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Partial Sums of Libera Integral Operator of univalent Functions

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وزارة التعليم العالي والبحث العلمي

جامعة القادسية

كلية التربية

قسم الرياضيات

المجموع الجزئي لمؤشر (*Libera*) التكاملي للدوال متعددة التكافؤ

بحث مقدم إلى

لجنة مناقشة البحوث في قسم الرياضيات–كلية التربية–جامعة القادسية

وهو جزء من متطلبات نيل درجة البكالوريوس

في الرياضيات

إعداد الطالب

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بإشراف الست

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2018 م　　　　1439 هـ

بسم الله الرحمن الرحيم

﴿ هُوَ ٱلَّذِى جَعَلَ ٱلشَّمْسَ ضِيَآءً وَٱلْقَمَرَ نُورًا وَقَدَّرَهُۥ
مَنَازِلَ لِتَعْلَمُوا۟ عَدَدَ ٱلسِّنِينَ وَٱلْحِسَابَ ۚ مَا خَلَقَ ٱللَّهُ
ذَٰلِكَ إِلَّا بِٱلْحَقِّ ۚ يُفَصِّلُ ٱلْـَٔايَٰتِ لِقَوْمٍ يَعْلَمُونَ ٥ ﴾

صدق الله العظيم

سورة يونس

Abstract

We introduce the concept of power series (Taylor, Maclaurin and Laurent's) with the testing of convergence of power series in chapter one. The concept of univalent functions is introduced in chapter two, where the analytic function $f(Z)$ is univalent under conditions that we get it in the same chapter. The main result is that if $f(Z) \in B(\alpha)$, then $F_n(Z) \in B(\alpha)$ if $\frac{1}{4} \leq \alpha < 1$.

شكر وتقدير

لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهوداً كبيرة في بناء جيل الغد لتبعث الأمة من جديد...

وقبل أن نمضي نقدم أسمى آيات الشكر والامتنان والتقدير والمحبة إلى الذين حملوا أقدس رسالة في الحياة ...

إلى الذين مهدوا لنا طريق العلم والمعرفة...

إلى جميع أساتذتنا الأفاضل...

"كن عالما... فإن لم تستطع فكن متعلما ، فإن لم تستطع فأحب العلماء ،فإن لم تستطع فلا تبغضهم"

وأخص بالتقدير والشكر

الست زينب عودة إتبينة

التي أقول لها بشراك قول رسول الله (ﷺ)

"إن الحوت في البحر، والطير في السماء، ليصلون على معلم الناس الخير"

الإهداء

يا من احمل اسمك بكل فخر

يا من أفتقدك ولن أنساك

يا من يرتعش قلبي لذكرك

يا من أودعتني لله أهديك هذا البحث أبي

إلى حكمتي ... وعلمي

إلى أدبي ... وحلمي

إلى طريقي ... المستقيم

إلى طريق ... الهداية

إلى ينبوع الصبر والتفاؤل والأمل

إلى كل من في الوجود بعد الله ورسوله أمي الغالية

إلى سندي وقوتي وملاذي بعد الله

إلى من آثروني على نفسهم

إلى من علموني علم الحياة

إلى من أظهروا لي ما هو أجمل من الحياة إخوتي

إلى من كانوا ملاذي وملجئي

إلى من تذوقت معهم أجمل اللحظات

إلى من سأفتقدهم . . . وأتمنى أن يفتقدوني

إلى من جعلهم الله أخوتي بالله . . . ومن أحببتهم بالله

طلاب قسم الرياضيات

إلى من يجمع بين سعادتي وحزني

إلى من لم أعرفهم . . . ولن يعرفوني

إلى من أتمنى أن أذكرهم . . . إذا ذكروني

إلى من أتمنى أن تبقى صورهم . . . في عيوني

محمد

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Introduction

Any function which is analytic at point to must have a Taylor series about to. For it f is analytic at Z_0, it is analytic throughout some neighborhood $|Z - Z_0| < \epsilon$ of that point. If limit f limit to be analytic at a point Z_0 but is analytic at some point in every neighborhood of Z_0, then Z_0 is called a singular point, or singularity of f.

A necessary, but by no name sufficient, condition for a function f to be analytic in domain D is clearly the continuity of f throughout D. Satisfaction of the Couchy–Riemann equations is also necessary but not sufficient. Sufficient conditions for analyticity in D if we suppose that:-

i. The differentiable power derivatives of the function u and v with respect to x to y exist everywhere in the neighborhood.
ii. Those power derivatives are continuous at $Z_0 = (x_0, y_0)$ and satisfy the Couchy–Riemann equations,

$$u_x = v_y \; ; \; u_y = -v_x$$

At $Z_0 = (x_0, y_0)$. Then $f^{'}(Z_0)$ exists, its value being

$$f^{'}(Z_0) = u_x + v_x$$

When f is analytic everywhere inside a circle centered at Z_0, convergence of its Taylor series about Z_0 to $f(Z)$ for each point Z within that circle. If true an constants at $n = 0,1,2, \ldots$ so that

$$f(Z)=\sum_{n=0}^{\infty}a_n(Z-Z_0)^n$$

For all points Z interior to some circle centered at Z_0, then the power series must be the Taylor series for f about Z_0, when

$$a_n=\frac{f^{(n)}(Z_0)}{n!}$$

Are the coefficients in Taylor series. We use the formula in Taylor's theorem find the Maclaurin Series expansions of some fairly smile function of a function f fails to be analytic at a point Z_0, one can not apply Taylor's theorem at the point. It is often possible, however, to find a series representation for $f(Z)$ involving both positive and negative powers of $Z-Z_0$ which is Laurent Series. Since we say limit the series lower every to a function $f(Z)=S$ (S sum of the power series); therefore we can say that this function is univalent in a domain $D\subset\mathbb{C}$ if $f(Z_1)\neq f(Z_2)$ for all $\{(Z_1,Z_2)\subset D\}$ within $Z_1\neq Z_2$. A necessary condition for analytic function $f(Z)$ to be univalent in D is $f^{'}(Z)\neq 0$ in D. The Libera integral equator F of a function f which is analytic in a domain $D\subset\mathbb{C}$ is introduced by R.J. Libera as (1965) in the present research we introduce two chapter, chapter one contain two suctions and chapter two contains two suctions.

الملخص

قدمنا مفاهيم متسلسلات القوى (تايلر، ماكلورين، لورانت) مع اختبارات تقارب تلك المتسلسلات في الفصل الأول. مفهوم الدوال وحيدة التكافؤ قدمناه في الفصل الثاني، حيث الدالة التحليلية $f(Z)$ تكون دالة وحيدة التكافؤ تحت شروط قدمناها بنفس الفصل. النتائج الأساسية أنه إذا كانت $f(Z) \in B(\alpha)$، فأن $F_n(Z) \in B(\alpha)$ إذا كانت $\frac{1}{4} \le \alpha < 1$.

Chapter One
Types of Power Series and Convergence

(1.1) Basic Definitions of power Series

Theorem (1.1.1)

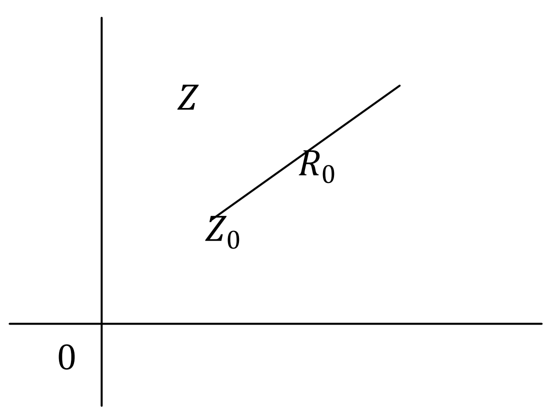
Suppose that a function f is analytic throughout a disk $|Z - Z_0| < R_0$ centered at Z_0 and with radius R_0. Then $f(Z)$ has the power series representation by:

1. $$f(Z) = \sum_{n=0}^{\infty} a_n (Z - Z_0)^n \qquad ; \ |Z - Z_0| < R_0$$

Where

2. $$a_n = \frac{f^{(n)}(Z_0)}{n!} \qquad ; \quad n = 0,1,2,\ldots$$

That is series 1 converge to $f(Z)$ when Z lies in the stated open disk.



This is the expansion of $f(Z)$ into a **Taylor Series** about the point Z_0. Series 1 can of course be written

3. $f(Z) = f(Z_0) + \frac{f^{'}(Z_0)}{1!}(Z - Z_0)^1 + \frac{f^{''}(Z_0)}{2!}(Z - Z_0)^2 + \cdots \; ; |Z - Z_0| < R_0$

Example (1.1.2)

$$f(Z) = \frac{2}{2-Z} \; ; \qquad f(0) = 1$$

$$f^{'}(Z) = \frac{2}{(2-Z)^2} \; ; \qquad f^{'}(0) = \frac{1}{2}$$

$$f^{''}(Z) = \frac{4}{(2-Z)^3} \; ; \qquad f^{''}(0) = \frac{1}{2}$$

$$f^{'''}(Z) = \frac{12}{(2-Z)^4} \; ; \qquad f^{''}(0) = \frac{3}{4}$$

$\vdots$

$$f(Z) = 1 + \frac{f^{'}(0)\,Z}{1!} + \frac{f^{''}(0)\,Z^2}{2!} + \cdots$$

$$= 1 + Z + \frac{Z^2}{4} + \frac{3Z^3}{3! \times 4} + \cdots$$

$$= 1 + \frac{Z}{2^0} + \frac{Z^2}{2^2} + \frac{Z^3}{2^3} + \frac{Z^4}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{Z^n}{2^n}$$

$$\therefore \; \frac{2}{2-Z} = \sum_{n=0}^{\infty} \frac{Z^n}{2^n}$$

Definition (1.1.3)

When we take $Z_0 = 0$ in Taylor series which case f is assumed to be analytic throughout a disk $|Z| < R_0$, where

$$f(Z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} Z^n \qquad ; \ |Z| < R_0$$

Example (1.1.4)

$$f(Z) = e^{Z-i} \quad ; \ when \ \ Z = i$$

$f(Z) = e^{Z-i}$; $\qquad f(i) = 1$

$f^{'}(Z) = e^{Z-i}$; $\qquad f^{'}(i) = 1$

$f^{''}(Z) = e^{Z-i}$; $\qquad f^{''}(0) = 1$

$\vdots$

$$f(Z) = 1 + \frac{f^{'}(0)\, Z}{1!} + \frac{f^{''}(0)\, Z^2}{2!} + \cdots$$

$$= 1 + \frac{(Z-i)}{1!} + \frac{(Z-i)^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(Z-i)^n}{n!}$$

$$\therefore \ \ e^{Z-i} = \sum_{n=0}^{\infty} \frac{(Z-i)^n}{n!}$$

Definition (1.1.5)

Laurent Series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (Z - Z_0)^n \qquad \ldots (*)$$

As (*) is a doubly infinite sum we define

$$\sum_{n=1}^{\infty} a_{-n} (Z - Z_0)^{-n} + \sum_{n=0}^{\infty} a_n (Z - Z_0)^n = \sum{}^{-} + \sum{}^{+}$$

Example (1.1.6)

Let $f(Z) = e^Z + e^{\left(\frac{1}{Z}\right)}$. Recall that

$$e^Z = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \qquad ; for\ all\ \ Z \in \mathbb{C}$$

Hence

$$e^{\left(\frac{1}{Z}\right)} = \sum_{n=0}^{\infty} \frac{Z^{-n}}{n!} \qquad ; for\ all\ \ Z \neq 0$$

$$f(Z) = \sum_{n=-\infty}^{\infty} a_n Z^n = \cdots + \frac{1}{n!\,Z^n} + \cdots + \frac{1}{2!\,Z^n} + \frac{1}{Z} + 2 + Z + \frac{Z^2}{2} + \cdots + \frac{Z^n}{n!} + \cdots$$

Where

$$a_n = \frac{1}{n!} \ \ ; for\ n \geq 1\ , \qquad a_0 = 2\ , \qquad a_{-n} = \frac{1}{n!} \ \ ; for\ n \geq 1$$

This expansion if valid for all $Z \neq 0$, i.e. $R_1 = 0, R_2 = \infty$.

(1.2) Testing for convergence of power Series

Definition (1.2.1)

Let f be a function whose domain of definition contains a neighborhood $|Z - Z_0| < \epsilon$ of a point Z_0. The derivative of f at Z_0 is the limit

$$f'(Z_0) = \lim_{Z \to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0}$$

And the function f is said to be differentiable at Z_0 when $f'(Z_0)$ exist by expressing.

The variable Z in definition (1.1.1) in terms of the new complex variable

$$\Delta Z = Z - Z_0 \qquad ; Z \neq Z_0$$

Example (1.2.2) $f(Z) = Z^2$

$$f'(Z) = \lim_{\Delta Z \to 0} \frac{(Z + \Delta Z_0)^2 - Z^2}{\Delta Z} = \lim_{\Delta Z \to 0} (2Z + \Delta Z) = 2Z$$

Definition (1.2.3)

A function of f the complex variable Z is analytic at a point Z_0 if it has a derivative at each point in some neighborhood of Z_0 it follows that f is analytic at point Z_0.

A single valued function which is defined in the domain D and is differentiable for all the point of D is called the analytic in the domain D.

The term Holomorphic function is also used to denote analytic function in domain D.

Proposition (1.2.4)

Let $Z \in \mathbb{C}$, suppose that

$$\lim_{n\to\infty} \frac{|Z_{n+1}|}{|Z_n|} = L$$

if $L < 1$ Then $\sum_{n=0}^{\infty} Z_n$ is absolutely convergent

and if $L > 1$ Then $\sum_{n=0}^{\infty} Z_n$ is diverges.

Proposition (1.2.5)

Let $Z \in \mathbb{C}$, suppose that

$$\lim_{n\to\infty} |Z_n|^{\frac{1}{n}} = L$$

If $L < 1$ Then $\sum_{n=0}^{\infty} Z_n$ is absolutely convergent and,

If $L > 1$ Then $\sum_{n=0}^{\infty} Z_n$ is diverges.

Example $(\mathbf{1.2.6})$: $-$Consider the series $\sum_{n=0}^{\infty} \frac{i^n}{2^n}$

Here $Z_n = \frac{i^n}{2^n}$ we can use the ratio test to show that this series converges absolutely indeed not that

$$\frac{|Z_{n+1}|}{|Z_n|} = \left|\frac{i^{n+1}}{2^{n+1}} . \frac{2^n}{i^n}\right| = \left|\frac{i}{2}\right| = \frac{1}{2}$$

Hence $$\lim_{n\to\infty} \frac{|Z_{n+1}|}{|Z_n|} = \frac{1}{2} < 1$$

An so by the ratio test the series converges absolutely. We could also have used the root test to show that this series converges absolutely. To see this note that

$$|Z_n|^{\frac{1}{n}} = \left|\frac{i^n}{2^n}\right|^{\frac{1}{n}} = \left(\frac{1}{2^n}\right)^{\frac{1}{n}} = \frac{1}{2}$$

Hence $$\lim_{n\to\infty} |Z_n|^{\frac{1}{n}} = \frac{1}{2} < 1$$

And so by the root test the series converges absolutely.

Definition (1.2.7)

A series of the form

$$\sum_{n=0}^{\infty} a_n (Z - Z_0)^n \quad \text{where} \quad a_n \in \mathbb{C}, \quad Z \in \mathbb{C}$$

Is called a power series at Z_0.By changing variable and replacing $Z - Z_0$ by Z we need only consider power series at 0 i.e. power series of the form

$$\sum_{n=0}^{\infty} a_n Z^n \qquad \text{where} \quad a_n \in \mathbb{C}, \quad Z \in \mathbb{C}$$

When does a power series a converge. Let

$$R = \sup\left\{r \geq 0/\text{There exist } \mathrm{Z} \in \mathbb{C} \text{ such that } |Z| = r \text{ and } \sum_{n=0}^{\infty} a_n Z^n \text{ converge}\right\}$$

We allow $R = \infty$ if no finite supermom exists.

Theorem (1.2.8)

Let $\sum_{n=0}^{\infty} a_n Z^n$ be a power series and let R be define as $(**)$, then

1. $\sum_{n=0}^{\infty} a_n Z^n$ converges absolutely for $|Z| < R$.

2. $\sum_{n=0}^{\infty} a_n Z^n$ diverges for $|Z| > R$.

Definition (1.2.9)

The number R given in the theorem is called the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n Z^n$$

We will call the set $\{Z \in \mathbb{C} \,/\, |Z| < R\}$ the disk of convergence.

Proposition (1.2.10)

Let $\sum_{n=0}^{\infty} a_n Z^n$ be a power series :

i. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exist, then

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

ii. If $\lim_{n\to\infty} |Z_n|^{\frac{1}{n}}$ exist, then

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Here we interpret $\frac{1}{0}$ as ∞ and $\frac{1}{\infty}$ as 0.

Proof: (**i**) Suppose that $\left|\frac{a_{n+1}}{a_n}\right|$ converges to a limit say L, as $n \to \infty$, i. e.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}\,Z^{n+1}}{a_n\,Z^n}\right| = L|Z|$$

By the ratio test the power series $\sum_{n=0}^{\infty} a_n Z^n$ converges for $L|Z| < 1$ and diverges for $L|Z| > 1$. Hence the radius of convergence

$$R = \frac{1}{L}$$

(**ii**) Suppose that $|a_n|^{\frac{1}{n}} \to L$ an $n \to \infty$ by the root test.

The power series $\sum_{n=0}^{\infty} a_n Z^n$ convergence if:

$$\lim_{n\to\infty}|a_n Z^n|^{\frac{1}{n}} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}|Z| = L|Z| < 1$$

And diverges if:

$$\lim_{n\to\infty}|a_n Z^n|^{\frac{1}{n}} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}|Z| = L|Z| > 1$$

Hence the radius of convergence

$$R = \frac{1}{L}$$

Example (1.2.11)

$\sum_{n=0}^{\infty} \frac{Z^n}{n}$ here $a_n = \frac{1}{n}$ in this case:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \to 1 = \frac{1}{R}$$

As $n \to \infty$. Hence the radius of convergence to 1.

Lemma (1.2.12)

Let $f(Z) = \sum_{n=0}^{\infty} a_n Z^n$ have radius of convergence R. Then

$$g(Z) = \sum_{n=1}^{\infty} n a_n Z^{n-1}$$

Converges for $|Z| < R$.

Proof: $-$ Let $|Z| < R$ and choose r such that $|Z| < r < R$, then $\sum_{n=1}^{\infty} a_n r^n$ converges absolutely. Hence the summands must be bounded. So there exists $k > 0$ such that $|a_n r^n| < k$ for all $n \geq 0$.

Let $q = \frac{|Z|}{r}$ and note that $0 < q < 1$ then

$$|n a_n Z^{n-1}| = n|a_n| \left|\frac{Z}{r}\right|^{n-1} r^{n-1} < n\frac{k}{r} q^{n-1}$$

But $\sum_{n=1}^{\infty} n q^{n-1}$ converges to $(1-q)^{-2}$. Hence by the comparison test $\sum_{n=0}^{\infty} |n a_n Z^{n-1}|$ converges. Hence $\sum_{n=0}^{\infty} n a_n Z^{n-1}$ converges absolutely and so converges.

Proposition (1.2.13)

Let $f(Z) = \sum_{n=0}^{\infty} a_n Z^n$ have radius of convergence R. Then all of the higher derivatives $f', f'', f''', \dots, f^{(k)}, \dots$ of f exist for Z within the disc of convergence moreover,

$$f^{(k)}(Z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n Z^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n Z^{n-k}$$

Chapter Two

Univalent Functions and Libera Integral

(2.1) Univalent Functions

Definition (2.1.1)

$w_0 \subset \mathbb{C}$ be a domain; that is, in open and connected non–empty sub set of the complex plane. A function $f: D \to \mathbb{C}$ is analytic at Z_0 if it is complex differentiable at every point in some neighborhood of $Z_0 \in D$. We say that f is analytic on D if f is analytic at Z_0 for every $Z_0 \in D$.

Definition (2.1.2)

A function $f: D \to \mathbb{C}$ is called univalent on D (or schlict or one–to–one) if $f(Z_1) \neq f(Z_2)$ for all $Z_1, Z_2 \in D$ with $Z_1 \neq Z_2$.

Example (2.1.3)

$f(Z) = Z + \frac{Z^2}{2} + \frac{Z^3}{3}$ is univalent in $|Z| < 1$.

Since $Z_1 = \frac{1}{2}, Z_2 = \frac{i}{2} \Rightarrow Z_1 \neq Z_2$

$$f\left(\frac{1}{2}\right) = \frac{2}{3} \neq f\left(\frac{i}{2}\right) = \frac{11i - 3}{24}$$

Definition (2.1.4)

A function $f: D \to \mathbb{C}$ is called Locally univalent at Z_0 if f is univalent in some neighborhood of Z_0.

Lemma (2.1.5)

Let $f: D \to \mathbb{C}$, if f analytic on D, then $f'(Z_0) \neq 0$ if and only if f Locally univalent at Z_0.

Remark (2.1.6)

It is also important to note that an analytic function may be Locally univalent throughout a domain although it need not be univalent in that domain.

Example (2.1.7)

Let $f: D \to \mathbb{C}$ given by $f(Z) = Z^2$, where

$$D = \left\{Z \in \mathbb{C}: 1 < |Z| < 2, 0 < \arg(Z) < \frac{3\pi}{2}\right\}$$

It is clear that f is analytic on D and Locally univalent at every $Z_0 \in D$, since $f'(Z_0) = 2Z_0 \neq 0$ for all $Z_0 \in D$ but f is not univalent on D.

Since $Z_1 = \dfrac{3}{2\sqrt{2}} + \dfrac{3}{2\sqrt{2}}i \neq Z_2 = \dfrac{-3}{2\sqrt{2}} - \dfrac{3}{2\sqrt{2}}i$

$f(Z_1) = f(Z_2) = \dfrac{9}{4}i$

Definition (2.1.8)

A function $f: D \to \mathbb{C}$ which is both analytic on D and univalent on D is called conformal mapping on D.

Example (2.1.9)

Let $f: D \to D \cap H$ be a function given by $f(Z) = Z^2$, where

$$D = \{Z \in \mathbb{C}: 0 < |Z| < 1, \mathrm{Im}\{Z\} > 0, Re\{Z\} > 0\}$$

$$\text{and} \quad H = \left\{Z \in \mathbb{C}: 0 < |Z| < 1, 0 < \arg(Z) < \frac{\pi}{2}\right\}$$

We note that f is conformal, since f is analytic and univalent on D and onto $D \cap H$.

Example (2.1.10)

Let $\mathrm{g}: D \to D$ be a function given by $\mathrm{g}(Z) = Z^2$, where

$$D = \{Z \in \mathbb{C}: 0 < |Z| < 1, \mathrm{Im}\{Z\} > 0, Re\{Z\} > 0\}$$

Note that f is conformal onto the disk D,

Since f analytic, but not univalent; for instance

$$f\left(\frac{1}{2}\right) = f\left(\frac{-1}{2}\right) = \frac{1}{4}$$

And $f'(0) = 0$ which means that there is no neighborhood of 0 in which f is univalent.

(2.2) Univalent Function defined by Libera Integral Operator

Definitions (2.2.1)

(1) Let $E = \{Z: |Z| < 1\}$ be a unite disk and

$$A = \left\{ f: f(A) = Z + \sum_{k=2}^{\infty} a_k Z^k \text{ , } Z \in E \text{ is analytic in } E \text{ and normalized by } f(0) = 0, f^{'}(0) = 1 \right\}$$

(2) we say that f is bounded if there exists $M > 0$ such that

$$|f^{'}(Z)| \leq M \text{ , } \forall Z \in N$$

(3) for every $\theta \in R$, the map

$$g(Z) = e^{-i\theta} + f\left(e^{i\theta} Z\right) = Z + e^{i\theta} a_2 Z^2 + e^{i2\theta} a_3 Z^3 + \cdots$$

Belongs to S. This property implies that for every integral $k \geq 1$ the set $\{f^k(0): f \in S\}$ is invariant under the rotation about 0.

Definition (2.2.2)

Let

$$B(\alpha) = \left\{ f: f(Z) = Z + \sum_{k=2}^{\infty} a_k Z^k \text{ such that } R\left(f^{'}(Z)\right) > \alpha in\ \epsilon, where\ 0 \leq \alpha < 1 \right\} \subset A$$

The functions in $B(\alpha)$ are called functions of bounded turning.

Definition (2.2.3)

The Libera integral operator F of $f \in A$ is given by

$$F(Z) = \frac{2}{Z}\int_{0}^{Z} f(t)dt = Z + \sum \frac{2}{(k+1)} a_k Z^k$$

The n–th partial sums $F_n(Z)$ of the Libera integral operator $F(Z)$ are given by

$$F_n(Z) = Z + \sum_{k=2}^{\infty} \frac{2}{(k+1)} a_k Z^k$$

Definition (2.2.4)

For function Q analytic in E the convolution function $P * Q$ takes values in the convex hull of the image on E under Q, where $P(Z)$ analytic in E.

The operator (*) stands for the Hadamard prod net or convolution of two power series

$$f(Z) = \sum_{k=1}^{\infty} a_k Z^k \qquad \text{and} \qquad g(Z) = \sum_{k=1}^{\infty} b_k Z^k$$

Denoted by:

$$(f * g)(Z) = \sum_{k=1}^{\infty} a_k b_k Z^k$$

Lemma (2.2.5)

Let θ be a real number and M and k be natural numbers. Then:

$$\frac{1}{3}+\sum_{k=2}^{M}\frac{\cos(k\theta)}{k+2}\geq 0$$

Lemma (2.2.6)

$$R\left(\sum_{k=1}^{M}\frac{Z^k}{k+2}\right)>-\frac{1}{3}, \quad \forall Z\in E.$$

Proof:- for $0\leq r<1$ and for $0\leq\theta\leq\pi$ write

$$Z=re^{i\theta}=r(\cos\theta+i\sin\theta)$$

By Demoiver's Law, we have

$$R\left(\sum_{k=1}^{M}\frac{Z^k}{k+2}\right)=\sum_{k=1}^{M}\frac{r^k\cos(k\theta)}{k+2}>\sum_{k=1}^{M}\frac{\cos(k\theta)}{k+2}$$

By lemma **2.2.5**, we conclude that $\sum_{k=1}^{M}\frac{\cos(k\theta)}{k+2}$ is grater than or equal to $-\frac{1}{3}$

Lemma (2.2.7)

Let $P(Z)$ be analytic in E, $P(0)=1$, and $R(P(Z))>\frac{1}{2}$ in E.

Theorem (2.2.8):- If $\frac{1}{4} \le \alpha < 1$ and $f \in B(\alpha)$, then

$$F_n \in B\left(\frac{4\alpha - 1}{3}\right)$$

Proof:- let $f \in B(\alpha)$ and $\frac{1}{4} \le \alpha < c_1$. Since $R(f^{'}(Z)) > \alpha$ we have

$$R\left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k Z^{k-1}\right) > \frac{1}{2}$$

Applying the convolution properties of power series to $f_n^{'}(Z)$. We may write

$$f_n^{'}(Z) = 1 + \sum_{k=2}^{n} \frac{2k}{k+1} a_k Z^{k-1}$$

$$= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k Z^{k-1}\right) * \left(1 + (1-\alpha) \sum_{k=2}^{n} \frac{4}{k+1} Z^{k-1}\right)$$

$$= P(Z) * Q(Z)$$

From lemma (2.2.6), for $M = n - 1$ we obtain

$$R\left(\sum_{k=2}^{n} \frac{Z^{k-1}}{k+1}\right) > -\frac{1}{3}$$

Thus we have

$$R\big(Q(Z)\big) = R\left(1 + (1-\alpha) \sum_{k=2}^{n} \frac{4Z^{k-1}}{k+1}\right) > \frac{4\alpha - 1}{3}$$

On the other hand, since

$$R\left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k Z^{k-1}\right) > \frac{1}{2}$$

Therefore by lemma (2.2.7),

$$R\big(F_n^{'}(Z)\big) > \frac{4\alpha - 1}{3}$$

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