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On Partial sum of Cap like and star like univalent function

A research

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الإهداء

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار إلى بطاعتك.. ولا تطيب اللحظات إلا بذكرك.. ولا تطيب الآخرة إلا بعفوك.. ولا تطيب الجنة إلا برويتك

الله جل جلاله

الى من بلغ الرسالة وأدى الأمانة.. ونصح الأمة.. الى نبي الرحمة ونور العالمين..

سيدنا محمد صلى الله عليه واله وسلم

الى من كلله الله بالهيبة والوقار.. الى من علمني العطاء بدون انتظار.. الى من أحمل أسمه بكل افتخار.. ارجو من الله أن يمد في عمرك لترى ثماراً قد حان قطافها بعد طول انتظار وستبقى كلماتك نجوم أهدي بها اليوم وفي الغد والى الأبد..

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الى ملاكي في الحياة.. الى معنى الحب والى معنى الحنان والتفاني.. الى بسملة الحياة وسر الوجود.. الى من كان دعائها سر نجاحي وحنانها بلسم جراحى إلى أغلى الحبايب

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الذي نقول له بشراك قول رسول الله صلى الله عليه واله وسلم:

((ان الحوت في البحر، والطير في السماء، ليصلون على معلم الناس الخير))

كما انني اتوجه بخالص الشكر، الى من علمنا التفاؤل والمضي الى الامام، الى من رعانا وحافظ علينا، الى من وقف الى جانبنا....

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شكرا وتقديرا لحضرة الاستاذة الفاضلة **زينب عوده ثبينه** على ما بذلته من سعة صدر وكرم طبعها ورحابة خاطرها وارشاد وتوجيه وتسديد لأفكاري

فجزاه الله خير جزاء المحسنين

الباحث

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

وَقُلْ اَعْمَلُوا فَسَيَرَنَ اللَّهُ عَمَلَكُمْ وَرَسُولُهُ
وَالْمُؤْمِنِينَ وَسَتَرْدُونَ إِلَىٰ عَالَمِ الْغَيْبِ وَالشَّهَادَةِ
فَيُنَبِّئُكُمْ بِمَا كُنْتُمْ تَعْمَلُونَ

صدق الله العلي العظيم

سورة التوبة

الآية (١٠٥)

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Introduction

We know that a complex -valued function is said to be regular or analytic in a domain D if it has a uniquely determined derivative at each point of D if $f(z_1) \neq f(z_2)$ for all $\{z_1, z_2\} \subset D$ with $z_1 \neq z_2$

A necessary condition for analytic function $f(z)$ to be univalent in D is $f'(z) \neq 0$. This condition is not sufficient

Riemann mapping theorem is one of the most remarkable results for complex analytic states the any simple connected results for complex analytic states the any simple connected proper sup set of \mathbb{C} can be mapping conformally onto the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ that is, if $D \subset \mathbb{C}$ is simple connected and $z_0 \in D$, then there exists a unique conformal transformation

$$f: D \rightarrow D \text{ with } f(z_0) = z_0 \text{ and } f'(z_0) > 0$$

therefore, statement about univalent functions on arbitrary simple connected domains can be translated to statements about univalent function on the unit disk, that is an analytic, and one-to-one function $f(z)$ of a power series (Taylor series) is called univalent in 1916, Carathéodory proved that $|a_2| \leq 2$ for every $f(z)$ in u when u if $f(z) = f(z)$ is analytic and univalent in $D = \{z \in \mathbb{C} : |z| < 1 \text{ with the condition } f(0) = 0, f'(0) = 1\}$ whose Taylor expansion about the Origin is $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let is also showed that $|a_2| = 2$. For the function $f(z) = z(1 - kz)^{-2}$, $|k| = 1$, which is known as KOEBE'S function. Note that similarity of KOEBE'S function is analytic in the open unit disk and $f'(z) = (-z)(1 - kz)^{-3}(-k) + 1(-z)(1 - kz)^{-2}$ implies $f'(0) = 0(-z)(1 - k0)^{-3}(-k) + 1(1 - k0)^{-2} = 1$. Clearly $f(0) = 0$, so KOEBE'S function is in u .

A function which is analytic and univalent in the open unit disc $\{z : |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$ is said to be car. Like function if

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, |z| < 1.$$



And it is called star like if $Re \left[\frac{zf''(z)}{f'(z)} \right] > 0, |z| = r < 1$.

Also it is called cup like of order α if $Re \left[z + \frac{zf''(z)}{f'(z)} \right] > \alpha(c), 0 \leq \overline{\alpha(c)} < 1, |z| = r < 1$.

And it is called star like of order α if $Re \left[z + \frac{zf'(z)}{f'(z)} \right] > \alpha(c), 0 \leq \alpha(c) < 1, |z| < c \leq 1$.

In our studied we introduce two chapter in chapter one contain two section and chapter two so that two section.



CHAPTER ONE

UNIVALENT FUNCTIONS



Section one

Basic definitions

Definition 1.1.1

Let $D \subset \mathbb{C}$ be a domain non-empty subset of the complex plane ,we say that A function $f: D \rightarrow \mathbb{C}$ is analytic at z_0 if it is complex differentiable at every point in some neighborhood of z_0 .We say that f is analytic on D if f is analytic at z_0 for every $z_0 \in D$.

Example 1.1.2

Let $f: D \rightarrow \mathbb{C}$ be a function given by $f(z) = e^z$, where D is any open subset of \mathbb{C} .note that f is analytic at every $z_0 \in D$.

Definition 1.1.3

A function $f(z)$ is said to be a univalent in a domain D if $f(z_1) \neq f(z_2)$ for all $\{z_1, z_2\} \subset D$ with $z_1 \neq z_2$.

A necessary condition for analytic function $f(z)$ to be univalent in D is $f'(z) \neq 0$ in D .

Example 1.1.4

Let $f(z)$ be a function given by $f(z) = z + \frac{z^2}{2}$.

$$z_1 = 1 - i \neq z_2 = i$$

$$f(z_1) = f(1 - i) = 1 - i + \frac{(1 - i)^2}{2} = \frac{2 - 4i}{2} = 1 - 2i$$

$$f(z_2) = f(i) = i - \frac{i^2}{2} = \frac{2i + 1}{2} \neq f(z_1)$$



Thus $f(z)$ is univalent.

Example 1.1.5

Let $f(z)$ be a function given by $f(z) = e^z$. Note that $f(z)$ is univalent since for

$z_1 = 2\pi \neq z_2 = 4\pi$ we have

$$f(z_1) = e^{z_1} = e^{2\pi} = \cos 2\pi + i \sin 2\pi = 1$$

$$f(z_2) = e^{z_2} = e^{4\pi} = \cos 4\pi + i \sin 4\pi = 1 = f(z_1)$$

Thus $f(z)$ is not univalent.

Definition 1.1.6

A function $f: D \rightarrow \mathbb{C}$ which is both analytic on D and univalent on D is called conformal on D . We will often refer to such an f as a conformal mapping of D .

A word of notation is needed here. If we are interested in both the domain and the range of a conformal mapping f , then we will write this explicitly as "let $f: D \rightarrow D'$ be a conformal transformation." That is, $f: D \rightarrow D'$ is a conformal mapping of D which is onto D' ; i. e.,

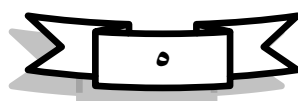
$$f(D) = D'.$$

Example 1.1.7

It is important to remember that the underlying domain is an integral part of the definition of a univalent function (or a conformal mapping). Suppose that

$$D = \{z \in \mathbb{C} : 0 < |z| < 1, y > 0, x > 0\} = \{z \in \mathbb{C} : 0 < |z| < 1,$$

$$0 < \theta < \pi/2\}$$



Which is that part of the unit disk in the first quadrant. The function $f(z) = z^2$ then maps D conformally onto $\mathbb{D} \cap \mathbb{H} = \{z \in \mathbb{C} : 0 < |z| < 1, y > 0\}$. That is, $f : D \rightarrow \mathbb{D} \cap \mathbb{H}$ is analytic and univalent on D , and onto $\mathbb{D} \cap \mathbb{H}$. However, the function $g(z) = z^4$ does NOT map \mathbb{D} conformally onto the unit disk \mathbb{D} , although $g(\mathbb{D}) = \mathbb{D}$. While $g : \mathbb{D} \rightarrow \mathbb{D}$ is analytic, it is not univalent. For instance, $g(1/3) = g(-1/3) = 1/81$. In fact, $g'(0) = 0$ which means that there is no neighbourhood of 0 in which g is univalent.

Example 1.1.8

The koebe function which is given by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots$$

And maps the unit disk to the complement of the ray $(-\infty, -1/4)$.

This can be verified by writing

$$k(z) = \frac{1}{4} \left(\frac{1+z}{1-z} \right)^2 - \frac{1}{4}$$

Which is analytic and univalent function, so k is conformal.

Definition 1.1.9

n - th Partial sum of the function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is $S_n(z, f) = z + \sum_{k=2}^n a_k z^k$.



Section two

Theory of Univalent functions

Theorem 1.2.1 (Riemann Mapping Theorem)

Let $D \subsetneq \mathbb{C}$ be a simply connected domain, and let $z_0 \in D$ be any given point. Then there exists a unique analytic, one-to-one function

$f: D \rightarrow \mathbb{D}$ Which maps D onto \mathbb{D} and has the properties that

$$f(z_0) = 0 \text{ and } f'(z_0) > 0.$$

Since the inverse image of a conformal map is also conformal, the Riemann mapping theorem implies that any two simply connected domains (neither of which is \mathbb{C} itself) are conformally equivalent. That is, if $D \subsetneq \mathbb{C}$ and $D' \subsetneq \mathbb{C}$ are simply connected, $z \in D$, and $w \in D'$, then there exists a unique conformal transformation $f: D \rightarrow D'$ with $f(z) = w$ and $f'(z) > 0$.

Theorem 1.2.2 (Bieberbach)

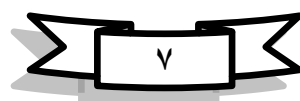
If $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in S$, then $|a_2| \leq 2$. Where S is a family of analytic functions.

Proof

Suppose that $f \in S$. Apply a square root transformation and invert it to give

$$g(z) = [f(z^{-2})]^{-1/2} = z - \frac{a_2}{2} z^{-1} + \dots$$

So that $g \in \mathcal{L}$. It therefore follows from the previous corollary (Corollary 2.2.7) that



$$|b_1| = \left| \frac{a_2}{2} \right| \leq 1$$

And so $|a_2| \leq 2$ as required.

Example 1.2.3

if $f \in S$, then $|a_2^2 - a_3| \leq 1$

By using (1.7) and corollary 1.1.7

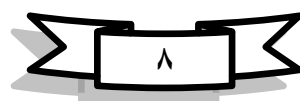
Bieberbach conjectured that *if $f \in S$, then the coefficients a_n of f satisfied $|a_n| \leq n$* . This problem, known as Bieberbach's conjecture, was finally proved by **L. de Branges [2] in 1985**.

Theorem 1.2.4 (Bieberbach Conjecture-de Branges' Theorem)

if $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in S$,

Then $|a_n| \leq n$ for all $n \geq 2$.

The final major geometric result for univalent functions $f \in S$ of this section is a theorem due to Koebe. In 1907, Koebe [7] showed that the images $f(\mathbb{D})$ of all functions $f \in S$ contained a common disk $\{|w| < r\}$ for some $r \leq 1/4$. it follows from Bieberbach's theorem that $r = 1/4$. for obvious reasons, is known as the Koebe one-quarter theorem.



Theorem 1.2.5: (Area theorem). *If*

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

belongs to Σ , then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

With the equality if and only if $g \in \tilde{\Sigma}$.

The above theorem is the basis of a theory of univalent functions, parts of which we shall present in this section. The reason-for the name area theorem comes. From the proof.

Proof

For $r > 1$, let C_r denote the image of the circle $|z| = r$ under g . Each C_r is a simple, closed, and smooth curve. Let E_r denote the bounded connected component of

Theorem 1.2.6: (Growth Theorem)

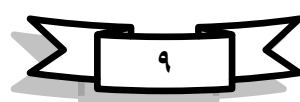
For each $f \in S$,

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r.$$

Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof

An upper bound on $|f'(z)|$, as in gives an upper bound on $|f(z)|$. That is, fix $z = re^{i\theta} \in \mathbb{D}$. Observe that



$$f(z) = \int_0^r f'(pe^{i\theta}) dp.$$

Then,

$$|f(z)| \leq \int_0^r |f'(pe^{i\theta})| dp \leq \int_0^r \frac{1+p}{(1-p)^3} dp = \frac{r}{(1-r^2)}.$$

However, since we are working in dimension 2, a lower bound on $|f'|$ does not give a lower bound $|f|$. Let z be an arbitrary point in \mathbb{D} . We consider two possibilities:

$$(i) |f(z)| \geq 1/4$$

$$(ii) |f(z)| \leq 1/4$$

Assume that (ii) occurs. Since for all $r \in (0,1)$, $r/(1+r^2) \leq 1/4$, we trivially have $r/(1+r^2) \leq |f(z)|$.

Now assume that (i) occurs. By the Koebe 1/4-Theorem, the radial line rz , for $r \in [0,1]$ is contained in the image of f . As f is one-to-one, the pre-image of this radial line, is a simple smooth curve in \mathbb{D} connecting 0 to z . Let C denote this curve. We have

$$f(z) = \int_C f'(w) dw.$$

By the definition of C , for any point w on C , $f'(w)dw$ has the same argument as the argument of z . Thus,

$$|f(z)| = \left| \int_C f'(w) dw \right| = \int_C |f'(w)| |dw| \geq \int_0^r \frac{1-p}{(1+p)^3} dp = \frac{r}{(1+r)^2}.$$

It follows from the above arguments that an inequality in either side of Equation (1.1.10) implies the equality in the corresponding side of Equation



(1.1.6), which by Theorem (1.2.5) implies that f is a suitable rotation of the Koebe function.

Also, as in the proof of the previous theorem, suitable rotations of the Koebe function lead to the equality on either side of Equation. Thus, the bounds in the theorem are sharp.

It is possible to prove a distortion estimate involving both of $|f(z)|$ and $|f'(z)|$.

Theorem 1.2.7 (Distortion Theorem)

For each $f \in S$, we have

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad r = |z| < 1.$$

Moreover, one of the equalities hold at some $z \neq 0$, if and only if f is a suitable rotation of the Koebe function.

In order to prove the above theorem we need a lemma on calculating derivatives with respect to the polar coordinates.

Lemma 1.2.8

There is a continuous branch of $\log f'(z)$ defined on \mathbb{D} that maps 0 to 0.

Moreover, for all $z = re^{i\theta}$ in \mathbb{D} we have

$$\frac{zf''(z)}{f'(z)} = r \frac{\partial}{\partial r} (\log |f'(z)|) + ir \frac{\partial}{\partial r} (\arg f'(z)).$$

Proof. Recall that $f'(0) = 1$, and since f is univalent on \mathbb{D} , for all $z \in \mathbb{D}$, $f'(z) \neq 0$. Thus, by Proposition 5.26, there is a continuous branch of $\log f'(z)$ defined on \mathbb{D} which maps 0 to 0.



Let $u(z) = u(re^{i\theta})$ be an arbitrary holomorphic function defined on some open set $U \subset \mathbb{C}$. Using the relation $z = r \cos \theta + ir \sin \theta$ we have

$$r \frac{\partial u}{\partial r} = r \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial r} = r \frac{\partial u}{\partial z} \cdot (\cos \theta + i \sin \theta) = z \cdot \frac{\partial u}{\partial z}.$$

Applying the above formula to the function $\log f'(z)$, and using $\log z = \log |z| + i \arg z$, we obtain the desired relation

$$\frac{zf''(z)}{f'(z)} = z \cdot \frac{\partial}{\partial z} (\log f'(z)) = r \frac{\partial}{\partial r} (\log f'(z)) = r \frac{\partial}{\partial r} (\log |f'(z)|) + ir \frac{\partial}{\partial r} (\arg f'(z)).$$

Proof of Theorem (1.2.5). Note that inequality $|w - c| < R$ implies $c - R \leq \operatorname{Re} w \leq c + R$. In particular, by Equation (6.5), for $|z| = r$, we have

$$\frac{2r^2}{1 - r^2} - \frac{4r}{1 - r^2} \leq \operatorname{Re} \left(\frac{(zf''(z))}{(f'(z))} \right) \leq \frac{2r^2}{1 - r^2} + \frac{4r}{1 - r^2},$$

Which simplifies to

$$\frac{2r^2 - 4r}{1 - r^2} \leq \operatorname{Re} \left(\frac{(zf''(z))}{(f'(z))} \right) \leq \frac{2r^2 + 4r}{1 - r^2}.$$

By Lemma 6.8, there is a continuous branch of $\log f'(z)$ defined on \mathbb{D} that maps 0 to 0. Moreover, the relation in the lemma and the above-inequality implies that

$$\frac{2r^2 - 4r}{1 - r^2} \leq \frac{\partial}{\partial r} \log |f'(re^{i\theta})| \leq \frac{2r^2 + 4r}{1 - r^2}.$$

Now we fix θ and integrate the above equation from 0 to R to obtain

$$\log \frac{1 - R}{(1 - R)^3} \leq \log |f'(re^{i\theta})| \leq \log \frac{1 + R}{(1 - R)^3}.$$

Above we have used the explicit calculation



$$\int_0^R \frac{2r+4}{1-r^2} dr = \int_0^R \frac{3}{1-r} + \frac{2}{1+r} dr = -3 \log(1-r) + \log(1+r) \Big|_{r=0}^{r=R} = \log \frac{1+R}{(1-R)^3}.$$

As the map $x \mapsto e^x$ is monotone, Equation (1.1.9) implies the desired inequality in the theorem.

Assume that for some $z = Re^{i\theta} \in \mathbb{D}$, $z \neq 0$, we have an equality in Equation 6.6. Then, we must have the corresponding equality in Equation (1.1.9) for R . The latter condition implies the corresponding equality in Equation (6.8) and then in Equation (1.2.7), for all $r \in (0, R)$. Now let r tend to 0 from above, to obtain one of the equalities

$$\operatorname{Re}(e^{i\theta} f''(0)) = +4, \quad \text{or} \quad \operatorname{Re}(e^{i\theta} f''(0)) = -4.$$

Recall that since $f \in S$, by Theorem (1.2.4), $|f''(0)| \leq 4$. Therefore, by the above equation we must have $|f''(0)| = 4$. By the same theorem, we conclude that f must be a rotation of the Koebe function. For the Koebe function $k(z) = z/(1 - z)^2$, we have

$$k'(z) = \frac{1+z}{(1-z)^3},$$

So we have the right-hand equality at every $z = r \in (0, 1)$

On the other hand, for the function $h(z) = e^{i\pi} k(e^{-i\pi} z)$, where k is the Koebe function we have

$$h'(z) = k'(e^{-i\pi} z) = \frac{1-z}{(1+z)^3},$$

So we have the left-hand equality at any $z \in (0, 1)$. this finishes the proof of the if and only if statement.

Theorem 1.2.9 (combined growth-distortion Theorem)

For each $f \in S$



$$\frac{1-r}{1+r} \leq \left| \frac{z f'(z)}{f'(z)} \right| \leq \frac{1+r}{1-r}, |r| = r.$$

Moreover, for each $z \in \mathbb{D}$ with $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

It is not possible to conclude the above theorem as a combination of the bounds in Theorems (1.2.4) and (1.2.7). But the proof is obtained from applying the Beiberbach Theorem (1.2.4) to a suitable disk automorphism applied to f . As we have already seen this technique we skip the proof of the above theorem.



CHAPTER TWO

CAP LIKE AND

STAR LIKE

Section one

Partial sum of Cap like and star like

Definition 2.1.1

A function $f(z)$ that is analytic in the open unit disc $\{z: |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$ is said to be cap. Like function if

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, |z| < 1.$$

Example 2.1.2

The function $(z) = z + 2z^2 + 4z^3$ is cap like for $\{z: |z| < 1\}$.

Definition 2.1.3

A function $f(z)$ that is analytic in the open unit disc $\{z/|z| < 1\}$ and univalent in the open disc $\{z/|z| < c \leq 1\}$ with $f(0) = 0$, $f'(0) = 1$ is said to be star like function in the open disc $\{z/|z| < c \leq 1\}$ if

$$\operatorname{Re} \left[\frac{zf'(z)}{f'(z)} \right] > 0, |z| < c.$$

Theorem 2.1.4

Let $f(z)$ is analytic in the open unit disc $\{z/|z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$, and $f(z)$ is univalent in the disc $\{z/|z| < 1\}$, then $s_n(z, f)$ is analytic in disc $\{z/|z| < 1\}$ with $s_n(0, f) = 0$, $s'_n(0, f) = 1$, and is univalent function in $|z| < 1 - 3^{-1}\sqrt{6} < 1$ for all integers $n = 2, 3, 4, \dots$.

Proof



Let $f(z)$ is analytic in the open unit disc $\{z/|z| < 1\}$ with $f(0) = 0, f'(0) = 1$

$$\Rightarrow f(z) = z + \sum_{k=2}^{\infty} a_k z_1^k \Rightarrow s_n(z, f) = z + \sum_{k=2}^n a_k z_2^k.$$

$\Rightarrow s_n(z, f)$ is analytic in the open unit disc $\{z/|z| < 1\}$ with $s_n(0, f) = 0, s'_n(0, f) = 1,$

Let $z_1 \neq z_2$. Then $z_1^k \neq z_2^k$ then $a_k z_1^k \neq a_k z_2^k$ ($k = 2, 3, 4, \dots$). But the inequality

$z_1 + \sum_{k=2}^n a_k z_1^k \neq z_2 + \sum_{k=2}^n a_k z_2^k$ Mayor may not hold. So we can do some work.

Since

$$z_1 \neq z_2, \text{ we have } z_1 - z_2 \neq 0 \Rightarrow |z_1 - z_2| \neq 0 \Rightarrow 0 < |z_1 - z_2|.$$

Let $\rho = |z_1| \leq |z_2| = r < 1 \Rightarrow 0 \leq r - \rho = |z_2| - |z_1| \leq |z_1 - z_2|$ by triangle inequality.

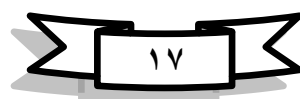
Consider

$$\begin{aligned} s_n(z_1, f) - s_n(z_2, f) &= [z_1 + \sum_{k=2}^n a_k z_1^k] - [z_2 + \sum_{k=2}^n a_k z_2^k] \\ &= z_1 - z_2 + [\sum_{k=2}^n a_k z_1^k - \sum_{k=2}^n a_k z_2^k] = z_1 - z_2 + \sum_{k=2}^n a_k [z_1^k - z_2^k] \end{aligned}$$

By triangle inequality,

$$\begin{aligned} |z_1 - z_2 + \sum_{k=2}^n a_k [z_1^k - z_2^k]| &\geq |z_1 - z_2| - |\sum_{k=2}^n a_k [z_1^k - z_2^k]| \\ \Rightarrow |s_n(z_1, f) - s_n(z_2, f)| &\geq |z_1 - z_2| - |\sum_{k=2}^n a_k [z_1^k - z_2^k]| \geq r - \rho - |\sum_{k=2}^n a_k [z_1^k - z_2^k]| \end{aligned}$$

Again by triangle inequality and by BEIRBERBACH conjecture $|a_k| \leq k$ since $f(z) \in U,$



$$\begin{aligned}
|\sum_{k=2}^n a_k [z_1^k - z_2^k]| &\leq \sum_{k=2}^n |a_k [z_1^k - z_2^k]| = \sum_{k=2}^n |a_k| |z_1^k - z_2^k| \\
&\leq \sum_{k=2}^n k [|z_1^k| + |z_2^k|] \\
&= \sum_{k=2}^n k [|z_1|^k + |z_2|^k] \leq \sum_{k=2}^n k [r^k + r^k]
\end{aligned}$$

Since $|z_1| \leq |z_2| = r$.

$$\begin{aligned}
i.e. \sum_{k=2}^n a_k [z_1^k - z_2^k] &\leq \sum_{k=2}^n k 2 r^k = 2r = \sum_{k=2}^n k 2 r^{k-1} = 2r \sum_{k=2}^n \frac{d}{dr} r^k \\
&= 2r \frac{d}{dr} \sum_{k=2}^n r^k.
\end{aligned}$$

$$i.e. \sum_{k=2}^n a_k [z_1^k - z_2^k] \leq 2r \frac{d}{dr} [-1 - r + \sum_{k=0}^n r^k] = 2r \frac{d}{dr} [-1 - r + \frac{1-r^{n+1}}{1-r}]$$

$$i.e. |\sum_{k=2}^n a_k [z_1^k - z_2^k]| \leq 2r[0 - 1 + \frac{(1-r)[0-(n-1)r^n] - (1-r^{n+1})(-1)}{(1-r)^2}]$$

$$|\sum_{k=2}^n a_k [z_1^k - z_2^k]| \leq 2r[-1 + \frac{-(n+1)(1-r)r^{n+1} + r^{n+1}}{(1-r)^2}] \quad , i.e.$$

$$\Rightarrow - \left| \sum_{k=2}^n a_k [z_1^k - z_2^k] \right| \geq -2r[-1 + \frac{-(n+1)(1-r)r^{n+1} + r^{n+1}}{(1-r)^2}]$$

$$i.e. -|\sum_{k=2}^n a_k [z_1^k - z_2^k]| \geq 2r[1 + \frac{(n+1)(1-r)r^{n+1} - r^{n+1}}{(1-r)^2}]$$

$$\begin{aligned}
i.e. -|\sum_{k=2}^n a_k [z_1^k - z_2^k]| &\geq 2r[1 + \frac{(n+1)(1-r)r^{n+1} - r^{n+1}}{(1-r)^2} - \frac{1}{(1-r)^2}] \geq 2r[1 + \\
&0 - \frac{1}{(1-r)^2}]
\end{aligned}$$

Since $0 < r < 1 \Rightarrow -r > -1 \Rightarrow 1 - r > 1 - 1 = 0$.

Thus we have

$$\begin{aligned}
|s_n(z_1, f) - s_n(z_2, f)| &\geq r - \rho - |\sum_{k=2}^n a_k [z_1^k - z_2^k]| \\
&\geq r - \rho + 2r[1 - \frac{1}{(1-r)^2}] = 3r - \rho - \frac{2r}{(1-r)^2} = r[3 - \frac{2}{(1-r)^2}] - \rho.
\end{aligned}$$

Observe that



$$r[3 - \frac{2}{(1-r)^2}] - \rho > 0 \Leftrightarrow r[3 - \frac{2}{(1-r)^2}] > \rho \geq 0$$

Consider

$$\begin{aligned} 0 \leq \rho < r \left[3 - \frac{2}{(1-r)^2} \right] &\Rightarrow 3 - \frac{2}{(1-r)^2} > 0 \Rightarrow 3 > \frac{2}{(1-r)^2} \\ \Rightarrow (1-r) > \frac{3}{2} &\Rightarrow 1-r > \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{\sqrt{3}} \Rightarrow 1 - \frac{\sqrt{6}}{3} > r \Rightarrow r < 1 - \frac{\sqrt{6}}{3} \\ &= c \end{aligned}$$

$$i.e. 0 \leq \rho = |z_1| \leq |z_2| = r < c = 1 - \frac{\sqrt{6}}{3} = \frac{3-\sqrt{6}}{3} < 1$$

$$\Rightarrow |s_n(z_1, f) - s_n(z_2, f)| > r[3 - \frac{2}{(1-r)^2}] - \rho > 0$$

$$\Rightarrow |s_n(z_1, f) - s_n(z_2, f)| \neq 0 \Rightarrow s_n(z_1, f) - s_n(z_2, f) \neq 0.$$

Hence $s_n(z, f)$ is a univalent function in the disc $|z| < c$ for all n .

Section two

Partial Sum of Cap like and Star like of order α

Definition 2.2.1

A function $f(z)$ that is analytic in the open unit disc $\{z/|z| < 1\}$ with $f(0) = 0, f'(0) = 1$ is said to be caplike function if

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha(c), \quad |z| < 1.$$

Definition 2.2.2

A function $f(z)$ that is analytic in the open unit disc $\{z/|z| < 1\}$ and univalent in the open disc $\{z/|z| < c \leq 1\}$ with $f(0) = 0, f'(0) = 1$ is said

To be star like function in the open disc $\{z/|z| < c \leq 1\}$ if

$$\left[\frac{zf'(z)}{f(z)}\right] > \alpha(c), \quad |z| < 1.$$

Theorem 2.2.3

Let $L(z) = z(1 - z)^{-1}$ then $s_n(z, L)$ ($n = 2, 3, 4, \dots$) is cap like function in disc $|z| < 0.25$

Proof

$$L(z) = z(1 - z)^{-1} = \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k+1} = \sum_{k-1=0}^{\infty} z^{k-1+1} = \sum_{k=0}^{\infty} z^k = z + \sum_{k=2}^{\infty} a_k z^k$$



Where $a_k = 1$ ($k = 2, 3, 4, \dots$). Then we have

$$\begin{aligned} s_n(z, L) &= z + \sum_{k=2}^n z^k = \sum_{k=1}^n z^k = \sum_{k+1=1}^n z^{k+1} = z L z k + 1 \\ &= z \sum_{k=0}^{n-1} z^{k+1} = z \frac{1 - z^n}{1 - z} \end{aligned}$$

$$\Rightarrow s'_n(z, L) = \frac{(1 - z)[1 - (n + 1)z^n] - (0 - 1)[z - z^{n+1}]}{(1 - z)^2}$$

$$\frac{1 - z - (n + 1)z^n + (n + 1)z^{n+1} + z - z^{n+1}}{(1 - z)^2} = \frac{1 - (n + 1)z^n + nz^{n+1}}{(1 - z)^2}$$

$$\Rightarrow \log s'_n(z, L) = \log [1 - (n + 1)z^n + nz^{n+1}] - 2 \log(1 - z)$$

By taking the derivative on both sides, we have

$$\begin{aligned} \frac{s''_n(z, L)}{s'_n(z, L)} &= \frac{0 - (n + 1)nz^{n-1} + n(n + 1)z^n}{1 - (n + 1)z^n + nz^{n+1}} - 2 \frac{-1}{1 - z} \\ &= \frac{(n + 1)nz^{n-1}[-1 + z]}{1 - (n + 1)z^n + nz^{n+1}} + \frac{2}{1 - z} \end{aligned}$$

$$\Rightarrow \frac{s''_n(z, L)}{s'_n(z, L)} = \frac{(n + 1)nz^n[-1 + z]}{1 - (n + 1)z^n + nz^{n+1}} + \frac{2z}{1 - z} = \frac{N(z)}{D(z)} + \frac{2z}{1 - z}$$

$$\begin{aligned} \Rightarrow 1 + \frac{s''_n(z, L)}{s'_n(z, L)} &= 1 + \frac{(n + 1)nz^n[-1 + z]}{1 - (n + 1)z^n + nz^{n+1}} + \frac{2z}{1 - z} \\ &= \frac{(n + 1)nz^n[-1 + z]}{1 - (n + 1)z^n + nz^{n+1}} + \frac{1 + z}{1 - z} \end{aligned}$$

To simplify the notations, put

$$N(z) = (n + 1)nz^n[-1 + z], \quad D(z) = 1 - (n + 1)z^n + nz^{n+1},$$

$$\frac{1 + z}{1 - z} = w = u + iv$$

$$\Rightarrow 1 + z \frac{s''_n(z, L)}{s'_n(z, L)} = \frac{N(z)}{D(z)} + w$$



$$\begin{aligned} \operatorname{Re} \left[1 + z \frac{s''_n(z, L)}{s'_n(z, L)} \right] &= \operatorname{Re} \left[\frac{N(z)}{D(z)} + w \right] = \operatorname{Re} \left[\frac{N(z)}{D(z)} \right] + \operatorname{Re} w \\ &= \operatorname{Re} \left[\frac{N(z)}{D(z)} \right] + u \dots\dots\dots (1) \end{aligned}$$

We have

$$w = \frac{1+z}{1-z} \Leftrightarrow w - wz = 1 + z \Leftrightarrow w - 1 = z + wz \Leftrightarrow w - 1 = (1+w)z$$

Consider

$$|z| = \frac{1}{4} \Leftrightarrow \left| \frac{w-1}{w+1} \right| = \frac{1}{4} \Leftrightarrow 4|w-1| = |w+1|$$

$$\Leftrightarrow 4|u+iv-1| = |u+iv+1| \Leftrightarrow 16|u+iv-1|^2 = |u+iv+1|^2$$

$$\begin{aligned} \Leftrightarrow 16[(u-1)^2 + v^2] &= [(u+1)^2 + v^2] \Leftrightarrow 16[u^2 - 2u + 1 + v^2] \\ &= [u^2 + 2u + 1 + v^2] \end{aligned}$$

$$\Leftrightarrow 16u^2 - 32u + 16 + 16v^2 = u^2 + 2u + 1 + v^2$$

$$\Leftrightarrow 15u^2 - 34u + 15 + 15v^2 = 0$$

$$\begin{aligned} \Leftrightarrow u^2 - \frac{34}{15}u + 1 + v^2 &= 0 \Leftrightarrow u^2 - 2\frac{17}{15}u + \left(\frac{17}{15}\right)^2 - \left(\frac{17}{15}\right)^2 + 1 + v^2 \\ &= 0 \end{aligned}$$

$$\Leftrightarrow \left(u - \frac{17}{15}\right)^2 + v^2 = \frac{289}{225} - 1 = \frac{289 - 225}{225} = \frac{64}{225} = \left(\frac{8}{15}\right)^2.$$

$$\operatorname{Max}\left(u - \frac{17}{15}\right)^2 = \max\left[\left(\frac{8}{15}\right)^2 - v^2\right] = \left(\frac{8}{15}\right)^2 \text{ i.e. max will exist at } v = 0$$

$$\Rightarrow \left(u - \frac{17}{15}\right)^2 = \left(\frac{8}{15}\right)^2 \Rightarrow u - \frac{17}{15} = \pm \frac{8}{15} \Rightarrow u = \frac{17}{15} \pm \frac{8}{15}$$

$$\Rightarrow u = \frac{17}{15} + \frac{8}{15} = \frac{25}{15} \text{ or } u = \frac{17}{15} - \frac{8}{15} = \frac{9}{15}.$$



Hence it is clear that the Mobius (Bilinear) transformation

$$w = \frac{1+z}{1-z}$$

Maps the circle $|z| = 4^{-1}$ in xy -plane into the circle

$$\left(u - \frac{17}{15}\right)^2 + v^2 = \left(\frac{8}{15}\right)^2$$

In uv -plane such that the line segment AB on u -axis ($v = 0$) is a diameter where

$$A = \left(\frac{9}{15}, 0\right) = \left(\frac{3}{5}, 0\right), \text{ and } B = \left(\frac{25}{15}, 0\right) = \left(\frac{5}{3}, 0\right).$$

$$\Rightarrow \frac{9}{15} \leq u \leq \frac{25}{15} \text{ or } \frac{3}{5} \leq u \leq \frac{5}{3} \dots \dots \dots (2).$$

Observe that

$$\begin{aligned} |N(z)| &= |(n+1)nz^n[-1+z]| = (n+1)n|z^n| |-1+z| \leq (n+1)n|z|^n [1+|z|] \\ \Rightarrow |N(z)| &\leq (n+1)n|z|^n [1+|z|] \leq (n+1)n(4^{-1})^n [1+4^{-1}] \\ &= (n+1)n4^{-n-1}[4+1] \\ \Rightarrow |N(z)| &\leq 5(n+1)n4^{-n-1} \text{ For } |z| = 4^{-1} \end{aligned}$$

Consider

$$|nz^{n+1} - (n+1)z^n| \leq |nz^{n+1}| + |-(n+1)z^n| = n|z|^{n+1} + (n+1)|z|^n$$

$$\text{Put } |z| = 4^{-1} \Rightarrow |nz^{n+1} - (n+1)z^n| \leq n4^{-n-1} + (n+1)4^{-n} < 1$$

$$\Rightarrow -|nz^{n+1} - (n+1)z^n| \geq -n4^{-n-1} - (n+1)4^{-n} > -1$$

$$\Rightarrow 1 - |nz^{n+1} - (n+1)z^n| \geq 1 - n4^{-n-1} - (n+1)4^{-n} > 1 - 1 = 0$$

But

$$|D(z)| = |1 + nz^{n+1} - (n+1)z^n| \geq 1 - (n+1)4^{-n} < 1 - |nz^{n+1} - (n+1)z^n| \geq 0$$



$$\Rightarrow \frac{1}{|D(z)|} \leq \frac{1}{1 - n4^{-n-1} - (n+1)4^{-n}} \text{ for } |z| = 4^{-1}$$

Thus we have, for $|z| = 4^{-1}$

$$\left| \frac{N(z)}{D(z)} \right| = \frac{|N(z)|}{|D(z)|} \leq \frac{5(n+1)n4^{-n-1}}{1 - n4^{-n-1} - (n+1)4^{-n}} = \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} \dots\dots\dots (3).$$

Observe that

$$n = 2 \Rightarrow \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} = \frac{5(2+1)2}{4^{2+1} - 2 - (2+1)4} = \frac{10(3)}{64 - 2 - 12} = \frac{10(3)}{50} = \frac{3}{5}.$$

$$\begin{aligned} \frac{5(n+1)n}{4^{n+1} - n - (n+1)4} &\leq \frac{3}{5} \Leftrightarrow \frac{25}{12} \leq \frac{4^{n+1} - n - (n+1)4}{4(n+1)n} \\ &= \frac{4^n}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \dots\dots\dots (4). \end{aligned}$$

$$\frac{1}{4(n+1)} < 1, \quad \frac{1}{n} < 1 \Rightarrow -\frac{1}{4(n+1)} > -1, \quad -\frac{1}{n} > -1 \Rightarrow -\frac{1}{4(n+1)} - \frac{1}{n} > -1 - 1$$

For any $n = 2, 3, 4, \dots$

$$n = 3 \Rightarrow \frac{4^n}{n(n+1)} = \frac{4^3}{3(3+1)} = \frac{64}{12} > \frac{25}{12}$$

Observe that for all integers k

$$\left\{ \begin{aligned} \frac{4^{k+1}}{(k+1)(k+2)} &> \frac{4^k}{k(k+1)} \Leftrightarrow \frac{4^k 4}{(k+1)(k+2)} > \frac{4^k}{k(k+1)} \Leftrightarrow \frac{4}{k+2} > \frac{1}{k} \\ \Leftrightarrow 4k &> k+2 \quad \Leftrightarrow 3k > 2 \end{aligned} \right.$$

Since $3k > 2$ for all integers $k \geq 1$ we have

$$\frac{4^{k+1}}{(k+1)(k+2)} > \frac{4^k}{k(k+1)} > \dots > \frac{4^3}{3(3+1)} = \frac{64}{12}.$$

Thus we have

$$\frac{4^n}{n(n+1)} \geq \frac{64}{12} (n = 3, 4, 5, \dots) \Leftrightarrow \frac{4^n}{(n+1)n} - \frac{1}{4(n+1)} - \frac{1}{n} \geq \frac{64}{12} - 2 \Rightarrow \frac{40}{12} > \frac{25}{12}$$



$$N(z) = (n+1)nz^n[-1+z], \quad D(z) = 1 - (n+1)z^n + nz^{n+1},$$

$$\frac{1+z}{1-z} = w = u + iv$$

$$\Rightarrow 1 + z \frac{s''_n(z, L)}{s'_n(z, L)} = \frac{N(z)}{D(z)} + w$$

$$\begin{aligned} \operatorname{Re} \left[1 + z \frac{s''_n(z, L)}{s'_n(z, L)} \right] &= \operatorname{Re} \left[\frac{N(z)}{D(z)} + w \right] = \operatorname{Re} \left[\frac{N(z)}{D(z)} \right] + \operatorname{Re} w \\ &= \operatorname{Re} \left[\frac{N(z)}{D(z)} \right] + u \end{aligned}$$

It remains to show that 0.25 is maximal radius. This is seen for $s_2(z, L) = z + z^2$. then

$$1 + z \frac{s''_2(z, L)}{s'_2(z, L)} = 1 + z \frac{0+2}{1+2z} = \frac{1+4z}{1+2z}$$

Has singularity at $z = 0.25$ and thus analytic with $|z| < 0.25$.

Clearly $s_n(z, L) = z + \sum_{k=2}^n z^k$ is analytic with $|z| < 0.25$, and $s_n(0, L) = 0$,

Since $s'_n(z, L) = z + \sum_{k=2}^n k z^{k-1}$, we $s'_n(0, L) = z + \sum_{k=2}^n 0^{k-1} = 1$.

Hence $s_n(z, L)$ is cap like function in the open disk $|z| < 0.25$.

Definition 2.2.4

Hadamard product (or convolution) of two analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in the open disk $|z| < r$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in the open disk $|z| < d$ is denoted by $f * g$ and is defined as an analytic function

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k \text{ In the open disk } |z| < rd.$$

Theorem 2.2.5

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is cap like function. Then $s_n(z, f) = z + \sum_{k=2}^n a_k z^k$ are cap like function in the open disc $|z| < 0.25$.

Proof.



$$L(z) = z(1-z)^{-1} = z \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k+1} = \sum_{k-1=0}^{\infty} z^{k+1-1} = \sum_{k=1}^{\infty} z^k = z + \sum_{k=2}^{\infty} a_k z^k.$$

Where $a_k = 1$ ($k = 2, 3, 4, \dots$). Then by Theorem 4,

$$s_n(z, L) = z + \sum_{k=2}^n z^k \text{ is cap like function in the open disc } |z| < 0.25.$$

Replace z by $0.25z$ in $s_n(z, L)$, then

$$s_n(0.25z, L) = 0.25z + \sum_{k=2}^n (0.25)^k z^k \text{ is also cap like function in the open disc } |0.25z| < 0.25 \text{ i.e. } |z| < 1.$$

By hypothesis, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is cap like function in the open disc $|z| < 1$.

$$\text{Convolution of } f(z), s_n(0.25z, L) \text{ is } f(z) * s_n(0.25z, L) = 0.25z + \sum_{k=2}^n (0.25)^k z^k.$$

By Theorem (2.2.5), convolution of two caps like functions is cap like function

$$\text{Hence } f(z) * s_n(0.25z, L) \text{ is cap like function in the open disc } |z| < 1$$

Replace z by $4z$ in $f(z) * s_n(0.25z, L)$, then

$$f(4z) * s_n(0.25 \times 4z, L) = 0.25 \times 4z + \sum_{k=2}^n a_k (0.25)^k (4z)^k = z + \sum_{k=2}^n a_k z^k$$

Is cap like function in the open disc $|4z| < 1$ i.e. $|z| < 0.25$.



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