Ministry of Higher Education and Scientific Research University of Qadisiyah College of Education Department of Mathematics



# THE ENUMREATION OF THE GRAPH POSET

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By

Mohammed Hassan Nahi

Supervisor

Dr. Khalid Shea Kharallha Al'Dzhabri

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#### Abstract

Any binary relation  $\sigma \subseteq X$  (where *X* is an arbitrary set) generates a characteristic function on the set  $X^2$ : If  $(x, y) \in \sigma$ , then  $\sigma(x, y) = 1$ , otherwise  $\sigma(x, y) = 0$ . In terms of characteristic functions on the set of all binary relations of the set *X* we introduced the concept of a binary of reflexive relation of adjacency and determined the algebraic system consisting of all binary relations of a set *X* and all unordered pairs of various adjacent binary relations. If *X* is finite set then this algebraic system is a graph " a graph of graphs" in this work we investigated some features of the structures of the graph G(X) of partial orders. In this work we study new concept support set of partial sets and proved some features of this concept.

#### Abstract

Any binary relation  $\sigma \subseteq X$  (where *X* is an arbitrary set) generates a characteristic function on the set  $X^2$ : If  $(x,y) \in \sigma$ , then  $\sigma(x,y)=1$ , otherwise  $\sigma(x,y)=0$ . In terms of characteristic functions on the set of all binary relations of the set *X* we introduced the concept of a binary of reflexive relation of adjacency and determined the algebraic system consisting of all binary relations. If *X* is finite set then this algebraic system is a graph " a graph of graphs" in this work we investigated some features of the structures of the graph G(X) of partial orders.

#### Introduction

A graph is a symbolic representation of a network and of its connectivity. It implies an abstraction of the reality so it can be simplified as a set of linked nodes. Graph theory is a branch of mathematics concerned about how networks can be encoded and their properties measured. It has been enriched in the last decades by growing influences from studies of social and complex networks. The origins of graph theory can be traced to Leonhard Euler who devised in 1735 a problem that came to be known as the "Seven Bridges of Konigsberg". In this problem, someone had to cross once all the bridges only once and in a continuous sequence, a problem the Euler proved to have no solution by representing it as a set of nodes and links. This led the the foundation of graph theory and its subsequent improvements. In transport geography most networks have an obvious spatial foundation, namely road, transit and rail networks, which tend to be defined more by their links than by their nodes. This it is not necessarily the case for all transportation networks. For instance, maritime and air networks tend to be more defined more by their nodes than by their links since links are often not clearly defined. A telecommunication system can also be represented as a network, while its spatial expression can have limited importance and would actually be difficult to represent. Mobile telephone networks or the Internet, possibly to most complex graphs to be considered, are relevant cases of networks having a structure that can

be difficult to symbolize. However, cellular phones and antennas can be represented as nodes while the links could be individual phone calls. Servers, the core of the Internet, can also be represented as nodes within a graph while the physical infrastructure between them, namely fiber optic cables, can act as links. Consequently, all transport networks can be represented by graph theory in one way or the other. The following elements are fundamental at understanding graph theory: Graph. The fundamental concept of graph theory is the graph, which (despite the name) is best thought of as a mathematical object rather than a diagram, even though graphs have a very natural graphical representation. A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. Vertices are also known as nodes, points and (in social networks) as actors, agents or players. Edges are also known as lines and (in social networks) as ties or links. An edge e =(u,v) is defined by the unordered pair of vertices that serve as its end points. Two vertices u and v are *adjacent* if there exists an edge (u,v)that connects them. An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*. As an example, the graph depicted in Figure 1 has vertex set  $V = \{a, b, c, d, e, f\}$ and edge set  $E = \{(a,b), (b,c), (c,d), (c,e), (d,e), (e,f)\}$ .



Figure 1.

When looking at visualizations of graphs such as Figure 1, it is important to realize that the only information contained in the diagram is adjacency; the position of nodes in the plane (and therefore the length of lines) is arbitrary unless otherwise specified. Hence it is usually dangerous to draw conclusions based on the spatial position of the nodes. For example, it is tempting to conclude that nodes in the middle of a diagram are more important than nodes on the peripheries, but this will often – if not usually – be a mistake. When used to represent social networks", "we typically use each line to represent instances of the same social relation, so that if (a,b) indicates a friendship between the person located at node a and the person located at node b, then (d,e) indicates a friendship between d and e. Thus, each distinct social relation that is empirically measured on the same group of people is represented by separate graphs, which are likely to have different structures (after all, who talks to whom is not the same as who dislikes whom).

Every graph has associated with it an adjacency matrix, which is a binary  $n \times n$  matrix A in which  $a_{ij} = 1$  and  $a_{ji} = 1$  if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown in Figure 2.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Figure 2. Adjacency matrix for graph in Figure 1.

Examining either Figure 1 or Figure 2, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*. The extent to which a graph is complete is indicated by its density, which is defined as the number of edges divided by the number possible. If self-loops are excluded, then the number possible is n(n-1)/2. If self-loops are allowed, then the number possible is n(n+1)/2. Hence the density of the graph in Figure 1 is 6/15 = 0.40.

A *clique* is a *maximal complete subgraph*. A *subgraph* of a graph *G* is a graph whose points and lines are contained in *G*. A complete subgraph of *G* is a section of *G* that is complete (i.e., has density = 1). A maximal complete subgraph is a subgraph of *G* that is complete and is maximal in the sense that no other node of *G* could be added to the subgraph without losing the completeness property. In Figure 1, the nodes  $\{c,d,e\}$  together with the lines connecting them form a clique. Cliques have been seen as a way to represent what social scientists have called primary groups.

# **CHAPTER ONE**

# BASIC DEFINITIONS AND FACTS

# **1.1. Partial Orderings**

# **Definition 1.1.1**

Let *R* be a relation on A. Then R is a *partial order* 

iff R is

- Reflexive
- Antisymmetric

and

- Transitive
- (A, R) is called a partially ordered set or a *poset*.

# Note 1.1.2

It is not required that two things be related under a partial order . That's the partial part of it .

If two objects are always related in a poset, it is called a total order or linear order. In this case (A, R) is called a chain.

# Examples 1.1.3

Examples:

1.  $(Z \le )$  is a poset. In this case either  $a \le b$  or  $b \le a$  so

two things are always related. Hence,  $\leq$  is a total order and

 $(Z, \leq)$  is a chain.

2. If S is a set then  $(P(S), \cup)$  is a poset. It may not be

the case that  $A \cup B$  or  $B \cup A$ . Hence,  $\cup$  is not a total

order.

3. (Z+, 'divides') is a poset which is not a chain.

**Definition 1.1.4** Let *R* be a total order on *A* and suppose  $S \cup A$ . An element *s* in *S* is a *least element* of *S* iff *sRb* for every *b* in *S*. Similarly for *greatest* element.

## Note 1.1.5

This implies that  $\langle a \rangle$ ,  $s \rangle$  is not in *R* for any a unless a = s.

(There is nothing smaller than s under the order R).

# **Definition 1.1.6**

A chain (A, R) is well – ordered iff every subset of A has a least element.

## Examples 1.1.7

- $(Z, \leq)$  is a chain but not well ordered .*Z* does not have least element.
- $(N, \leq)$  is well ordered.
- $(N, \ge)$  is not well ordered.

## **Definition 1.1.8**

Given two posets (A1, R1) and (A2, R2) we construct an *induced* partial order R on  $A1 \times A2$ :

$$< x1$$
,  $y1 > R < x2$ ,  $y2 > iff$   
•  $x1 R x2$   
or  
•  $x1 = x2$  and  $y1 R2 y2$ .

### Example 1.1.9

Let A1 = A2 = Z + and R1 = R2 = 'divides'.

Then

- <2, 4 > R < 2, 8 > since x1 = x2 and y1 R2 y2.
- <2, 4 > is not related under R to <2, 6 > since x1 = x2 but 4 does not divide 6.
- <2, 4 > R < 4, 5 > since x1 R1 x2 (Note that 4 is not related to 5)

This definition extends naturally to multiple Cartesian products of partially ordered sets :

 $A1 \times A2 \times A3 \times \ldots \times An$ .

## Example 1.1.10

Using the same definitions of Ai and Ri as above,

- <2, 3, 4, 5 > R < 2, 3, 8, 2 >since x1 = x2, y1 = y2 and 4 divides8.
- <2 ,3, 4,5 > is not related to <3 ,6, 8,10 > since 2 does not divide 3 .

We apply this ordering to strings of symbols where is an underlying 'alphabetical' or partial order ( which is a total order in this case ).

# Example 1.1.11

Let  $AI = \{a, b, c\}$  and suppose *R* is the natural alphabetical order on *A*:

a R b and b R c.

Then

- Any shorter string is related to any longer string( comes before it in the ordering ).
- If two strings have the same length then use the induced partial order from the alphabetical order :

AabcR abac

#### 1.2 Walks, Trails, Paths, Circuits, Connectivity, Components

**Remark 1.2.1***There are many different variations of the following terminologies. We will adhere to the definitions given here.* A *walk* in the graph G = (V,E) is a finite sequence of the form vi0, ej1, vi1, ej2, . . . , ejk , vik , which consists of alternating vertices and edges of G. The walk starts at a vertex. Vertices vit–1 and vit are end vertices of ejt (t = 1, . . . , k). vi0 is the *initial vertex* and vik is the *terminal vertex*. k is the *length* of the walk. A zero length walk is just a single vertex vi0 . It is allowed to visit a vertex or go through an edge more than once. A walk is *open* if vi0 6= vik . Otherwise it is *closed*.

**Example 1.2.2** *In the graph* 



*the walk* v2, e7, v5, e8, v1, e8, v5, e6, v4, e5, v4, e5, v4 *is open. On the other hand, the walk* v4, e5, v4, e3, v3, e2, v2, e7, v5, e6, v4 *is closed.* A walk is a *trail* if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v.

**Example 1.2.3** (Continuing from the previous example) The walk in the graph v1, e8, v5, e9, v1, e1, v2, e7, v5, e6, v4, e5, v4, e4, v4 is a trail. A trail is a path if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a *circuit*. For simplicity, we will assume in the future that a circuit is not empty, i.e. its length  $\geq$  1. We identify the paths and circuits with the subgraphs induced by their edges.

**Example 1.2.4** (*Continuing from the previous example*) *The walk* v2, e7, v5, e6, v4, e3, v3 *is a path and the walk* v2, e7, v5, e6, v4, e3, v3, e2, v2 *is a circuit*. The walk starting at u and ending at v is called an u–v

*walk.* u and v are *connected* if there is a u–v walk in the graph (then there is also a u–v path!). If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a u–v walk and a v–w walk, then there is also a u–w walk. A graph is *connected* if all the vertices are connected to each other. (A trivial graph is connected by convention.)

Example 1.2.5 *The graph:* 



*is not connected*. The subgraph G1 (not a null graph) of the graph G is a *component* of G if:

1. G1 is connected and

2. Either G1 is trivial (one single isolated vertex of G) or G1 is not trivial and G1 is the subgraph induced by those edges of G that have one end vertex in G1. Different components of the same graph do not have any common vertices because of the following theorem.

**Theorem 1.2.6** *If the graph* G *has a vertex* v *that is connected to a vertex of the component* G1 *of* G, *then* v *is also a vertex of* G1.

Proof. If v is connected to vertex v' of G1, then there is a walk in G

v = vi0, ej1, vi1, ..., vik-1, ejk, vik = v'. Since v' is a vertex of G1, then (condition #2 above) ejk is an edge of G1 and vik-1 is a vertex of G1. We continue this process and see that v is a vertex of G1.





The components of G are G1, G2, G3 and G4.

**Theorem 1.2.8** Every vertex of G belongs to exactly one component of G. Similarly, every edge of G belongs to exactly one component of G. Proof. We choose a vertex v in G. We do the following as many times as possible starting with  $V1 = \{v\}$ : If v' is a vertex of G such that v' / $\in$  V1 and v' is connected to some vertex of V1, then V1  $\leftarrow$  V1  $\cup$  {v'}.

Since there is a finite number of vertices in G, the process stops eventually. The last V1 induces a subgraph G1 of G that is the component of G containing v. G1 is connected because its vertices are connected to v so they are also connected to each other. Condition #2 holds because we can not repeat . By Theorem 1.2.6 v does not belong to any other component. The edges of the graph are incident to the end vertices of the components.

We now that theorem 1.2.8 divides a graph into distinct components. The proof of the theorem gives an algorithm to do that. We have to repeat what we did in the proof as long as we have free vertices that do not belong to any component. Every isolated vertex forms its own component. A connected graph has only one component, namely, itself. A graph G with n vertices, m edges and k components has the *rank* 

$$\rho(G) = n - k.$$

The *nullity* of the graph is

$$\mu(G) = m - n + k.$$

We see that  $\rho(G) \ge 0$  and  $\rho(G) + \mu(G) = m$ . In addition,  $\mu(G) \ge 0$ 

#### **1.3 Graph Operations**

The *complement* of the simple graph G = (V,E) is the simple graph G = (V,E), where the edges in E are exactly the edges not in G.

### Example 1.3.1



**Example 1.3.2** The complement of the complete graph Kn is the empty graph with n vertices. Obviously, G = G. If the graphs G = (V,E) and G' = (V',E') are simple and  $V' \subseteq V$ , then the difference graph is G - G' = (V,E''), where E'' contains those edges from G that are not in G' (simple graph).

Example 1.3.3



Here are some binary operations between two simple graphs G1 = (V1,E1) and G2 = (V2,E2):

- The *union* is  $G1 \cup G2 = (V1 \cup V2, E1 \cup E2)$  (simple graph).
- The *intersection* is  $G1 \cap G2 = (V1 \cap V2, E1 \cap E2)$  (simple graph).

• The *ring sum* G1 $\oplus$ G2 is the subgraph of G1UG2 induced by the edge set E1 $\oplus$ E2 (simple graph). *Note!* The set operation  $\oplus$  is the *symmetric difference*, i.e. E1  $\oplus$  E2 = (E1 – E2) U (E2 – E1). Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative.

**Example 1.3.4** For the graphs:





We have





**Remark 1.3.5** The operations  $\cup$ ,  $\cap$  and  $\bigoplus$  can also be defined for more general graphs other than simple graphs. Naturally, we have to "keep track" of the multiplicity of the edges:

 $\cup$ : The multiplicity of an edge in G1  $\cup$  G2 is the larger of its multiplicities in G1 and G2.

 $\cap$ : The multiplicity of an edge in G1  $\cap$  G2 is the smaller of its multiplicities in G1 and G2.

 $\oplus$ : The multiplicity of an edge in G1  $\oplus$  G2 is |m1 - m2|, where m1 is its multiplicity in G1 and m2 is its multiplicity in G2.

(We assume zero multiplicity for the absence of an edge.) In addition, we can generalize the difference operation for all kinds of graphs if we take account of the multiplicity. The multiplicity of the edge e in the

$$m_1 - m_2 = \begin{cases} m_1 - m_2, \text{ if } m_1 \ge m_2 \\ 0, \text{ if } m_1 < m_2 \end{cases}$$

difference G - G' is

(also known as the proper difference), where m1 and m2 are the multiplicities of e in G1 and G2, respectively.

If v is a vertex of the graph G = (V,E), then G - v is the subgraph of G induced by the vertex set  $V - \{v\}$ . We call this operation the *removal of a vertex*.

**Example 1.3.6** (*Continuing from the previous example*)



Similarly, if e is an edge of the graph G = (V,E), then G - e is graph (V,E'), where E' is obtained by removing e from E. This operation is known as *removal of an edge*. We remark that we are not talking about removing an edge as in Set Theory, because the edge can have non unit multiplicity and we only remove the edge once.

**Example 1.3.7** (*Continuing from the previous example*)



If u and v are two distinct vertices of the graph G = (V,E), then we can *short-circuit* the two vertices u and v and obtain the graph (V ',E'), where  $V ' = (V - \{u, v\}) \cup \{w\}$  (w / $\in$  V is the "new" vertex)

and  $E' = (E - \{(v', u), (v', v) | v' \in V \}) \cup \{(v', w) | (v', u) \in E \text{ or } (v', v) \in E \}$ 

 $\cup \{(w,w) | (u, u) \in E \text{ or } (v, v) \in E\}$ 

(Recall that the pair of vertices corresponding to an edge is not ordered). *Note!* We have to maintain the multiplicity of the edges. In particular, the edge (u, v) becomes a loop.

**Example 1.3.8** (*Continuing from the previous example*) *Short-circuit* v3 *and* v4 *in the graph* G1:



In the graph G = (V,E), *contracting* the edge e = (u, v) (not a loop) means the operation in which we first remove e and then short-circuit u and v. (Contracting a loop simply removes that loop.)

**Example 1.3.9** (*Continuing from the previous example*) We contract the edge e3 in G1 by first removing e3 and then short-circuiting v2 and v3.



**Remark 1.3.10** If we restrict short-circuiting and contracting to simple graphs, then we remove loops and all but one of the parallel edges between end vertices from the results.

# **CHAPTER TWO**

# THE ENUMERATION OF THE GRAPHS OF POSET

#### 2.1 Adjacency of binary relations

**Definition 2.1.1** Let  $B \notin \{0,1\}$  –Boolean set, X – arbitrary set, and  $X^2 \notin X \times X$  – a direct product. The function  $X^2 \rightarrow B$ , will be called *characteristic*. Any subset  $\sigma \subseteq X^2$ , called a binary relation (or relation) on the set X, generates characteristic function

$$\chi_R: X^2 \to B$$
,  $\chi_R(x, y) \, \stackrel{\text{def}}{\Longrightarrow} \left\{ \begin{array}{ll} 1, & \text{if } (x, y) \in R, \\ 0, & \text{if } (x, y) \notin R. \end{array} \right.$ 

Next, the function  $\chi_R(\cdot,\cdot)$  will be denoted by R(x, y). On the other hand, any characteristic function  $\chi : X^2 \rightarrow B$  generates a binary relation  $R_{\chi} \subseteq X^2$ such that  $(x, y) \in R_{\chi}$  if  $\chi(x, y) = 1$ . Obviously, the map  $R \rightarrow R(\cdot, \cdot)$  is a bijection between the set of binary relations and the set of characteristic functions.

On the set of  $2^{x^2}$  all sets of binary relations on the set *x* we introduce a binary reflexive adjacency.

**Definition 2.1.2** Let  $X = Y \cup Z$  – the disjoint union of two subsets (allowed, that either  $Y = \emptyset$  or  $Z = \emptyset$ ). Suppose that the relation  $\sigma \subseteq X^2$  such that  $\sigma(x, y) = 0$  for all  $(x, y) \in Y \times Z$ . It generates the relation  $\tau \subseteq X^2$  such that

1) 
$$\tau(x, y) = 1 - \sigma(y, x)$$
 for all  $(x, y) \in Y \times Z$ , (1)

2) 
$$\tau(x, y) = 0$$
 for all  $(x, y) \in Z \times Y$ , (2)

3)  $\tau(x, y) = \sigma(x, y)$  for all  $(x, y) \in Y^2 \cup Z^2$ . (3)

The relation  $\tau$  is called *adjacent* with a relation  $\sigma$ .

**Remark 2.1.3** From the definition it follows that if the relation  $\tau$  adjacent with a relation  $\sigma$ , then  $\sigma$  adjacent with a relation  $\tau$ , and this fact we write in the form of a diagram  $\sigma \xleftarrow{Y \times Z} \tau$ :



Here and elsewhere in the diagrams we mark for the value of the characteristic functions at those points which are known a priori. For example, in the block  $Y \times Z$  for the relation  $\sigma$  we write = *generalized* ? zero, and this means that

 $\sigma(x, y) = 0$  for all  $(x, y) \in Y \times Z$ ,

And in the same block for the relation  $\tau$  we write  $1-\sigma(x, y)$  for all  $(x, y) \in Y \times Z$ .

For example,  $X = \{1, ..., 6\}$ ,  $Y = \{1, 2\}$ ,  $Z = \{3, 4, 5, 6\}$ ,

1 0 0 1	0		1 0 0 1	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$
0 0 1 0 0 0 1 1	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$=\sigma \xleftarrow{Y \times Z} \tau =$	0	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$

#### 2.2 Adjacency of the partial orders

Let V(X) is the collection of all partial orders set define on the set X. In the other words, the relation  $\sigma \subseteq X^2$  belongs in the set V(X), if satisfies the following axioms:

- 1) reflexivity:  $(x, x) \in \sigma$ ;
- 2) transitivity: if  $(x, y) \in \sigma$ ,  $(y, z) \in \sigma$ , then  $(x, z) \in \sigma$ ;
- 3) antisymmetry: if  $(x, y) \in \sigma$ ,  $(y, x) \in \sigma$ , then x = y.

In the terms of the characteristic we have:  $\sigma \in V(X)$  if and only if

- 1)  $\sigma(x,x) = 1$  for all  $x \in X$ ; (4)
- 2)  $\sigma(x, y)\sigma(y, z) \le \sigma(x, z)$  for all  $x, y, z \in X$ ; (5)

3)  $\sigma(x, y)\sigma(y, x) = \delta_{xy}$  for all  $x, y \in X$  (where  $\delta_{xy}$  – Kronecker (6) symbol).

symbol).

**Theorem 2.2.1** Let  $\sigma$   $u \tau$  – are adjacent relations( i.e  $\sigma \xleftarrow{Y \times Z} \tau$ ). Inclusion  $\sigma \in V(X)$  hold if and only if  $\tau \in V(X)$ .

**Proof.** By symmetry, it suffices to prove this implication  $\sigma \in V(X) \Rightarrow \tau \in V(X)$ .

Let  $\sigma \in V(X)$ .

1. Since  $\tau(x,x) = \sigma(x,x) = 1$ , then the reflexivity relation  $\tau$  obviously.

2. Its clear that,  $\tau(x, y)\tau(y, x) = \sigma(x, y)\sigma(y, x)$  for any  $x, y \in X$ , which proves that antisymmetry relations  $\tau$ .

3. Transitivity. Let  $x, z, y \in X$  such that  $\tau(x, y) = \tau(y, z) = 1$ , in the first suppose that  $y \in Y$ . Since  $\tau(\zeta, y) = 0$  for all  $\zeta \in Z$ , then  $x \in Y$ . If  $z \in Y$ , then  $\sigma(x, y) = \tau(x, y) = 1$  and  $\sigma(y, z) = \tau(y, z) = 1$ , and since  $\sigma \in V(X)$ , then  $\sigma(x, z) = 1$ , therefore  $\tau(x, z) = 1$ . If  $z \in Z$ , then  $\sigma(x, y) = \tau(x, y) = 1$  and  $\sigma(z, y) = 1 - \tau(y, z) = 0$ , and since  $\sigma \in V(X)$ , then by (5)  $\sigma(z, x) = \sigma(z, x)\sigma(x, y) \le \sigma(z, y) = 0$ , hence,  $\sigma(z, x) = 0$ , and therefore  $\tau(x, z) = 1$ .

Now suppose that  $y \in Z$ . Since  $\tau(y,\eta) = 0$  for all  $\eta \in Y$ , then  $z \in Z$ . If  $x \in Z$ , then  $\sigma(x, y) = \tau(x, y) = 1$  and  $\sigma(y, z) = \tau(y, z) = 1$ , and since  $\sigma \in V(X)$ , then  $\sigma(x, z) = 1$ , and since  $\sigma \in V(X)$ , then by (5)  $\sigma(z, x) = \sigma(y, z)\sigma(z, x) \le \sigma(y, x) = 0$ , hence,  $\sigma(z, x) = 0$ , and therefore  $\tau(x, z) = 1$ . in all cases, we have the equality  $\tau(x, z) = 1$ .

Thus, the set *x* generates a pair  $\langle V(X), E(X) \rangle$ , where V(X) – is a set of vertices, consist of all partial orders of the set *x* and E(X) – is a set of edges, consist of all unordered distinct pairs of adjacent partial orders of the set *x*. The pair  $G(X) \not\ll V(X)$ ,  $E(X) \rangle$  will be called (undirected) graph of partial orders of the set *x*.

**Definition 2.2.2** The partial orders  $\sigma$  and  $\tau$  belong to the same connected component of the graph G(X), if there is a finite sequence of partial orders  $\sigma = \sigma_1, \sigma_2, K, \sigma_m = \tau$ , in which the relations  $\sigma_{k-1}$  and  $\sigma_k$  are adjacent for all k = 2, K, m. Let  $G_{\sigma}(X)$  is the connected component of the graph G(X), which contains the partial order  $\sigma$ .

#### 2.3 On the features of the structure of the graph of partial orders.

We fix the partial order  $\sigma \in V(X)$  and an element  $x \in X$ . For  $\sigma$  we have the representation:



 $I_x B I_x(\sigma) B\{y \in X : \sigma(x, y) = 1, \sigma(y, x) = 0\},$   $K_x B K_x(\sigma) B\{y \in X : \sigma(x, y) = \sigma(y, x) = \delta_{xy}\},$   $J_x B J_x(\sigma) B\{y \in X : \sigma(x, y) = 0, \sigma(y, x) = 1\}.$ Obviously,  $x \in K_x$ .

**Lemma 2.3.1** *The following statements holds:* 

- 1)  $\sigma(y,z) = 1$  for all  $(y,z) \in J_x \times I_x$ ;
- 2)  $\sigma(y,z) = 0$  for all  $(y,z) \in I_x \times (K_x \cup J_x)$ ;
- 3)  $\sigma(y,z) = 0$  for all  $(y,z) \in (K_x \cup I_x) \times J_x$ .

**Proof:** Obviously,  $K_x = \{ y \in X : \sigma(x, y) = \sigma(y, x) = \delta_{xy} \},\$ 

 $J_x = \{ y \in X : \sigma(x, y) = 0, \ \sigma(y, x) = 1 \}, \ I_x = \{ y \in X : \sigma(x, y) = 1, \ \sigma(y, x) = 0 \}.$ 

1. Since  $y \in J_x$ , then  $\sigma(y,x)=1$ , and since  $z \in I_x$ , then  $\sigma(x,z)=1$ , therefore

 $\sigma(y,z) = 1$ . In particular,  $(y,z) \in I_x \times J_x$  we have the equality  $\sigma(z, y) = 0$ .

2. Let  $(y,z) \in I_x \times K_x$ .

If z = x, then  $\sigma(y, z) = \sigma(y, x) = 0$  (since  $y \in I_x$ ).

Let  $z \neq x$ , and  $z \in K_x$ ,  $\sigma(x, z) = 0$ . Since  $y \in I_x$ , then  $\sigma(x, y) = 1$ , and then by (5)  $\sigma(y, z) = \sigma(x, y)\sigma(y, z) \le \sigma(x, z) = 0$  and therefore  $\sigma(y, z) = 0$  for all  $(y, z) \in I_x \times K_x$ . 3. Let  $(y, z) \in K_x \times J_x$ . If y = x, then  $\sigma(y, z) = \sigma(x, z) = 0$  (since  $z \in J_x$ ). Let  $y \neq x$ , and  $y \in K_x$ , then  $\sigma(y, x) = 0$ . Since  $z \in J_x$ , then  $\sigma(z, x) = 1$ , and by (5),  $\sigma(y, z) = \sigma(y, z)\sigma(z, x) \le \sigma(y, x) = 0$  therefore  $\sigma(y, z) = 0$  for all  $(y, z) \in K_x \times J_x$ Hence we can construct a sequence of adjacent of partial orders :



Which leads us to the partial order  $\sigma^x \in V(X)$ , that  $\sigma^x(x, y) = \sigma^x(y, x) = \delta_{xy}$  for all  $y \in X$  (in other words, if we interpret the partial order as relation  $\leq$ , then *x* is both a maximum and minimum element of a partial order  $\sigma^x$ ). thus, for a fixed partial order  $\sigma \in V(X)$  defined a map  $X \to G_{\sigma}(X)$ , associates to an element  $x \in X$  the partial order  $\sigma^x \in G_{\sigma}(X)$  (it may be that  $\sigma^x = \sigma^y$  at  $x \neq y$ ). We also note that this map is uniquely defined in the algorithm (7) are used uniquely defined sets  $I_x(\sigma)$ ,  $K_x(\sigma)$ ,  $J_x(\sigma)$ .

**Lemma 2.3.2** Suppose that the partial orders  $\sigma, \tau \in V(X)$  belong to the same connected component of the graph G(X), Then  $\sigma^x = \tau^x$  for any  $x \in X$ .

**Proof.** We can assume that  $\sigma$  and  $\tau$  – adjacent partial orders, then there is a disjoint union  $I \cup J = X$  such, that :  $\sigma \xleftarrow{I \times J} \tau$ .

without loss of generality, we can also assume that  $x \in J$  (if  $x \in J$  in the calculations presented below the relation  $\sigma$  and  $\tau$  changing places. For  $\sigma$  have the representation:



$$I_{1} = \{ y \in I : \sigma(x, y) = 0 \},\$$

$$I_{2} = \{ y \in I : \sigma(x, y) = 1 \},\$$

$$J_{1} = \{ y \in J : \sigma(x, y) = 1, \ \sigma(y, x) = 0 \},\$$

$$J_{2} = \{ y \in J : \sigma(x, y) = 0, \ \sigma(y, x) = 1 \},\$$

$$J_{3} = \{ y \in J : \sigma(x, y) = \sigma(y, x) = \delta_{xy} \}.$$

Its cleary that  $x \in J_3$ .

1. We fix  $y \in I_2 \cup J_1$ , to  $\sigma(x, y) = 1$ , since  $z \in I_1$ , then  $\sigma(x, z) = 0$  then by (5) we have  $\sigma(y, z) = \sigma(x, y)\sigma(y, z) \le \sigma(x, z) = 0$ . Thus  $\sigma(y, z) = 0$  for all  $(y, z) \in (I_2 \cup J_1) \times I_1$ . 2. Let  $(y, z) \in I \times (I_1 + I_1)$  And since  $y \in I$  then  $\sigma(y, z) = 1$  and since

2. Let  $(y,z) \in J_2 \times (I_2 \cup J_1)$ . And since  $y \in J_2$ , then  $\sigma(y,z) = 1$ , and since  $z \in I_2 \cup J_1$ , then  $\sigma(x,z) = 1$ , therefore  $\sigma(y,z) = 1$ . Thus ,  $\sigma(y,z) = 1$  for all  $(y,z) \in J_2 \times (I_2 \cup J_1)$ .

3. Due to of the antisymmetry  $\sigma$  for all  $(y,z) \in J_1 \times J_2$  have the equality  $\sigma(y,z) = 0$ .

4. Let 
$$(y,z) \in J_1 \times J_3$$
. if  $z = x$ , then  $\sigma(y,z) = \sigma(y,x) = 0$  (since  $z \in J_3$ ).  
Let  $z \neq x$ , and  $z \in J_3$  then  $\sigma(x,z) = 0$ . Since  $y \in J_1$ , then  $\sigma(x,y) = 1$ , then from  
(5)  $\sigma(y,z) = \sigma(x,y)\sigma(y,z) \le \sigma(x,z) = 0$ . Thus  $,\sigma(y,z) = 0$  for all  $(y,z) \in J_1 \times J_3$ .  
5. Let  $(y,z) \in J_3 \times J_2$ . If  $y = x$ , then  $\sigma(y,z) = \sigma(x,z) = 0$  (since  $z \in J_2$ ).  
Let  $y \neq x$ , and  $y \in J_3$ , TO  $\sigma(y,x) = 0$ . Since  $z \in J_2$ , then  $\sigma(z,x) = 1$ , then from (5),  
 $\sigma(y,z) = \sigma(y,z)\sigma(z,x) \le \sigma(y,x) = 0$  therefore  $\sigma(y,z) = 0$  thus  $\sigma(y,z) = 0$  for all  
 $(y,z) \in J_3 \times J_2$ .

Thus, for the adjacent of partial orders  $\sigma$  and  $\tau$  we have the representation:



(31)



We construct a sequence of two adjacent of partial orders:

$$\sigma \xleftarrow{(I_2 \cup J_1) \times (I_1 \cup J_2 \cup J_3)} \sigma' \xleftarrow{(I \cup J_1 \cup J_3) \times J_2} \sigma^x,$$

 $\tau \xleftarrow{J_1 \times (I \cup J_2 \cup J_3)}{\tau'} \tau' \xleftarrow{(I_2 \cup J_1 \cup J_3) \times (I_1 \cup J_2)}{\tau^x} \tau^x.$ 

				Ι			J								
				$I_1$		I <sub>2</sub>	$J_{1}$		$J_2$		<i>J</i> <sub>3</sub>				
σ =	I	Ι	1			*	0		0		0				
		Ι	2	0			0		0		0				
		J	1	0		*			0		0				
		J	2	*		1	1				M 1 M				
		J	3	L	0 L	LIL	L 1	L		0	N 0 N L 0 L 1 N 0 N	4 4 L O L 4	<i>←x</i>		
											$x\uparrow$				
					Ι		J								
					$I_1$	$I_2$	$J_1$	$J_2$		$J_3$					
σ' =			I	1		0	0	0		0					
		Ι	Ι	2	1—*		0	0		M O M					
				:		J	1	1	*		(	)	M O M		
	i.		J	2	*	0	0			N 1 N					
		J	J	3	LOL	0	0		0	LOL	M 0 M 1 L 0 L M 0 M	$\leftarrow x$			

(33)

 $x\uparrow$ 



(34)



 $x\uparrow$ 





Visual comparison  $\sigma^x$  and  $\tau^x$  shows their equality.

**Corollary 2.3.3.** In each connected component  $G_{\sigma}(X)$  of the graph G(X)for any  $x \in X$  there exists a unique  $\sigma^x \in V(X)$ , having the property, that  $\sigma^x(x,y) = \sigma^x(y,x) = \delta_{xy}$  for all  $y \in X$ .

**Remark 2.3.4** We fix  $x \in X$ . since from corollary (3.3) in the component  $G_{\sigma}(X)$  there unique partial order  $\sigma^x$  such that  $\sigma^x(x,y) = \sigma^x(y,x) = \delta_{xy}$  for all  $y \in X$  therefore the component  $G_{\sigma}(X)$  we can associate one-to- one partial order  $\sigma_x$ , define on the set  $X \setminus \{x\}$ , such that  $\sigma_x(y,x) = \sigma^x(y,x)$  for all  $y, z \in X \setminus \{x\}$ .

**Remark 2.3.5** If *card*  $X < \infty$  then there exist a one - to- one between the set  $V_0(X)$  and the set of all labeled of transitive graph define on the set X (see example [1, p28]) and there exist a one- to –one between these set and the set of  $T_0$  –topology define on the set X (see example [2, p256]) and the number of these topology denoted by  $T_0(n)$  and in the particular *card*  $V_0(X) = T_0(n)$ 

# **CHAPTER THREE**

# SUPPORT SET OF PARTIAL ORDERES

#### 3.1 Support of partial order.

**Definition 3.1.1** For a partial order  $\sigma \in V(X)$ .

The set  $S(\sigma) B\{y \in X : \sigma(y, x) = \delta_{xy} \text{ for all } x \in X\}$  is called (**support of partial order**)  $\sigma$  (or support set). a fact that we write in the form.



**Remake 3.1.2** Suppose that  $S(G_{\sigma}) = \{S(\tau) \subseteq X : \tau \in G_{\sigma}(X)\}$  the set of support of the partial order belong to the component  $G_{\sigma}(X)$  then:

1- 
$$\emptyset \notin S(G_{\sigma})$$
.

2- if 
$$\emptyset \neq \alpha \subseteq \beta \subseteq X$$
 and  $\beta \in S(G_{\sigma})$ , then  $\alpha \in S(G_{\sigma})$ .

3- if  $\alpha \subseteq X$  and  $|\alpha| \le 2$ , then  $\alpha \in S(G_{\alpha})$ .

**Remake 3.1.3** Suppose that *card* X=n then:

1-  $nT_0(n-1)$  different support sets of partial orders which contain one element.

2-  $\frac{1}{2}n(n-1)T_0(n-1)$  different support sets of partial orders which contain two element.

The proof of the following theorem in [3,4]

### **Theorem 3.1.4 For any** $n \ge 2$ then

$$T_0(n) = \frac{1}{2}n(n+1)T_0(n-1) + card\{\sigma \in V(\{1,...,n\}) : |S(\sigma)| \ge 3\}.$$

### **3.2 Examples of support sets.**

**Example 3.2.1 In the graph**  $G(\{1,2\})$  which have unique component which contains the partial order  $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \leftrightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \leftrightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \leftrightarrow \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$ .

**Example 3.2.2** The representation below of 3 connected component of the graph  $G(\{1,2,3\})$  contains 19 partial order  $T_0(2) = 3$ , and  $T_0(3) = 19$ :



We denote the graphs of the components  $K_1$ ,  $K_2 \bowtie K_3$ . It is clear that the component  $K_2$  and  $K_3$  are isomorphic if applied, for example, substitution  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$  to the elements of the component  $K_2$  we get the component  $K_3$ 

and  $S(K_1) = \{1, 2, 3\}, S(K_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},$  in the graph there is only one partial order, which  $|S(\sigma)| \ge 3$ .

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