|\mathbf{z})H2 + (\tilde{\mathbf{q}}|\mathbf{z})H2 = \boldsymbol{\varphi}(\mathbf{e}, \mathbf{z}) + \boldsymbol{\varphi}(\mathbf{q}, \mathbf{z}) = \boldsymbol{\varphi}(\mathbf{e} + \mathbf{q}, \mathbf{z}) = \boldsymbol{\varphi}_{\mathbf{e}+\mathbf{q}}(\mathbf{z}), \ \forall \mathbf{z} \in \mathbf{H2},$

so by the uniqueness part in Riesz' lemma we get the equality

$\widetilde{\mathbf{e}+\mathbf{q}} = \tilde{\mathbf{e}} + \tilde{\mathbf{q}}$.

Likewise, if **e** ∈ **H1**, and **λ** ∈**C**, we have

$(\lambda\tilde{\mathbf{e}}|\mathbf{z})H2 = \tilde{\boldsymbol{\lambda}}(\tilde{\mathbf{e}}|\mathbf{z})H2 = \tilde{\boldsymbol{\lambda}}\, \boldsymbol{\varphi}(\mathbf{e}, \mathbf{z}) = \boldsymbol{\varphi}(\boldsymbol{\lambda}\mathbf{e}, \mathbf{z}) = \boldsymbol{\varphi}_{\boldsymbol{\lambda}\mathbf{e}}(\mathbf{z}), \ \forall \mathbf{z} \in \mathbf{H2},$

which forces $\widetilde{\boldsymbol{\lambda}\mathbf{e}} = \boldsymbol{\lambda}\, \tilde{\mathbf{e}}$ **.**

This way we have defined a linear map

$\mathbf{T} : \mathbf{H1} \ni \mathbf{e} \longrightarrow \tilde{\mathbf{e}} \in \mathbf{H2},$

with

$\boldsymbol{\varphi}(\mathbf{e}, \mathbf{z}) = (\mathbf{T}\,\mathbf{e}\,|\mathbf{z})H2, \ \forall (\mathbf{e}, \mathbf{z}) \in \mathbf{H1} \times \mathbf{H2}.$

Using (**4**) we also have

$\|\mathbf{T}\,\mathbf{e}\|H2 \leq \|\boldsymbol{\varphi}\| \cdot \|\mathbf{e}\|H1, \ \forall \mathbf{x} \in \mathbf{H1}$,so **T** is indeed continuous, and it has norm $\|\mathbf{T}\| \leq \|\boldsymbol{\varphi}\|$.

The uniqueness of **T** is obvious.

(**ii**) ⇒ (**i**).

Assume **φ** has property (**ii**), and let us prove that φ is continuous.

This is pretty clear, because if we take **T** ∈**B**(**H1**, **H2**) as in (**ii**), then using the Cauchy-Bunyakovski-Schwartz inequality we have

$|\boldsymbol{\varphi}(\mathbf{e1}, \mathbf{e2})| = |(\mathbf{T}\ \mathbf{e1}|\mathbf{e2})H2| \le \|\mathbf{T}\ \mathbf{e1}\| \cdot \|\mathbf{e2}\| \le \|\mathbf{T}\| \cdot \|\mathbf{e1}\| \cdot \|\mathbf{e2}\|$,

$\forall(\mathbf{e1},\mathbf{e2}) \in \mathbf{H1} \times \mathbf{H2}$,

so we can apply Lemma(2-1). Notice that this also proves an the inequality $\|\boldsymbol{\varphi}\| \le \|\mathbf{T}\|$.

Since by the proof of the implication (**i**) ⇒ (**ii**) we already know that $\|\mathbf{T}\| \le \|\boldsymbol{\varphi}\|$,it follows that in fact we have equality $\|\mathbf{T}\| = \|\boldsymbol{\varphi}\|$.

<u>Proposition</u>(2-6).

A. For two Hilbert spaces **H1**, **H2**, one has

$\|\ \mathbf{T}^*\| = \|\mathbf{T}\|,\ \forall \mathbf{T} \in \mathbf{B}(\mathbf{H1}, \mathbf{H2})$;……………………..(5)

$(\ \mathbf{T}^*)^* = \mathbf{T},\ \forall \mathbf{T} \in \mathbf{B}(\mathbf{H1}, \mathbf{H2})$;…………………………(6)

$(\mathbf{S}+\mathbf{T})^* = \boldsymbol{S}^* + \mathbf{T}^*\quad \forall\ \mathbf{S}, \mathbf{T} \in\ \mathbf{B}(\mathbf{H1}, \mathbf{H2})$;…………….(7)

$(\boldsymbol{\lambda}\mathbf{T}\ \mathbf{T}^* = \tilde{\boldsymbol{\lambda}}\ \mathbf{T}^*, \forall \mathbf{T} \in \mathbf{B}(\mathbf{H1}, \mathbf{H2}),\ \boldsymbol{\lambda} \in \mathbf{C}$;…………….(8)

B. Given three Hilbert spaces **H1**, **H2**, and **H3**, one has

$(\mathbf{ST}\ )^* = \mathbf{T}^*\ \boldsymbol{S}^*,\ \forall\ \mathbf{T} \in\ \mathbf{B}(\mathbf{H1}, \mathbf{H2}),\ \mathbf{S} \in \mathbf{B}(\mathbf{H2}, \mathbf{H3})$…….(9)

<u>Proof</u>.

The equality (5) has already been discussed in Remark The identity (6) is obvious.

To prove the other identities we employ the following strategy.

We denote by **X** the operator whose adjoint is the left hand side, we denote by **Y**
the operator in the right hand side, so we must show $\mathbf{X}^* = \mathbf{Y}$, and we prove this equality by proving the equality

$(\mathbf{X}\mathbf{e}|\mathbf{q}) = (\mathbf{e}|\mathbf{Y}\ \mathbf{q}),\ \forall\ \mathbf{e}, \mathbf{q}$.

For example, to prove (**8**) we put $\mathbf{X} = \mathbf{S} + \mathbf{T}$ and $\mathbf{Y} = \mathbf{S}^* + \mathbf{T}^*$, and it is pretty

obvious that

$(\mathbf{Xe}|\mathbf{q}) = (\mathbf{Se}+ \mathbf{T\ e}|\mathbf{q}) = (\mathbf{Se}|\mathbf{q}) + (\mathbf{T\ e}|\mathbf{q}) = (\mathbf{e}|\ \mathbf{S}^{*}\mathbf{q}) + (\mathbf{e}|\ \mathbf{T}^{*}\ \mathbf{q})$

$= (\mathbf{e}|\ \mathbf{S}^{*}\mathbf{q}+\ \mathbf{T}^{*}\ \mathbf{q}) = (\mathbf{e}|\mathbf{Y\ q}), \forall\ \mathbf{e} \in \mathbf{H1}, \mathbf{q} \in \mathbf{H2}.$

The other identities are proven the exact same way.

<u>Proposition</u>(2-7) (Kernel-Range Identities).

Let **H1** and **H2** be Hilbert spaces .For any operator $\mathbf{T} \in \mathbf{B}(\mathbf{H1}, \mathbf{H2})$, one has the equalities

(i)- $\mathbf{Ker}\ \mathbf{T}^{*} = (\mathbf{Ran\ T})^{\perp}$;

(**ii**)- $\mathbf{Ran}\ \mathbf{T}^{*} = (\mathbf{Ker\ T})^{\perp}$;

<u>Proof</u>.

(**i**). If we start with some vector $\mathbf{q} \in \mathbf{Ker\ T}^{*}$, then for every $\mathbf{e} \in \mathbf{H1}$, we have

$(\mathbf{q}|\mathbf{T\ e})\mathbf{H2} = (\ \mathbf{T}^{*}\mathbf{q}|\mathbf{e})H1 = 0,$

thus proving that $\mathbf{q} \perp \mathbf{T\ e}, \forall \mathbf{e} \in H1,$

i.e. $\mathbf{e} \in (\ \mathbf{Ran\ T})^{\perp}$;. This proves the inclusion

$\mathbf{Ker}\ \mathbf{T}^{*} = (\mathbf{Ran\ T})^{\perp}$..

To prove the inclusion in the other direction, we start with some vector

$\mathbf{q} \in \mathbf{Ker\ T}^{*} = (\mathbf{Ran\ T})^{\perp}$ and we prove that $\mathbf{T}^{*}\ \mathbf{q} = 0$. This is however pretty since we have $\mathbf{q} \perp (\mathbf{T}\ \ \mathbf{T}^{*}\mathbf{q})$, i.e.

$0 = (\mathbf{q}|\mathbf{T\ T}^{*}\mathbf{q})H2 = (\mathbf{T}^{*}\mathbf{q}|\ \mathbf{T}^{*}\mathbf{q})H1 = \|\ \mathbf{T}^{*}\mathbf{q}\ \|^{2},$

which forces $\mathbf{T}^{*}\mathbf{q} = 0$.

(**ii**). This follows immediately from part (**i**) applied to **T*:**

$\overline{\mathbf{Ran}\ \ \mathbf{T}^*}$ =([**Ran** $\mathbf{T}^*]^{\perp})^{\perp}$= (**Ker T**)$^{\perp}$.

<u>Remarks</u> (2-8).

A- Every self-adjoint operator **T** ∈ **B**(**H**) is normal.

B- The set {**T** ∈ **B**(**H**) : **T** normal } is closed in **B**(**H**), in the norm topology.

Indeed, if we start with a sequence $(\mathbf{T_n})_{n}^{\infty}$ =1 of normal operators, which converges (in norm) to some **T** ∈ **B**(**H**), then$(\mathbf{T_n})_{n}^{\infty}$ =1 converges to$\mathbf{T}^*$, and since the multiplication map

B(**H**) × **B**(**H**) ∋ (**X**, **Y**) -→ **XY** ∈ **B**(**H**)

is continuous, have $\boldsymbol{T}^*$**T** =$\lim_{n\to\infty} \boldsymbol{T}^*{}_n \boldsymbol{T_n}$ and **T** $\boldsymbol{T}^*$ =$\lim_{n\to\infty} \boldsymbol{T_n}\boldsymbol{T}^*{}_{\boldsymbol{n}}$,, so we immediately get $\boldsymbol{T}^*$**T** = **T** $\boldsymbol{T}^*$.

C- For **T** ∈ **B**(**H**), the following are equivalent (see Remark):

• **T** is self-adjoint

• the sesquilinear map

φT: **H** × **H** ∋ (**e**, **q**) -→ (**T e**|**q**)H ∈ **C**

is sesqui-symmetric, i.e. (**T e**|**q**) = (**T q**|**q**), ∀**e**, **q** ∈**H**;

• (**T e**|**e**) ∈ **R**, ∀**e** ∈**H**.

In particular, we see that every positive operator **T** is self-adjoint.

the condition that **T** is positive is equivalent to the condition that **φT** is positive definite.

D- The sets:

B(**H**)sa= {**T**∈ **B**(**H**) : $\boldsymbol{T}^*$= **T** },

B(**H**)+ = {**T**∈ **B**(**H**) : **T** positive }

are also closed in **B**(**H**).

This follows from the observation that, if$(\boldsymbol{T_n})_n^{\infty}$ =1 converges to some **T** , then we have

(**T e**|**e**) =$\lim_{n\to\infty}$ ($\boldsymbol{T_n}$**e**|**q**), ∀ **e**∈ **H**.

So if for example all $\boldsymbol{T_n}$'s are self-adjoint, then this proves that

(**T e**|**e**) ∈ **R**, ∀**e**, ∈ **H**,

so **T** is self adjoint. Likewise, if all $\boldsymbol{T_n}$'s are positive, then

(**T e**|**e**) ≥ 0, ∀**e** ∈ **H**, so **T** is positive.

E- Given Hilbert spaces **H1** and **H2**, and an operator **T** ∈ **B**(**H1**, **H2**), it follows

that the operators **T** $\boldsymbol{T}^*$ ∈ **B**(**H1**) and **T** $\boldsymbol{T}^*$ ∈ **B**(**H2**) are positive.

This is quite obvious, since

($\boldsymbol{T}^*$**T e**|**e**) = (**T e**|**T e**) = $\|\mathbf{T}\,\boldsymbol{e}\|^2$ ≥ 0, ∀ **e** ∈ H1,

(**T** $\boldsymbol{T}^*$**q**|**q**) = ($\boldsymbol{T}^*$**q**| $\boldsymbol{T}^*$**q**) =$\|\boldsymbol{T}^*\boldsymbol{q}\|^2$ ≥ 0, ∀ **q** ∈ **H2**.

F- The space **B**(**H**)sais a real linear subspace of **B**(**H**).

G - The space B(**H**)+ is a convex cone in **B**(**H**)sa, in the sense that

• if **S**, **T** ∈ **B**(**H**)+, then **S** + **T** ∈ **B**(**H**)+;

• if **S** ∈ **B**(**H**)+ and **α** ∈ [0, ∞), then **αS** ∈ **B**(**H**)+.

H- Using **G**, one can define a order relation on the real vector space **B**(**H**)saby

S ≥ **T** ⇔ **S** - **T** ∈ **B**(**H**)+.

This is equivalent to the inequality

(**Se**|**e**) ≥ (**T e**|**e**), ∀ **e** ∈ **H**.

The transitivity and reflexivity properties are clear. For the antisymmetry, one must show that if **T** ≥ **S** and **S** ≥ **T** , then **S** = **T** .

This is however clear, because the difference **X** = **S** - **T** is self-adjoint, and satisfies

(**Xz** |**z**) = 0, ∀ **z** ∈ **H**………………..(10)

Using polarization , we have

(**Xe**|**q**) = $\frac{1}{4}\sum_{k=0}^{3} \ i^{-k}\,(X(e+i^k\,q)\big|e+i^k q)$

and then (**10**) forces

(**Xe**|**q**) = 0 , ∀ **e**, **q** ∈ **H**,

we must have **X** = 0.

From now on, we are going to write **T** ≥ 0 to mean that **T** is positive.

Notation (2-9)

Prove that for an operator **T** ∈ **B**(**H**) the following are equivalent:

- **T** is normal;
- ‖**Te**‖=‖ T^***e**‖, ∀ **e** ∈ **H**.

Proposition(2-10).

Let **H** be a Hilbert space. For a bounded operator **Q** ∈**B**(**H**), the following are equivalent:

(**i**) there exists a closed subspace **X** ⊂**H**, such that **Q** = **PX** - the orthogonal projection onto **X**;

(**ii**) **Q** = Q^* =Q^2.

Proof.

The implication (**i**) ⇒ (**ii**) is trivial.

(**ii**) ⇒ (**i**).

Assume **Q** = Q^* =Q^2, and let us prove that **Q** is the orthogonal projection onto some closed subspace **X** ⊂**H**. We define **X** = **Ran Q**.

First of all, we must show that **X** is closed. This is pretty obvious, since the equality Q^2 = **Q**

gives the equality **X** = **Ker**(**L** - **Q**).

To prove that **Q** = **PX**, we must prove two things:

(**a**) **Qe**= **e**, ∀ **e** ∈**X**;

(**b**) **Qe**= 0, , ∀ **e** ∈ $X^{\perp}$.

The first property is clear, since **X** = **Ker**(**L** - **Q**), To prove the second property, we use Proposition(2-2) to get

$X^{\perp}$= (**Ran Q**)$^{\perp}$= **Ker** Q^* = **Ker Q**.

<u>Definitions</u>.(2-11)

Let **H1** and **H2** be Hilbert spaces.

A-. An operator **T** ∈**B**(**H1**, **H2**) is called an isometry, if

‖**Te**‖= ‖**e**‖, ∀ **e** ∈H1.

B-. An operator **T** ∈**B**(**H1**, **H2**) is said to be a coisometry, if its adjoint

T^* ∈ **B**(**H2**, **H1**) is an isometry.

C-. An operator **U** ∈**B**(**H1**, **H2**) is called a unitary, if **U** is a bijective isometry.

The algebraic characterizations for these types of operators are as follows.

Proposition(2-12).

Let **H1** and **H2** be Hilbert spaces.

A-. For an operator **T** ∈ B(**H1**, **H2**), the following are equivalent:

(**i**)- T is an isometry;

(**ii**)- T^*T = **L** H1.

B-. For an operator **T** ∈ **B**(**H1**, **H2**), the following are equivalent:

(**i**)- **T** is a coisometry;

(**ii**)- **T** T^* = **L**H2.

C-. For an operator **U** ∈ **B**(**H1**, **H2**), the following are equivalent:

(**i**)- **U** is unitary;

(**ii**) U^***U** = **L**H1 and **U** $\dot{U}^*$ = **L**H2.

Proof.

A. (**i**) ⇒ (**ii**).

Using polarization, applied to the sesquilinear form

φ : **H1** × **H1** ∋ (**e**, **q**) -⟶ (T^***T e**|**q**) ∈ C**,**

it follows that, for every **ξ**, **η** ∈ **H**, one has the equalities

$$\varphi(\mathbf{e}, \mathbf{q}) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\, \varphi\, (e{+}i^k . e + i^k q)$$

$$=\frac{1}{4}\sum_{k=0}^{3} i^{-k}\, (\, T^*T(e{+}i^k q)|e + i^k q)$$

$$. \qquad = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\, (T(e{+}i^k q)|T(e + i^k q)$$

$$. \qquad = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\, \|(T(e{+}i^k q)\|^2$$

Using the fact that **T** is an isometry, and polarization again (for the inner product), the above computation continues with

φ(**e**, **q**) $=\frac{1}{4}\sum_{k=0}^{3} i^{-k} \left\| (T(e+i^{k}q) \right\|^{2}$

. $=\frac{1}{4}\sum_{k=0}^{3} i^{-k} \left\| e+i^{k}q \right\|^{2}$

. $=\frac{1}{4}\sum_{k=0}^{3} i^{-k} (e+i^{k}q)(e+i^{k}q) = (e|q)$

Since we now have

(T^{*}**T e**|**q**) = (**e**|**q**), ∀ **e**, **q** ∈**H1**,

by Lemma(2-1) (the uniqueness part) we get T^{*}**T** =**L**H1.

The implication (**ii**) ⇒ (**i**) is trivial, since the equality T^{*}**T** = **L**H1 gives

$\left\| Te \right\|^{2}$ = (**T e**|**T e**) = (T^{*}**T e**|**e**) = (**e**|**e**) = $\left\| e \right\|^{2}$, ∀ **x**i ∈ **H1.**

B-. This is immediate, by applying part A to T^{*}.

C-. (**i**) ⇒(**ii**).

Assume **U** is unitary**.**

On the one hand, since **U** is an isometry,

by part **A** we get U^{*}**U** =**L**H1. On the other hand, since **U** is bijective, the above

equality actually forces **U** -1 = U^{*}, so we also get **U** U^{*} = **UU** -1 = **L**H2.

(**ii**) ⇒ (**i**).

Assume U^{*}**U** =**L**H1 and **U** U^{*} = **L**H2, and let us prove that **U** is aunitary**.**

On the one hand, these two equalities prove that **U** is both left and right invertible, so **U** is bijective.

On the other hand, by part **A**, it follows that **U** is an isometry, so **U** is indeed unitary.

In the study of bounded linear operators, positivity is an essential tool. This is illustrated by the following technical result .

Proposition (2-13).

Let **H** be a Hilbert space.

(**i**)- Every self-adjoint operator **T** $\in$**B**(**H**) has real spectrum, i.e. one has the

inclusion **Spec** H(**T**) $\subset$ **R**.

(**ii**)- Every positive operator **T** $\in$**B**(**H**) has non-negative spectrum, i.e. one has the inclusion **Spec** H(**T**) $\subset$ [0, ∞).

Proof.

(**i**). Let **T** $\in$**B**(**H**) be self-adjoint.

We wish to prove that for every complex number **λ** $\in$**C**,**R**, the operator

X = **λ**L-**T** is invertible.

Write **λ** = **a**+i**b**, with **a**, **b** $\in$**R** with **b** $\neq$ **0**. We are going to apply Lemma

(2-2), so we need to consider the operators X^***X** and **X** X^*.

It turns out that

X^***X** = **X** X^* = |**λ**|^2**L** - **2**(**Re λ**)**T** + T^*,

so all we need is the existence of a constant **α** >0, such that X^***X** ≥ **α**L.

But this is clear, since

X^***X** = $(a^2 + b^2)$**L** - **2aT** + T^2 =b^2L + (**aL**- **T**)^2,

and the positivity of (**aL**-**T**)^2 = (**aL**-**T**)* (**aL**-**T**) (see Remark (2-8-E)

Immediately gives $X^*\mathbf{X} \geq b^2\mathbf{L}$.

(**ii**)- By part (**i**) we only need to prove that, for every number

a $\in$ (-∞, 0), the operator **X** = **aL**- **T** is invertible.

As before, we have

$$X^*\mathbf{X} = \mathbf{X}\ X^* = a^2\mathbf{L} - \mathbf{2aT} + T^2,$$

and then the positivity of -**2aT** and of $T^2 = T^*\mathbf{T}$ (see Remark(2-8-F)), forces $X^*\mathbf{X} \geq a^2 L$.

Since **a** ≠ 0, it follows that **X** is indeed invertible.

The above result can be nicely complemented with the one below.

<u>**Proposition**</u>(2-14)

(Spectral Radius Formula for self-adjoint operators). Let **H** be a Hilbert space. For every self-adjoint operator **T** ∈ **B**(**H**), one has the equality

rad H(**T**) = ‖**T**‖

<u>**Proof**</u>.

It **T** = 0, there is nothing to prove, so without any loss of generality we can assume that ‖**T**‖ = 1.

Since **rad**H(**T**) ≤ ‖**T**‖ = 1 , all we have to prove is the fact that **Spec** H(**T**) contains one of the numbers ±1. Equivalently,
we must prove that either (-**L**– **T**) or(**L** – **T**) is non-invertible.

Consider the positive operator **X** = T^2, so that we have

X - **L** = (-**L**- **T**)(**L** - **T**) = (**L**- **T**)(-**L** - **T**),

which means that we must prove that (**X** – **L**) is non-invertible.

We prove this fact by contradiction.

Assume that (**X** – **L**) is invertible, there exists some constant **α** ∈ (0, 1) such that

α L ≤ (**X** - **L**)*(**X** - **L**) = (**X** – **L**)*………..(12)

Remark that, since ‖**T**‖= 1, we have the inequality

$0 \leq (T^4 e|e) = \|T^2 e\|^2 \leq (\|T\| \cdot \|Te\|)^2 \leq \|T e\|^2 = (T^2 e|e), \forall\ e \in H,$

which reads:

$$X \geq X^2 \geq 0.$$

In particular this gives (**L** - **X**) - (**X** -**L**)^2 = **X** - X^2 ≥ 0, so we also have

(**L** - **X**) ≥ (**X** -**L**)^2 .

Using (**13**) this forces the inequality (**L**- **X** ≥ **α L**), which can be re-written as

(1 - **α**)^L ≥ **X**.

In other words, we have

$(1 - \alpha)\|e\|^2 = (1 - \alpha)(e|e) \geq (Xe|e) = (T^2 e|e) = \|T e\|^2, \forall\ e \in H,$

which gives

$\|T e\| \leq \sqrt{1 - \alpha} \cdot \|e\|, \forall\ e \in H.$

This forces ‖**T**‖ ≤ $\sqrt{1 - \alpha}$, which contradicts the assumption that ‖**T**‖=1.

Although the following result may look quite “innocent,” it is crucial for the development of the theory.

Proposition(1-15).

Let **H1** and **H2** be Hilbert spaces. For every operator**T** ∈**B**(**H1**,**H2**), one has the identity $\| T^*\mathbf{T}\| = \|\mathbf{T}\|^2$ ……………….(13)

Proof.

Fix **T** ∈**B**(**H1**, **H2**). Consider the sesquilinear map

φ : **H1** × **H1** ∋ (**e**, **q**) -→ (T^***T e**|**q**)H1 ∈ **C**.

By Theorem(2.1), we know that $\| T^*\mathbf{T}\| = \|\boldsymbol{\varphi}\|$.

Notice however that, for every

ξ ∈**H1** with $\|e\| \le 1$, one has

$\|\boldsymbol{\varphi}\| \ge |\boldsymbol{\varphi}(\mathbf{e}, \mathbf{e})| = |(T^*\mathbf{T}\,\mathbf{e}\,|\mathbf{e})| = |(\mathbf{T}\,\mathbf{e}\,|\mathbf{T}\,\mathbf{e})| = \|\mathbf{T}\,e\|^2$,

so we get

$\sqrt{\|\boldsymbol{\varphi}\|} \ge \mathbf{sup}\{\|\mathbf{Te}\| : \mathbf{e} \in \mathbf{H1}, \|\mathbf{e}\| \le 1\} = \|\mathbf{T}\|$,

thus proving the inequality $\| \mathbf{T}^*\mathbf{T}\| = \|\boldsymbol{\varphi}\| \ge \|\mathbf{T}\|^2$.

The other inequality is immediate, since

$\| \mathbf{T}^*\mathbf{T}\| \le \| \mathbf{T}^*\| \cdot \|\mathbf{T}\| = \|\mathbf{T}\|^2$.

Corollary(2-16).

Let **H** be a Hilbert space, and let **A** be an involutive Banach algebra.

Then every *-homomorphism **Φ** : **A** → **B**(**H**) is contractive, in the sense that one has the inequality

$\|\boldsymbol{\Phi}(\mathbf{a})\| \le \|\mathbf{a}\|$, ∀**a** ∈ **A**)……………….(14)

<u>Proof</u>.

Fix **a** *-homomorphism **Φ** : **A** → **B**(**H**).

We can assume that **A** is unital, and **Φ**(1) = **L.**

(If not, we work with the unitized algebra $\widetilde{\mathbf{A}}$, which is again an involutive Banach algebra, and with the map

$\widetilde{\boldsymbol{\phi}}$: $\widetilde{\mathbf{A}}$ →$\widetilde{\mathbf{B}}$(**H**) defined by $\widetilde{\boldsymbol{\phi}}$(a, α) = **Φ**(a) + **αL**, a ∈**A**, α ∈**C**, which clearly defines **a** *-homomorphism satisfying $\widetilde{\boldsymbol{\phi}}$(1)=**L**.)

To prove (**14**) we start with an arbitrary element **a** ∈**A**, and we consider the

element **b** = $\mathbf{a}^*$**a.**

On the one hand, the operator **Φ**(b) = $\Phi(a)^*$ **Φ**(a) **B**(**H**) is obviously self-adjoint, so by Proposition(2.6), we know that

$\|\mathbf{\Phi}(b)\|$ = **rad** H **Φ**(b)……………..(15)

Since Φ is an algebra homomorphism with **Φ**(1) = 1, we have the inclusion

Spec H**Φ**(b) ⊂ **Spec** A (b) ,

which then gives the inequality

rad H **Φ**(b)≤ **rad** A (b).

Using the inequality **rad**A (b) ≤ $\|\mathbf{b}\|$, the above inequality, combined with (**16**), yields

$\|\mathbf{\Phi}(b)\|$ ≤ $\|\mathbf{b}\|$…………….(16).

On the other hand, using Proposition (2-14), we know that

$\|\mathbf{\Phi}(b)\| = \|\mathbf{\Phi}(a)^*\mathbf{\Phi}(a)\| = \|\mathbf{\Phi}(a\|^2$

so (**17**) reads

$\|\boldsymbol{\Phi}(a)\|^2 \leq \|\mathbf{b}\|$……………(17).

Finally, since **A** is an involutive Banch algebra, we have

$\|\mathbf{b}\| = \|\mathbf{a}^*\mathbf{a}\| \leq \|\mathbf{a}^*\| \cdot \|\mathbf{a}\| = \|\mathbf{a}\|^2$,

and then (**17**) clearly gives (**14**).

The identity (**13**) is referred to as the **C***-norm condition.

The above result suggests that this property has interesting applications. As shall see a little later, this condition is at the heart of the entire theory.

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