Ministry of Higher Education
and Scientific Research
University of Qadisiyah
College of Education
Department of Mathematics

CERTAIN TYPE OF THE SUBGRAPHS OF THE GRAPH OF BINARY RELATIONS

Search submitted to the Mathematics Department Board As part of the requirements for a bachelor's degree

By

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بسم اللّه الرحمن الرحيم

" وما اوتيتم من العلم الا قليلاً "

صدق الله العلي العظيم
سورة الاسراء
الآية (٨٥)

Dedication

To ...... who gave me an idea and gave me a heart
My dear father

To the lamp of hope without fatigue or boredom
My mother is affectionate

To ... ... who walked with me towards science, step by step
My dear teacher and brothers and sisters

Give this humble effort

Contents

Abstract

Any binary relation $\sigma \subseteq X$ (where X is an arbitrary set) generates a characteristic function on the set X^2: If $(x, y) \in \sigma$, then $\sigma(x, y) = 1$, otherwise $\sigma(x, y) = 0$. In terms of characteristic functions on the set of all binary relations of the set X we introduced the concept of a binary of reflexive relation of adjacency and determined the algebraic system consisting of all binary relations of a set X and all unordered pairs of various adjacent binary relations. If X is finite set then this algebraic system is a graph " a graph of graphs" in this work we investigated some features of the structures of the graph $G(X)$ of partial orders. In this work we study new concept support set of partial sets and proved some features of this concept.

Introduction

Most of the work in function space topologies concerns continuous functions. In this connection see a remark by Kelley [3, p. 217]. As soon as we begin to consider function spaces of noncontiguous functions we come face to face with some extremely difficult problems. So in order to make a beginning, it is advisable to consider first a subfamily of noncontiguous functions which, in a certain sense, can be approximated by continuous functions. One such subfamily consists of almost continuous functions which were introduced by Stallings [6]. An almost continuous function is one whose graph can be approximated by graphs of continuous functions (see 2.3). The need to introduce a suitable topology for the function space of almost continuous functions arose when the author investigating the essential fixed points of such functions in his doctoral thesis [4] . The introduction of a new function space topology , called " the graph topology”, enabled him to tackle almost continuous functions. Let F denote an arbitrary subfamily of functions on a topological space X to a topological space Y and let F be given some topology. Most problems concerning F center round the following question, “what conditions on X and Y are sufficient to ensure that . F has a desired property” In this paper a few problems of the above nature are discussed.

This paper has a nonempty intersection with the author's doctoral thesis written under the supervision of Professor J. G. Hocking of Michigan State University. The author is grateful to his former colleague Professor D. E. Sanderson for valuable suggestions and comments. The referee suggested several improvements, supplied Example 5.1 and the references [1] and [5]

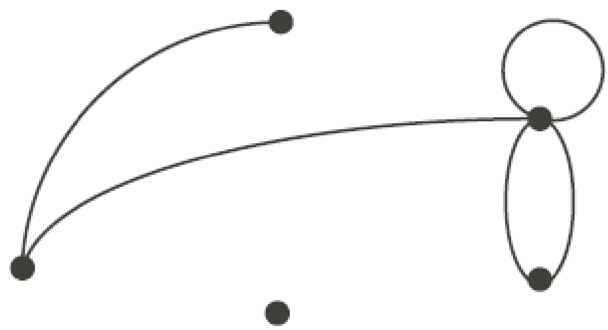
CHAPTER ONE

BASIC DEFINITIONS AN

1.1Graphs

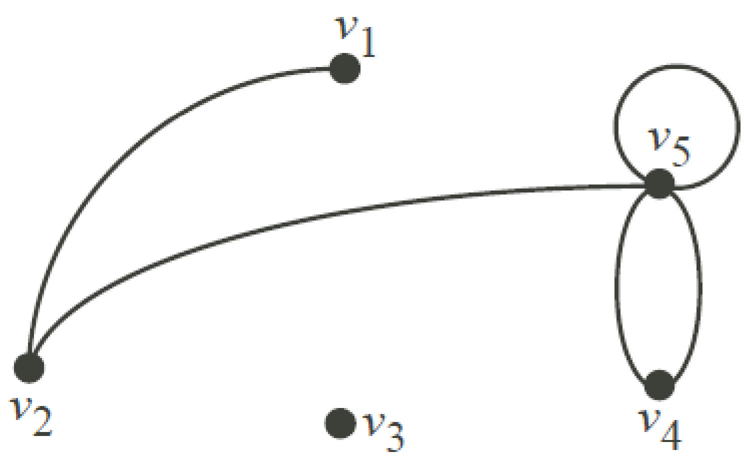
Definitions 1.1.1 Conceptually, graph is formed by vertices and edges connecting the vertices.

Example 1.1.2



Formally, graph is pair of sets (V,E),where V is the set of vertices and E is the set of edges , formed by pairs of vertices . E is multiuse ، in other words ,its elements can occur more than once so that every element has a multiplicity. Often, we label the vertices with letters) for example:a,b,c...،orv1,v2,...)or numbers 1,2,...Throughout this lecture material ,we will label the elements of V in this way.

Example. 1.1.3*(Continuing from the previous example) We label the vertices as follows:*

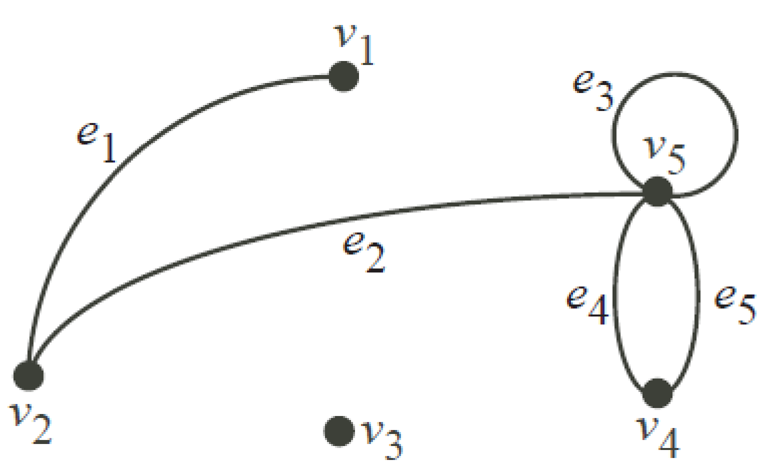


We have V = {v1, . . . , v5} *for the vertices and* E = {(v1, v2), (v2, v5), (v5, v5), (v5, v4), (v5, v4)}

for the edges. Similarly, we often label the edges with letters (for example: a, b, c, . . . or e1, e2, . . .) or numbers 1, 2, . . . for simplicity.

Remark 1.1.4 *The two edges* (u, v) *and* (v, u) *are the same. In other words, the pair is not* ordered.

Example 1.1.5 *(Continuing from the previous example) We label the edges as follows:*



So $E = \{e1, . . . , e5\}$.

We have the following terminologies:

1. The two vertices u and v are *end vertices* of the edge (u, v).
2. Edges that have the same end vertices are *parallel*.
3. An edge of the form (v, v) is a *loop*.
4. A graph is *simple* if it has no parallel edges or loops.
5. A graph with no edges (i.e. E is empty) is *empty*.
6. A graph with no vertices (i.e. V and E are empty) is a *null graph*.
7. A graph with only one vertex is *trivial*.
8. Edges are *adjacent* if they share a common end vertex.
9. Two vertices u and v are *adjacent* if they are connected by an edge, in other words, (u, v) is an edge.
10. The *degree* of the vertex v, written as d(v), is the number of edges with v as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
11. A *pendant vertex* is a vertex whose degree is 1.
12. An edge that has a pendant vertex as an end vertex is a *pendant edge*.
13. An *isolated vertex* is a vertex whose degree is 0.

Example 1.1.6 *(Continuing from the previous example)*

- v4 *and* v5 *are end vertices of* e5.
- e4 *and* e5 *are parallel.*
- e3 *is a loop.*

- *The graph is not simple.*
- $e1$ *and* $e2$ *are adjacent.*
- $v1$ *and* $v2$ *are adjacent.*
- *The degree of* $v1$ *is* 1 *so it is a pendant vertex.*
- $e1$ *is a pendant edge.*
- *The degree of* $v5$ *is* 5*.*
- *The degree of* $v4$ *is* 2*.*
- *The degree of* $v3$ *is* 0 *so it is an isolated vertex.*

In the future, we will label graphs with letters, for example:

G = (V,E). The *minimum degree* of the vertices in a graph G is denoted $\delta(G)$ (= 0 if there is an isolated vertex in G). Similarly, we write $\Delta(G)$ as the *maximum degree* of vertices in G.

Example 1.1.7 *(Continuing from the previous example)* $\delta(G) = 0$ *and* $\Delta(G) = 5$.

Remark 1.1.8 *In this course, we only consider* finite *graphs, i.e.* V *and* E *are finite sets.*

Since every edge has two end vertices, we get

Theorem 1.1.9 *The graph* G = (V,E)*, where* V = {v1, . . . , vn} *and* E = {e1, . . . , em}*, satisfies*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Corollary 1.1.10 *Every graph has an even number of vertices of odd degree.*

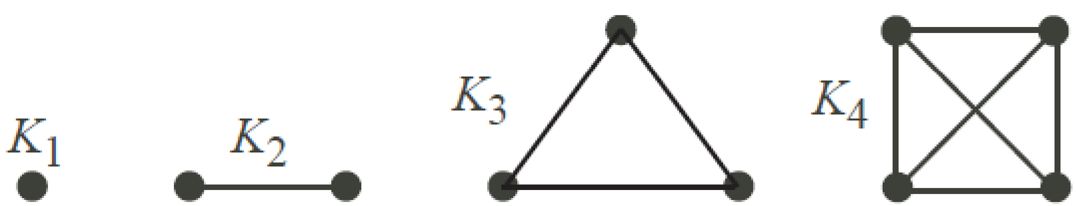
Proof. If the vertices v1, . . . , vk have odd degrees and the vertices vk+1, . . . , vn have even degrees,

then (Theorem 1.1)

$$d(v1) + \cdot\cdot\cdot + d(vk) = 2m - d(vk+1) - \cdot\cdot\cdot - d(vn)$$

is even. Therefore, k is even.

Example 1.1.11 *(Continuing from the previous example) Now the sum of the degrees is* 1 + 2 + 0 + 2 + 5 = 10 = 2 · 5*. There are two vertices of odd degree, namely* v1 *and* v5*.* A simple graph that contains every possible edge between all the vertices is called a *complete graph*. A complete graph with n vertices is denoted as Kn. The first four complete graphs are given as examples:

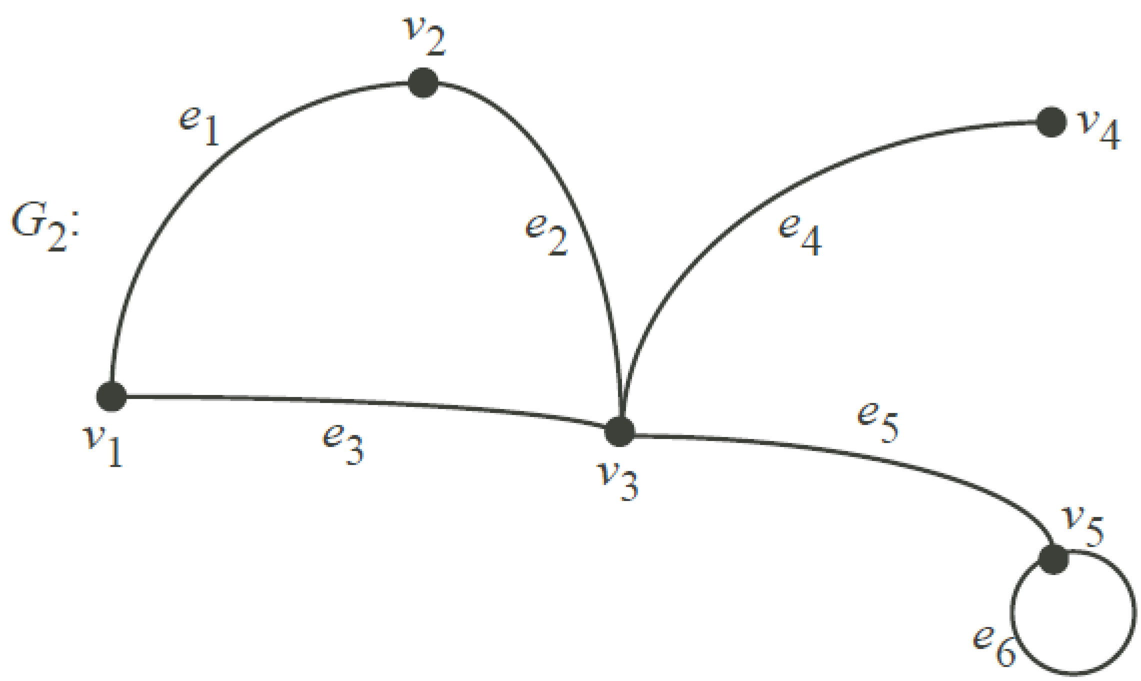


The graph G1 = (V1,E1) is a *subgraph* of G2 = (V2,E2) if:

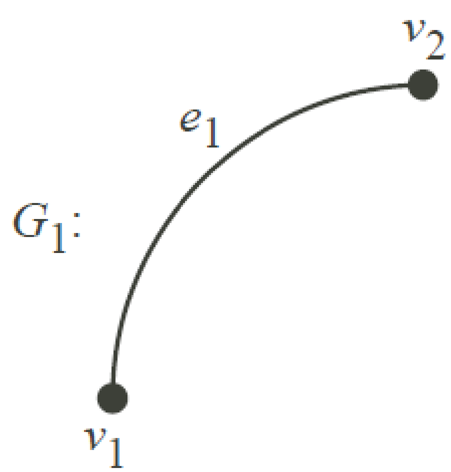
1. V1 ⊆ V2 and

2. Every edge of G1 is also an edge of G2.

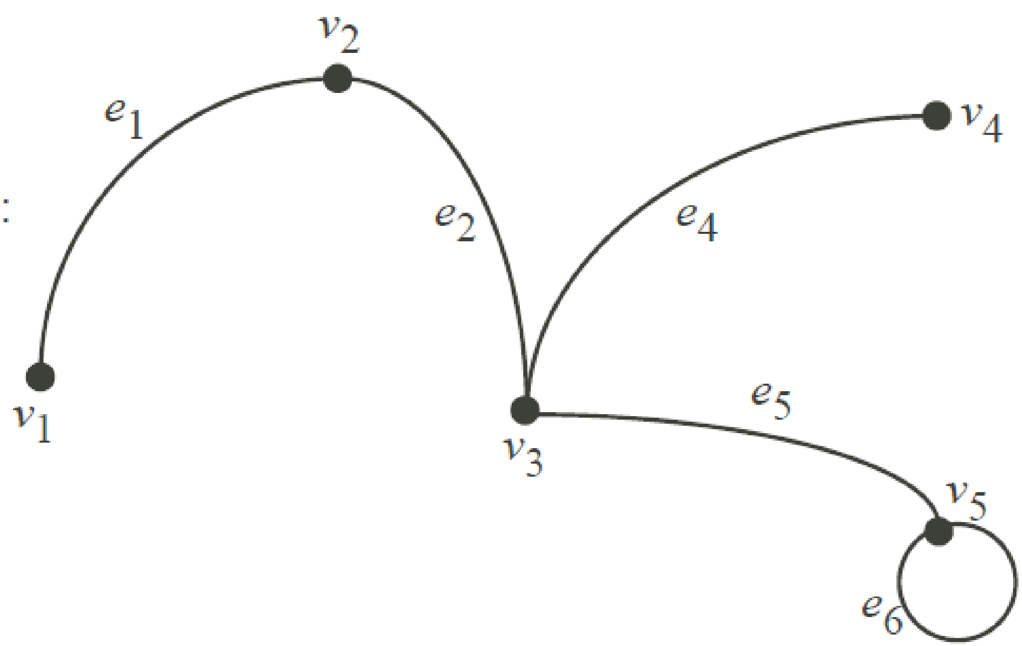
Example 1.1.12 *We have the graph*



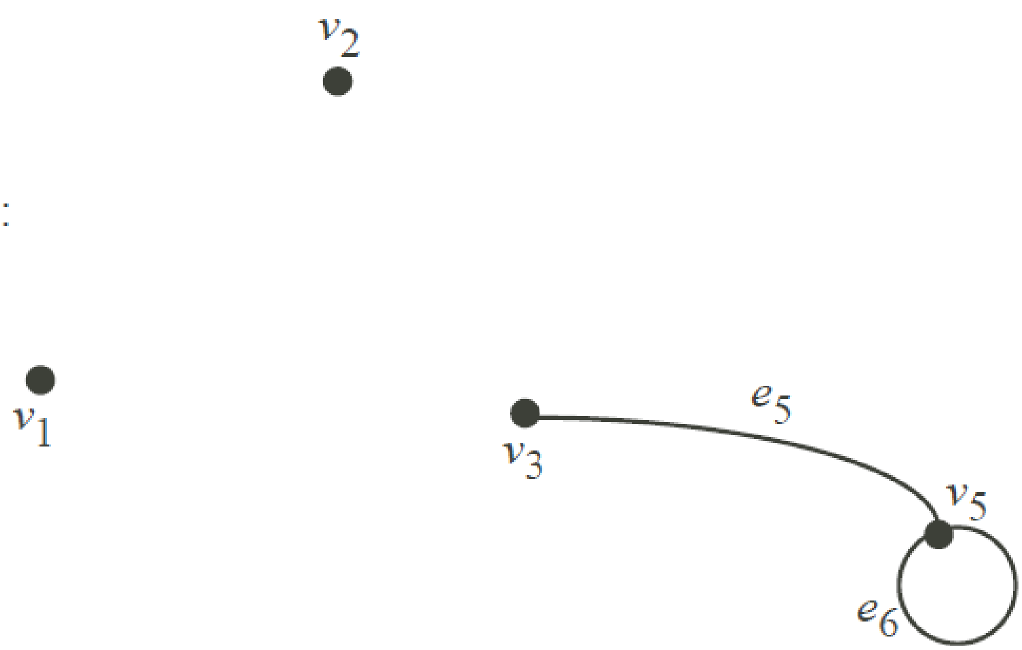
and some of its subgraphs are



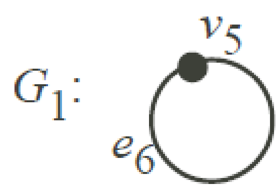
G_1:



G_1:



And

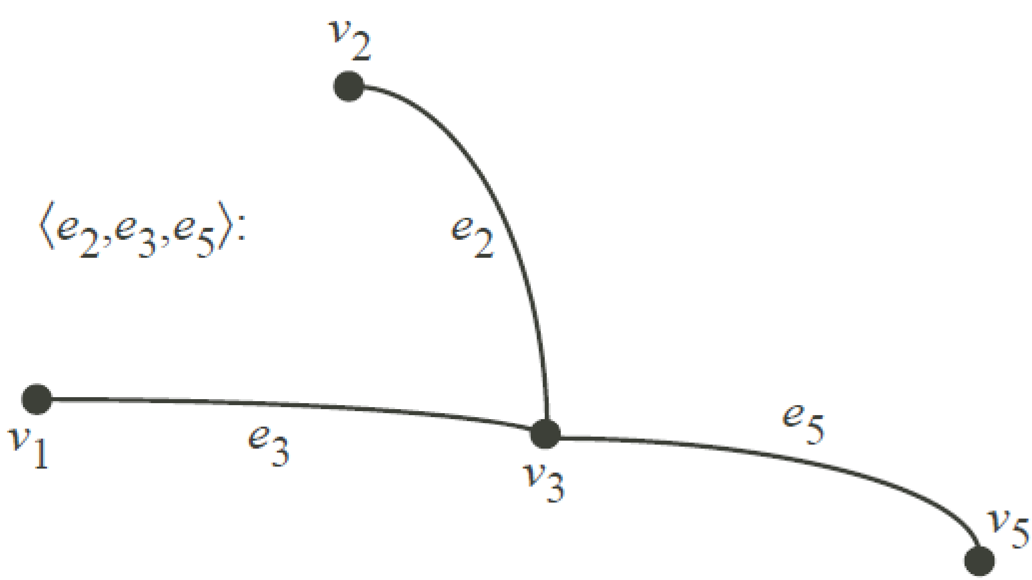


The *subgraph of* G = (V,E) *induced by the edge set* E1 ⊆ E is:

G1 = (V1,E1) =def. hE1i,

where V1 consists of every end vertex of the edges in E1.

Example 1.1.13 *(Continuing from above) From the original graph* G, *the edges* e2, e3 *and* e5 *induce the subgraph*



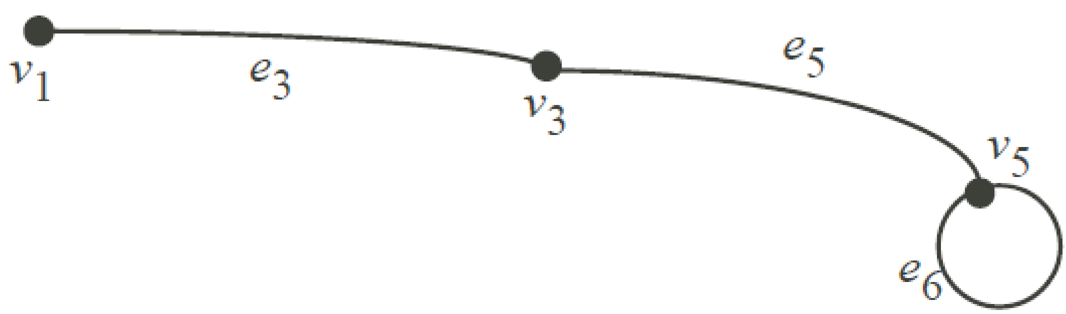
The *subgraph of* G = (V,E) *induced by the vertex set* V1 ⊆ V is:

G1 = (V1,E1) =def. hV1i,

where E1 consists of every edge between the vertices in V1.

Example 1.1.14 *(Continuing from the previous example) From the original graph* G, *the vertices* v1, v3 *and* v5 *induce the subgraph*

$\langle v_1, v_3, v_5 \rangle$:

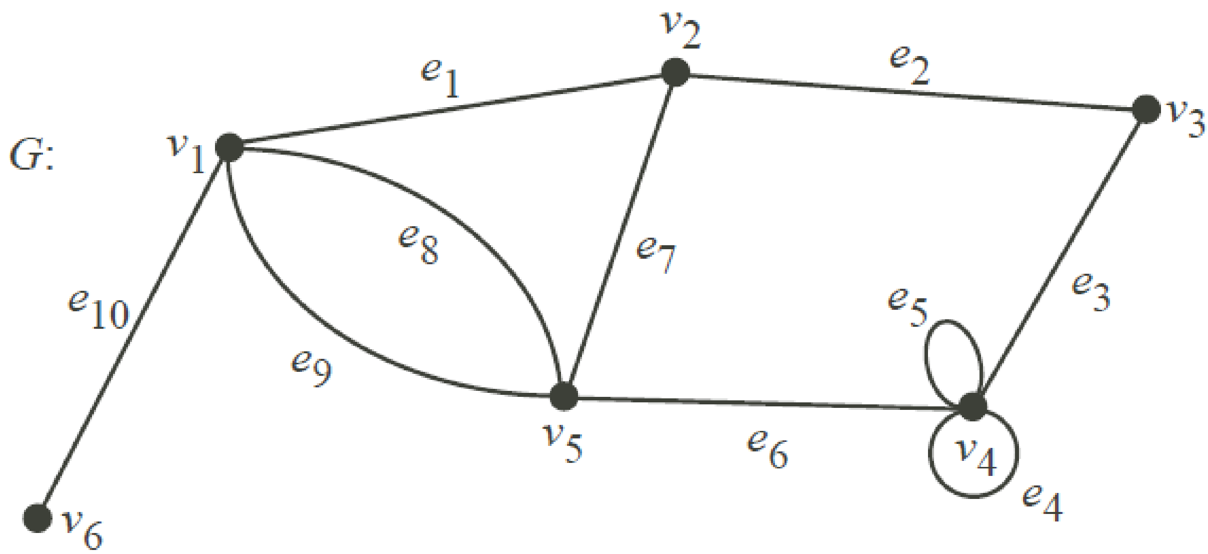


A complete subgraph of G is called a *clique* of G.

1.2 Walks, Trails, Paths, Circuits, Connectivity, Components

Remark 1.2.1*There are many different variations of the following terminologies. We will adhere to the definitions given here.* A *walk* in the graph G = (V,E) is a finite sequence of the form vi0 , ej1 , vi1, ej2, . . . , ejk , vik , which consists of alternating vertices and edges of G. The walk starts at a vertex. Vertices vit−1 and vit are end vertices of ejt (t = 1, . . . , k). vi0 is the *initial vertex* and vik is the *terminal vertex*. k is the *length* of the walk. A zero length walk is just a single vertex vi0 . It is allowed to visit a vertex or go through an edge more than once. A walk is *open* if vi0 6= vik . Otherwise it is *closed*.

Example 1.2.2 *In the graph*



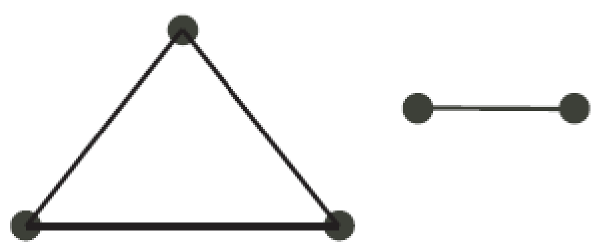
the walk v2, e7, v5, e8, v1, e8, v5, e6, v4, e5, v4, e5, v4 *is open. On the other hand, the walk* v4, e5, v4, e3, v3, e2, v2, e7, v5, e6, v4 *is closed.* A walk is a *trail* if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v.

Example 1.2.3 *(Continuing from the previous example) The walk in the graph* v1, e8, v5, e9, v1, e1, v2, e7, v5, e6, v4, e5, v4, e4, v4 *is a trail.*

A trail is a *path* if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same. A closed path is a *circuit*. For simplicity, we will assume in the future that a circuit is not empty, i.e. its length ≥ 1. We identify the paths and circuits with the subgraphs induced by their edges.

Example 1.2.4 *(Continuing from the previous example) The walk* v2, e7, v5, e6, v4, e3, v3 *is a path and the walk* v2, e7, v5, e6, v4, e3, v3, e2, v2 *is a circuit.* The walk starting at u and ending at v is called an u–v *walk*. u and v are *connected* if there is a u–v walk in the graph (then there is also a u–v path!). If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a u–v walk and a v–w walk, then there is also a u–w walk. A graph is *connected* if all the vertices are connected to each other. (A trivial graph is connected by convention.)

Example 1.2.5 *The graph:*



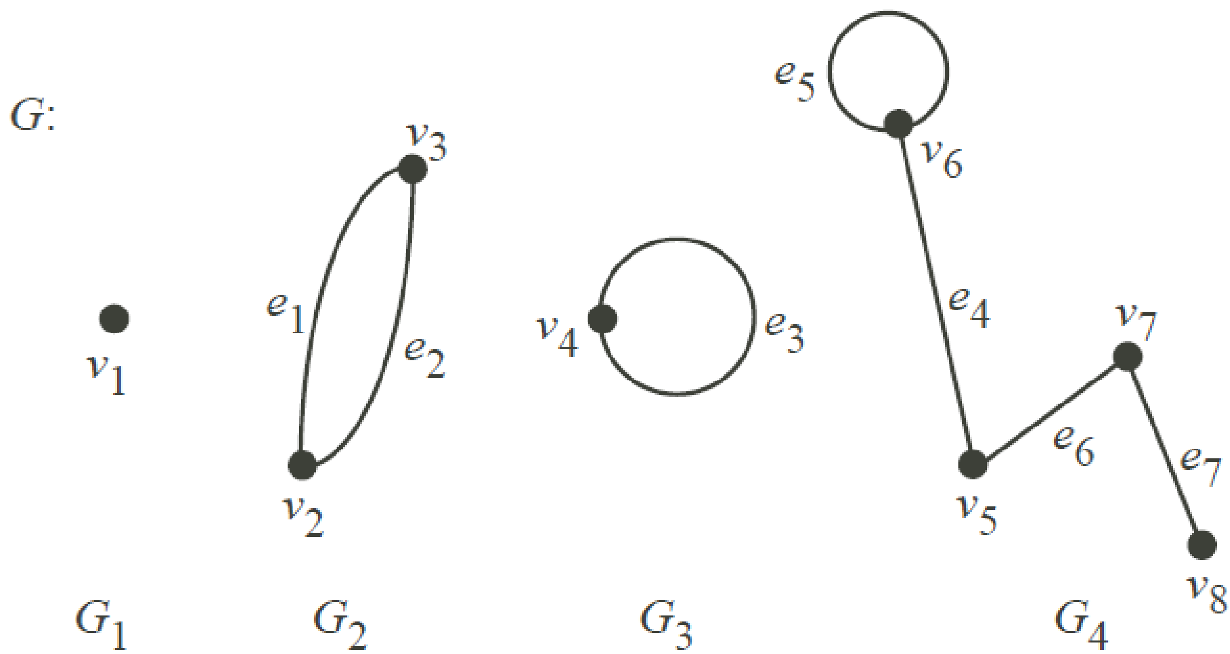
is not connected. The subgraph G1 (not a null graph) of the graph G is a *component* of G if:

1. G1 is connected and
2. Either G1 is trivial (one single isolated vertex of G) or G1 is not trivial and G1 is the subgraph induced by those edges of G that have one end vertex in G1. Different components of the same graph do not have any common vertices because of the following theorem.

Theorem 1.2.6 *If the graph* G *has a vertex* v *that is connected to a vertex of the component* G1 *of* G, *then* v *is also a vertex of* G1.

Proof. If v is connected to vertex v′ of G1, then there is a walk in G v = vi0 , ej1, vi1 , . . . , vik−1, ejk , vik = v′. Since v′ is a vertex of G1, then (condition #2 above) ejk is an edge of G1 and vik−1 is a vertex of G1. We continue this process and see that v is a vertex of G1.

Example 1.2.7



The components of G *are* G1*,* G2*,* G3 *and* G4*.*

Theorem 1.2.8 *Every vertex of* G *belongs to exactly one component of* G*. Similarly, every edge of* G *belongs to exactly one component of* G.

Proof. We choose a vertex v in G. We do the following as many times as possible starting with V1 = {v}: If v′ is a vertex of G such that v′ /∈ V1 and v′ is connected to some vertex of V1, then V1 ← V1 ∪ {v′}. Since there is a finite number of vertices in G, the process stops eventually. The last V1 induces a subgraph G1 of G that is the component of G containing v. G1 is connected because its vertices are connected to v so they are also connected to each other. Condition #2 holds because we can not repeat . By Theorem 1.2.6 v does not belong to any other component. The edges of the graph are incident to the end vertices of the components.

We now that theorem 1.2.8 divides a graph into distinct components. The proof of the theorem gives an algorithm to do that. We have to repeat what we did in the proof as long as we have free vertices that do not belong to any component. Every isolated vertex forms its own component. A connected graph has only one component, namely, itself. A graph G with n vertices, m edges and k components has the *rank*

$$\rho(G) = n - k.$$

The *nullity* of the graph is

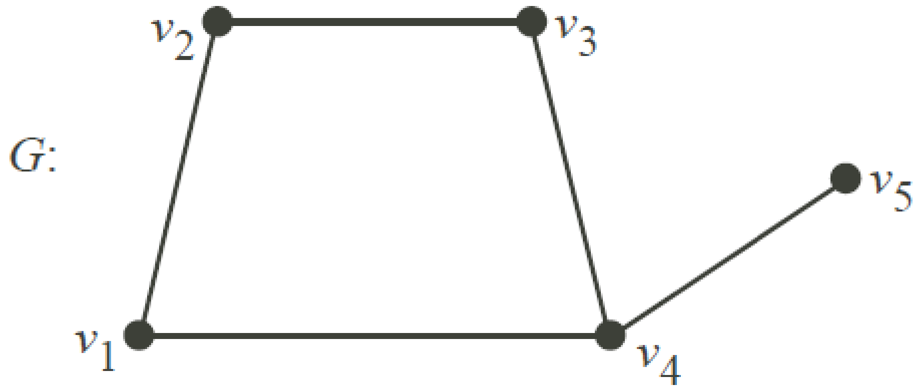
$$\mu(G) = m - n + k.$$

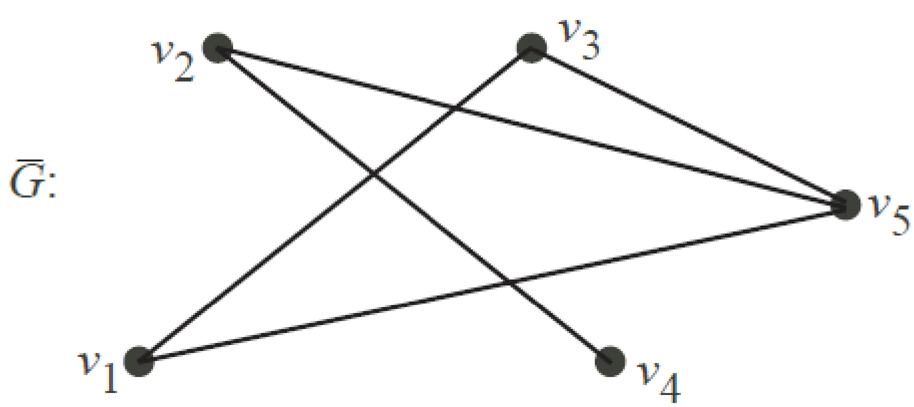
We see that $\rho(G) \geq 0$ and $\rho(G) + \mu(G) = m$. In addition, $\mu(G) \geq 0$

1.3 Graph Operations

The *complement* of the simple graph G = (V,E) is the simple graph G = (V,E), where the edges in E are exactly the edges not in G.

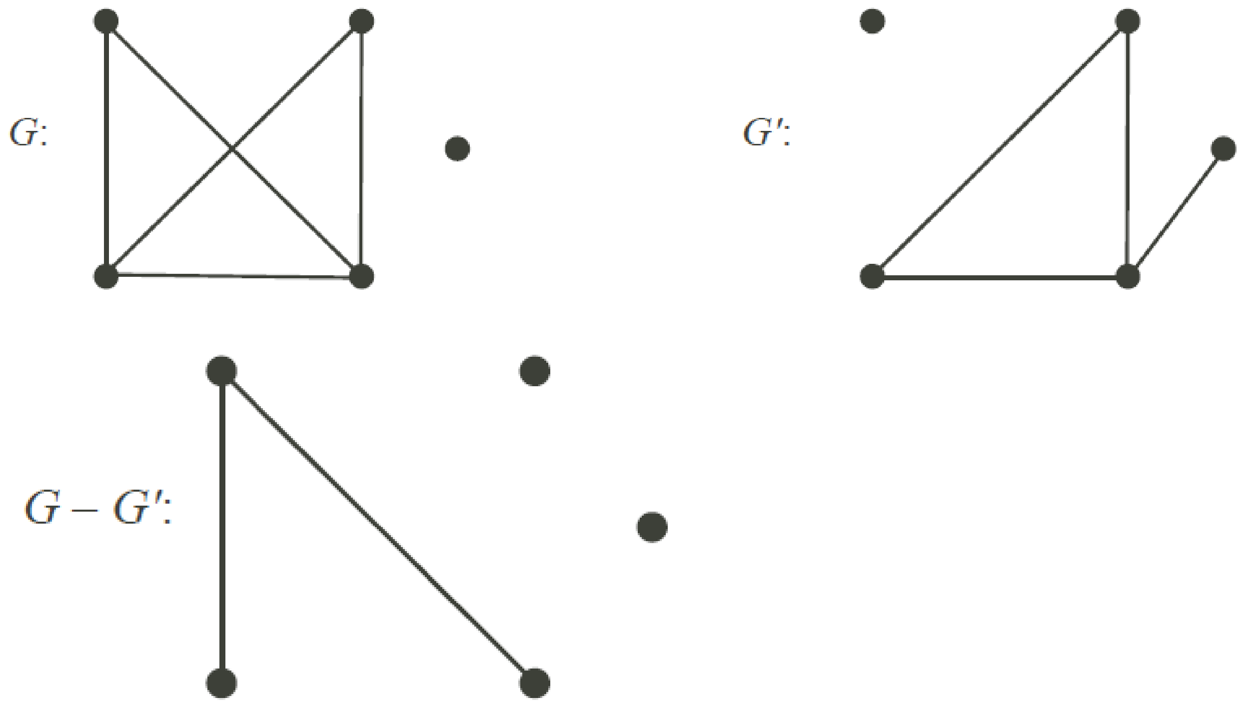
Example 1.3.1





Example 1.3.2 *The complement of the complete graph* Kn *is the empty graph with* n *vertices.* Obviously, G = G. If the graphs G = (V,E) and G′ = (V ′,E′) are simple and V ′ ⊆ V , then the *difference* graph is G − G′ = (V,E′′), where E′′ contains those edges from G that are not in G′ (simple graph).

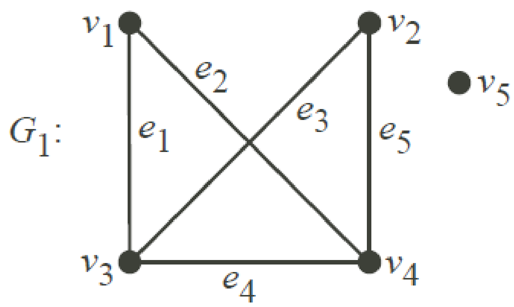
Example 1.3.3

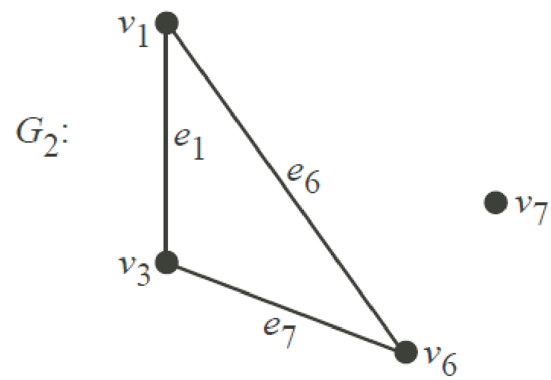


Here are some binary operations between two simple graphs G1 = (V1,E1) and G2 = (V2,E2):

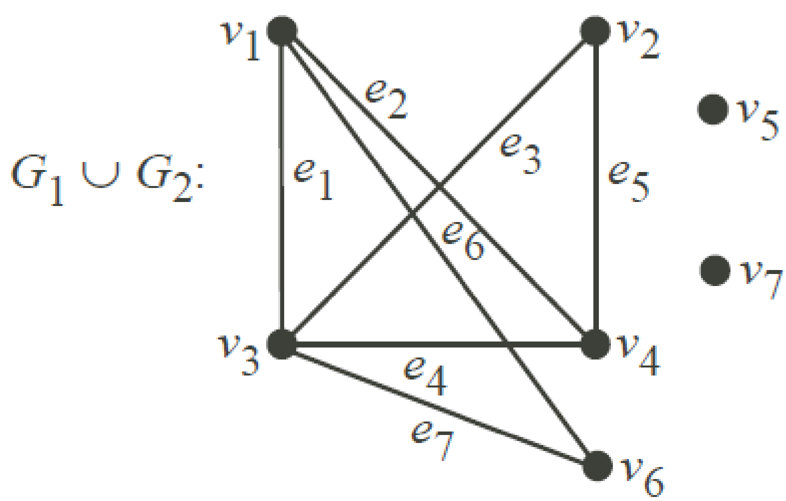
- The *union* is $G1 \cup G2 = (V1 \cup V2, E1 \cup E2)$ (simple graph).
- The *intersection* is $G1 \cap G2 = (V1 \cap V2, E1 \cap E2)$ (simple graph).
- The *ring sum* $G1 \oplus G2$ is the subgraph of $G1 \cup G2$ induced by the edge set $E1 \oplus E2$ (simple graph). *Note!* The set operation $\oplus$ is the *symmetric difference*, i.e. $E1 \oplus E2 = (E1 - E2) \cup (E2 - E1)$. Since the ring sum is a subgraph induced by an edge set, there are no isolated vertices. All three operations are commutative and associative.

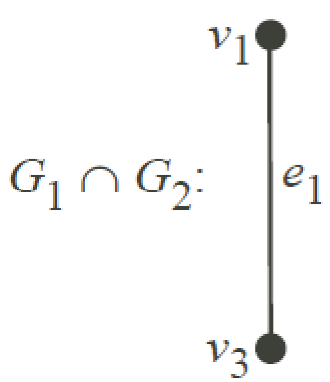
Example 1.3.4 *For the graphs*

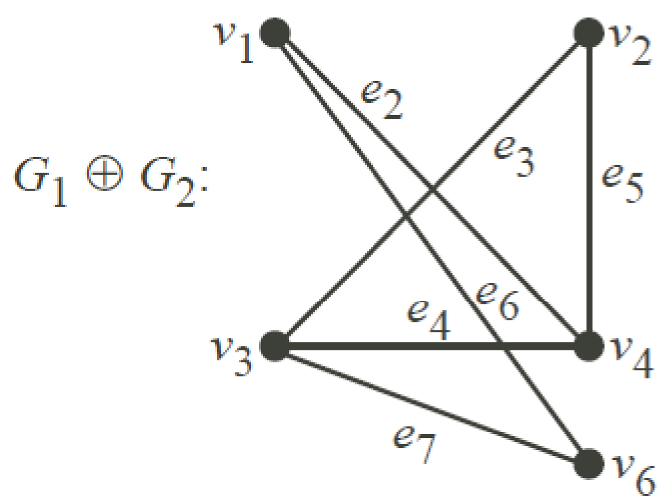




We have







Remark 1.3.5 The operations $\cup$, $\cap$ and $\oplus$ can also be defined for more general graphs other than simple graphs. Naturally, we have to ”keep track” of the multiplicity of the edges:

$\cup$: The multiplicity of an edge in $G1 \cup G2$ is the larger of its multiplicities in G1 and G2.

$\cap$: The multiplicity of an edge in $G1 \cap G2$ is the smaller of its multiplicities in G1 and G2.

$\oplus$: The multiplicity of an edge in $G1 \oplus G2$ is $|m1 - m2|$, where m1 is its multiplicity in G1 and m2 is its multiplicity in G2.

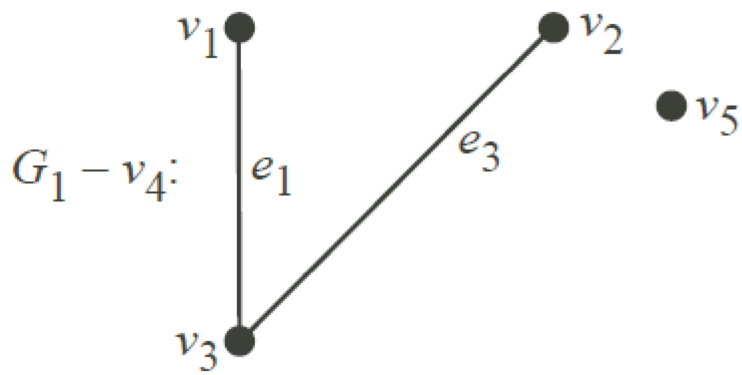
(We assume zero multiplicity for the absence of an edge.) In addition, we can generalize the difference operation for all kinds of graphs if we take account of the multiplicity. The multiplicity of the edge e in the difference $G - G'$ is

$$m_1 \dot{-} m_2 = \begin{cases} m_1 - m_2, \textit{ if } m_1 \geq m_2 \\ 0, \textit{ if } m_1 < m_2 \end{cases}$$

(also known as the proper difference*), where* m1 *and* m2 *are the multiplicities of* e *in* G1 *and* G2*, respectively*.

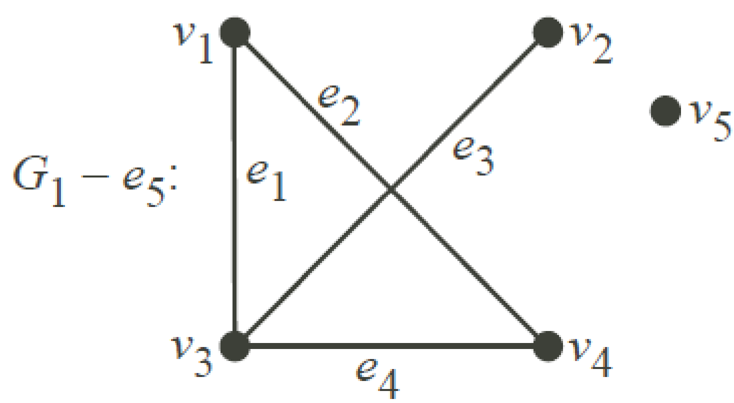
If v is a vertex of the graph G = (V,E), then G − v is the subgraph of G induced by the vertex set V − {v}. We call this operation the *removal of a vertex*.

Example 1.3.6 *(Continuing from the previous example)*



Similarly, if e is an edge of the graph G = (V,E), then G − e is graph (V,E′), where E′ is obtained by removing e from E. This operation is known as *removal of an edge*. We remark that we are not talking about removing an edge as in Set Theory, because the edge can have non unit multiplicity and we only remove the edge once.

Example 1.3.7 *(Continuing from the previous example)*

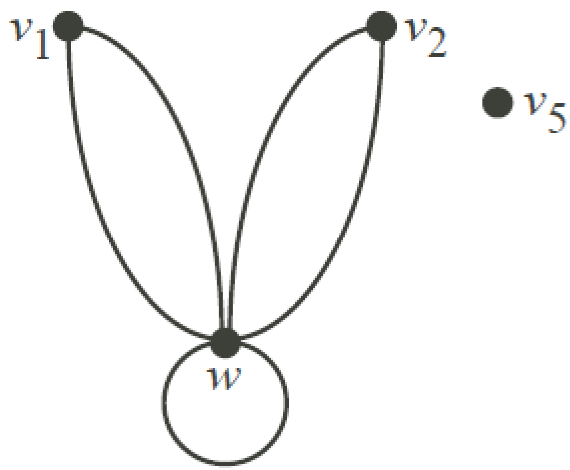


If u and v are two distinct vertices of the graph G = (V,E), then we can *short-circuit* the two vertices u and v and obtain the graph (V ′,E′), where V ′ = (V − {u, v}) ∪ {w} (w /∈ V is the ”new” vertex) andE′ = (E − {(v′, u), (v′, v) | v′ ∈ V }) ∪ {(v′,w) | (v′, u) ∈ E or (v′, v) ∈ E}

$\cup$ {(w,w) | (u, u) $\in$ E or (v, v) $\in$ E}

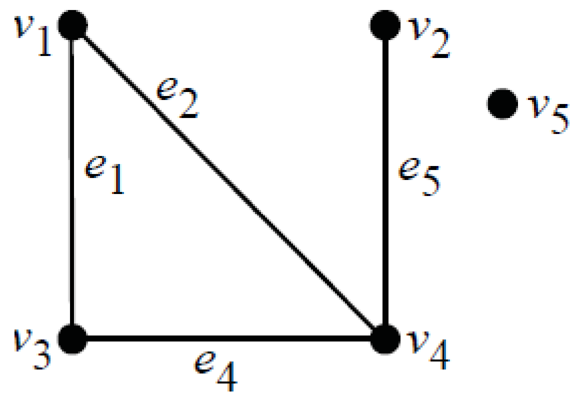
(Recall that the pair of vertices corresponding to an edge is not ordered). *Note!* We have to maintain the multiplicity of the edges. In particular, the edge (u, v) becomes a loop.

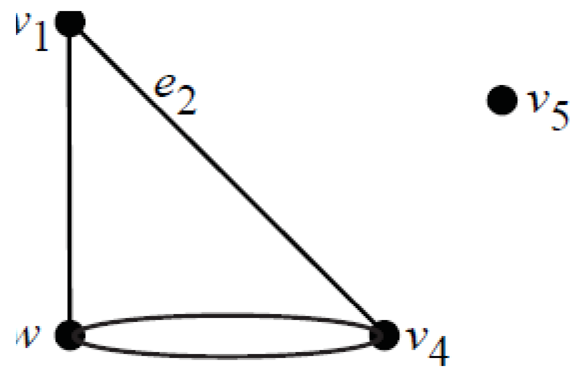
Example 1.3.8 *(Continuing from the previous example) Short-circuit* v3 *and* v4 *in the graph* G1*:*



In the graph G = (V,E), *contracting* the edge e = (u, v) (not a loop) means the operation in which we first remove e and then short-circuit u and v. (Contracting a loop simply removes that loop.)

Example 1.3.9 *(Continuing from the previous example) We contract the edge* e3 *in* G1 *by first removing* e3 *and then short-circuiting* v2 *and* v3.





Remark 1.3.10 If we restrict short-circuiting and contracting to simple graphs, then we remove loops and all but one of the parallel edges between end vertices from the results.

CHAPTER TWO

The Graphs of Reflexive-Transitive Relations

2.1 Adjacency of binary relations.

Definition 2.1.1 Let $B = \{0,1\}$ Boolean set, X- arbitrary set, and $X^2 = X \times X$ a direct product. The functions $X^2 \to B$ will be called characteristic. Any subset, $\sigma \subseteq X^2$, called a binary relation (or relation) on the set X generates characteristic function

$$\chi_R : X^2 \longrightarrow B, \quad \chi_R(x,y) = \begin{cases} 1, & if\ (x,y) \in R, \\ 0, & if\ \ (x,y) \notin R. \end{cases}$$

Next, the function $\chi_R(.,.)$ will be denoted by $R(.,.)$.

Definition 2.1.2 Let $X = Y \cup Z$- the disjoint union of two subsets (allowed that either $Y = \emptyset$ or $Z = \emptyset$). Suppose that the relation $\sigma \subseteq X^2$ such that $\sigma\ (x;\ y) = 0$ for all $(x;\ y) \in Y \times Z$: . It generates the relation $\tau \subseteq X^2$ such that

$$\tau(x,y) = 1 - \sigma(y,x) \quad (x,y) \in Y \times Z,$$

$$\tau(x,y) = 0 \quad (x,y) \in Z \times Y,$$

$$\tau(x,y) = \sigma(x,y) \quad (x,y) \in Y^2 \cup Z^2.$$

The relation τ is called adjacent with the relation σ.

Remark 2.1.3 From the definition above it follows that if the relation τ adjacent with a relation σ then σ adjacent with a relation τ, and this fact we write in the form of a diagram $\sigma \overset{Y \times Z}{\longleftrightarrow} \tau$ or

$$\begin{array}{c|c|c|} & Y & Z \\ \hline Y & & 0 \\ \hline Z & \sigma(x,y) & \\ \hline \end{array} = \sigma \xleftrightarrow{Y\times Z} \tau = \begin{array}{c|c|c|} & Y & Z \\ \hline Y & & 1-\sigma(y,x) \\ \hline Z & 0 & \\ \hline \end{array} .$$

And we note that in the block $Y \times Z$ for the relation σ we write≪ *generalized* ≫ *zero; i.e* (x; y) =0 for all $(x, y) \in Y \times Z$ *;* and in the same block for the relation τ we write $1 - \sigma(y, x)$*;* i.e $\tau(x, y) = 1 - \sigma(y, x)$ for all $(x, y) \in Y \times Z$.

Example 2.1.4 Let $X = \{1, \dots, 6\}$, $Y = \{1,2\}$, $Z = \{3,4,5,6\}$, then:

$$\left|\begin{array}{c|c} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \end{array}\right| = \sigma \xleftrightarrow{Y\times Z} \tau = \left|\begin{array}{c|c} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{smallmatrix} \\ \hline 0 & \begin{smallmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix} \end{array}\right| .$$

2.2 Reflexive-transitive relation.

Through *V(X)* the collection of all reflexive transitive relations defined on the set *X:* In the other words, the relation $\sigma \subseteq X^2$ belong in *V*(*X*)*;* if it satisfies axioms reflexivity((x; x)$\in \sigma$) and transitive (if $(x, y) \in \sigma, (y, z) \in \sigma$ then $(x, z) \in \sigma$). And in the terms of characteristic function we have : $\sigma \in V(X)$ if and only if

$$\sigma(x, x) = 1 \text{ for all } x \in X;$$

$$\sigma(x, y)\sigma(y, z) \leq \sigma(x, z) \text{ for all } x, y, z \in X.$$

For any $\sigma \in V(X)$ and $x \in X$ the set

$$U_\sigma(x) = \{y \in X : \sigma(x, y) = 1\} \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

Is not empty (since $x \in U_\sigma(x)$)

Proposition 2.2.1 Let $\sigma \in V(X)$ and $x, y \in X$: Then $y \in U_\sigma(x)$ if and only if $U_\sigma(y) \subseteq U_\sigma(x)$.

Proof: Let $y \in U_\sigma(x)$ then $\sigma(x, y) = 1$. If $z \in U_\sigma(y)$ then $\sigma(y, z) = 1$, therefore $\sigma(x, z) = 1$ and $z \in U_\sigma(x)$, and hence $U_\sigma(y) \subseteq U_\sigma(x)$ the proof of conversely its clear. ■

Remark 2.2.2 The relation $\sigma \in V(X)$ generates an equivalence relation on the set X : write $x \sim y$ (or $x \underset{\sigma}{\sim} y$) if and only if $U_\sigma(x) = U_\sigma(y)$: The equivalence class containing the element $x \in X$; denote by $[x]_\sigma$ (or $[x]$).

Proposition 2.2.3 Let $\sigma \in V(X)$ and $x, y \in X$: The following are holds:

1. $[x]_\sigma \subseteq U_\sigma(x)$;
2. If $y \in [x]_\sigma$, then $[y]_\sigma \subseteq U_\sigma(x)$; therefore

$$U_\sigma(x) = \bigcup_{[\xi]_\sigma \subseteq U_\sigma(x)} [\xi]_\sigma \,; \qquad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

3. $\sigma(\xi, \eta) = 1$ for all $(\xi, \eta) \in [x]_\sigma^2$;
4. $\sigma(\xi, \eta) = \sigma(x, y)$ for all $(\xi, \eta) \in [x]_\sigma \times [y]_\sigma$;
5. If $[x]_\sigma \neq [y]_\sigma$ then $\sigma(\xi, \eta)\sigma(\eta, \xi) = 0$ for all $(\xi, \eta) \in [x]_\sigma \times [y]_\sigma$;

Proof: 1) Suppose that $\xi \in [x]_\sigma$ then $x \sim \xi$ and $\xi \in U_\sigma(\xi) = U_\sigma(x)$. ■

2) From (1) above and proposition (1) we have $[y]_\sigma \subseteq U_\sigma(y) \subseteq U_\sigma(x)$. If $z \in \bigcup_{[\xi]_\sigma \subseteq U_\sigma(x)} [\xi]_\sigma$ then $z \in [\xi]_\sigma$ for some ξ, such that

$[\xi]_\sigma \subseteq U_\sigma(x)$, therefore $z \in U_\sigma(x)$. Conversely if $z \in U_\sigma(x)$ then $[z]_\sigma \subseteq U_\sigma(x)$ and $z \in [z]_\sigma \subseteq \bigcup_{[\xi]_\sigma \subseteq U_\sigma(x)} [\xi]_\sigma$. ∎

3) Since $\xi \sim \eta$, then $U_\sigma(\xi) = U_\sigma(\eta)$ and hence $\eta \in U_\sigma(\xi)$ and $\sigma(\xi, \eta) = 1$. ∎

4) If $\sigma(\xi, \eta) = 0$ for all $\xi \in [x]_\sigma$, $\eta \in [y]_\sigma$, then the proof its clear. Now suppose that $\sigma(z, w) = 1$ for some $z \in [x]_\sigma$ and $w \in [y]_\sigma$, then $w \in U_\sigma(z)$ therefore, $[w]_\sigma \subseteq U_\sigma(w) \subseteq U_\sigma(z)$, hence $\eta \in U_\sigma(z)$ for any $\eta \in [y]_\sigma$=$U_\sigma(w)$. Since $\xi \in [x]_\sigma$=$[z]_\sigma$, then we have $U_\sigma(\xi) = U_\sigma(z)$, therefore $\eta \in U_\sigma(\xi)$. And hence $\sigma(\xi, \eta) = 1$ for all $\xi \in [x]_\sigma$, $\eta \in [y]_\sigma$. ∎

5) We can prove that by a contradiction i.e $\sigma(\xi, \eta) = 1$, $\sigma(\eta, \xi) = 1$. Then $\eta \in U_\sigma(\xi)$, $\xi \in U_\sigma(\eta)$ and $U_\sigma(\eta) \subseteq U_\sigma(\xi) \subseteq U_\sigma(\eta)$, hence $\xi \sim \eta$, and this a contradiction since $[\xi]_\sigma = [x]_\sigma \neq [y]_\sigma = [\eta]_\sigma$. ∎

2.3 Graph of reflexive-transitive relations.

Theorem 2.3.1 Let σ and τ are adjacent relations defined on the set X,(i.e $\sigma \overset{Y \times Z}{\longleftrightarrow} \tau$). Then $\sigma \in V(X)$ if and only if $\tau \in V(X)$.

Proof: We will prove that this implication $\sigma \in V(X) \Longrightarrow \tau \in V(X)$.

Reflexivity. Let $\sigma \in V(X)$ since σ and τ are adjacent relations then $\sigma(x, x) = \tau(x, x) = 1$ for all $x \in X$, and this proof for reflexive relation τ.

Transitivity. Let $x, y, z \in X$ such that $\tau(x, y) = \tau(y, z) = 1$,

i) Suppose that $y \in Y$ sine $\tau(\xi, y) = 0$ for all $\xi \in Z$, then $x \in Y$, if $z \in Y$ then $\sigma(x, y) = \tau(x, y) = 1$ and $\sigma(y, z) = \tau(y, z) = 1$ and since $\sigma \in V(X)$ then $\sigma(x, z)$ =1, therefore $\tau(x, z) = 1$ and hence $z \in Z$ then $\sigma(x, y) = \tau(x, y) = 1$ and $\sigma(z, y) = 1 - \tau(y, z) = 0$ and since $\sigma \in V(X)$ then $\sigma(z, x) = \sigma(z, x)\sigma(x, y) \leq \sigma(z, y)$ and hence $\sigma(z, x) = 0$ therefore, $\tau(x, z) = 1$.

ii) Suppose that $y \in Z$. Since $\tau(y, \eta) = 0$ for all $\eta \in Y$ then $z \in Z$. If $x \in Z$, then $\sigma(x, y) = \tau(x, y) = 1$ and $\sigma(y, z) = \tau(y, z) = 1$, and since $\sigma \in V(X)$, then $\sigma(x, z) = 1$ therefore, $\tau(x, z) = 1$ .if $x \in Y$, then $\sigma(y, z) = \tau(y, z) = 1$ and $\sigma(y, x) = 1 - \tau(x, y) = 0$, and since $\sigma \in V(X)$, then $\sigma(z, x) = \sigma(y, z)\sigma(z, x) \leq \sigma(y, x) = 0$ hence $\sigma(z, x) = 0$ therefore $\tau(x, z) = 1$. Thus, in all cases we get the equality $\tau(x, z) = 1$. ■

Remarks 2.3.2 1) Thus, the set X generates a pair $< V(X), E(X) >$ where V (X) this is a set of ((vertices)) consist of the set of all reflexive - transitive relations and $E(X)$ a set of ((edges)), consist of the set of all unordered distinct pairs of adjacent

reflexive - transitive relations of the set X .The pair $G(X)$ =$<V$ (X);$E(X) >$ will be called (undirected) ((graph)) of reflexive - transitive relations of the set X.

2) We say that the reflexive - transitive relations σ and τ belong to the same connected component of the graph $G(X)$; if there is a finite sequence of reflexive

- transitive relations $\sigma = \sigma_1, \sigma_2, \dots, \sigma_m = \tau$; which the relations σ_{k-1} and σ_k are adjacent with all $k = 2, \dots, m$. We denote $G_\sigma(X)$ is the connected component of the graph $G(X)$; which that contains the reflexive - transitive relation σ.

2.4 The structure of the graph reflexive-transitive relations.

Let $\sigma \in V(X)$: Through $[X]_\sigma$denoted the set of all equivalence class of the set X; (i.e $[X]_\sigma = \{[x]_\sigma\}_{x\in X} = \{x\}_{x\in X}$). Due to the point (4) of proposition (2) we can define the following characteristic function $\bar{\sigma}$: $[x]_\sigma^2 \to B$ such that:

$\bar{\sigma}(\bar{x}, \bar{y}) = \sigma(\xi, \eta)$ where (ξ, η) for any order pair in the direct product $\bar{x} \times \bar{y}$ it is clear that:

$$\bar{\sigma}(\bar{x}, \bar{y}) = \sigma(x, y)\text{=1 for all } \bar{x} \in [X]_\sigma$$

$$\bar{\sigma}(\bar{x}, \bar{y})\bar{\sigma}(\bar{y}, \bar{z}) = \sigma(x, y)\sigma(y, z) \le \sigma(x, z) = \bar{\sigma}(\bar{x}, \bar{z})$$

$$\text{for all } \bar{x}, \bar{y}, \bar{z} \in [X]_\sigma;$$

$\bar{\sigma}(\bar{x}, \bar{y})\bar{\sigma}(\bar{y}, \bar{x}) = \sigma(x, y)\sigma(y, x) = \delta_{\bar{x}\bar{y}}$ for all $\bar{x}, \bar{y} \in [X]_\sigma$where $\delta_{\bar{x}\bar{y}}$ symbol Kronecker.

This means, that σ it generates a partial order $\bar{\sigma}$ on the set $[X]_\sigma$. Consequently, in accordance with the concept of graph of the partial order that σ generated a graph $G_0([X]_\sigma) =< V_0([X]_\sigma), E([X]_\sigma) >$ where $V_0([X]_\sigma) -$ this is a set of partial order defined on the set $[X]_\sigma$; and $E([X]_\sigma) -$ this is a set of edges, consist of unordered pairs of distinct sets of adjacent partial orders of the set $[X]_\sigma$ Thus, $\bar{\sigma} \in$

$V_0([X]_\sigma)$ the components of the graph $G_0([X]_\sigma)$*;* contains a partial order $\bar{\sigma}$ (denoted by $G_0^{\bar{\sigma}}([X]_\sigma)$.

Example 2.4.1 Let $X = \{1,2,3\}$, $\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\text{then } G_\sigma(X) = \langle \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \rangle$$

$$[X]_\sigma = \{\bar{1}, \bar{3}\} = \{\{1,2\},\{3\}\},\ \bar{\sigma} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$G_0([X]_\sigma) = G_0^{\bar{\sigma}}([X]_\sigma) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle$$

We say that the relations $\sigma, \tau \in V(X)$ are of the same type if $[X]_\sigma = [X]_\tau$.

Proposition 2.4.2 If σ , $\tau -$ are adjacent reflexive - transitive relations defined on the set *X;* then $[X]_\sigma = [X]_\tau$.

Proof: Since σ , $\tau -$ are adjacent then $\sigma \overset{Y\times Z}{\longleftrightarrow} \tau$, $X = Y \cup Z$. Let $x, y \in X$ By virtue of the symmetry of the proposition , it suffices to show the implication $x \sim^{\sigma} y \Longrightarrow x \sim^{\tau} y$ or $[x]_\sigma = [y]_\sigma \Longrightarrow [x]_\tau = [y]_\tau$. Suppose that $[x]_\sigma = [y]_\sigma$ and $[x]_\tau \neq [y]_\tau$ then according to (5) of proposition (2) we have $\tau(x,y)\tau(y,x) = 0$. And according to (3) of the same proposition we have the $\sigma(x,y) = \sigma(y,x) = 1$, therefore $(x,y) \notin Y \times Z$ and $(y,x) \notin Y \times Z$ and hence $(x,y),(y,x) \in Y^2 \cup Z^2$ and then $\tau(x,y) = \sigma(x,y) = 1$, $\tau(y,x) = \sigma(y,x) = 1$ and this a contradiction with $[x]_\tau \neq [y]_\tau$.Then $[x]_\tau = [y]_\tau$. ■

Remark 2.4.3 In the process of the proof, we showed, in particular, that if $\sigma \overset{Y\times Z}{\longleftrightarrow} \tau$, then for any $x \in X = Y \cup Z$ there is an alternative: either $\bar{x} \subseteq Y$, or $\bar{x} \subseteq Z$. In the other word $[X] = [X]_\sigma = [X]_\tau$ is representable as a disjoint union :

$$[X] = [Y] \cup [Z], \quad [Y] = \{\bar{x} \in [X]: \bar{x} \subseteq Y\}, \quad [Z] = \{\bar{x} \in [X]: \bar{x} \subseteq Z\}. \quad \dots\dots\dots\dots \dots\dots (3)$$

Remark 2.4.4 In the final analysis, we established that any relation $\sigma \in V(X)$ generated a connected component $G_\sigma(X)$ of the graph $G(X)$. The set $[X]_\sigma$ is the equivalence class of the partial order $\bar{\sigma} \in V_0([X]_\sigma)$, of the graph $G_0([X]_\sigma)$ and his connected components is $G_0^{\bar{\sigma}}([X]_\sigma)$. Furthermore, if $\tau \in G_\sigma(X)$, then $G_\tau(X) = G_\sigma(X)$ and $[X]_\tau = [X]_\sigma$.

In the following proposition we proved that $G_0^{\bar{\tau}}([X]_\tau) = G_0^{\bar{\sigma}}([X]_\sigma)$.

Proposition 2.4.5 Let σ and τ reflexive-transitive relations defined on the set X; and $\bar{\sigma}, \bar{\tau} -$ the partial orders generated by σ and τ defined on the sets $[X]_\sigma$ and $[X]_\tau$ respectively. Then σ and τ are adjacent relations if and only if $\bar{\sigma}$ and $\bar{\tau}$ are adjacent relations.

Proof: Since $\sigma, \tau \in V(X)$, then $\bar{\sigma} \in G_0^{\bar{\sigma}}([X]_\sigma)$, $\bar{\tau} \in G_0^{\bar{\tau}}([X]_\tau)$. Suppose that, σ and τ are adjacent relations then there exists $X = Y \cup Z$ such that $\sigma \overset{Y\times Z}{\longleftrightarrow} \tau$. According to the proposition (3) we have $[X]_\sigma = [X]_\tau$, and from the formula (3) we get:

$[X] = [X]_\sigma = [X]_\tau = [Y] \cup [Z]$. And from the definition (2) we get the following:

$$\bar{\sigma}(\bar{x},\bar{y}) = \sigma(x,y) = 0 \text{ for all } (\bar{x},\bar{y}) \in [Y] \times [Z],$$

$$\bar{\tau}(\bar{x},\bar{y}) = \tau(x,y) = 1 - \sigma(y,x) = 1 - \bar{\sigma}(\bar{y},\bar{x}) \text{ for all } (\bar{x},\bar{y}) \in [Y] \times [Z],$$

$$\bar{\tau}(\bar{x},\bar{y}) = \tau(x,y) = 0 \text{ for all } (\bar{x},\bar{y}) \in [Z] \times [Y],$$

$$\bar{\tau}(\bar{x},\bar{y}) = \tau(x,y) = \sigma(x,y) = \bar{\sigma}(\bar{x},\bar{y}) \text{ for all } (\bar{x},\bar{y}) \in [Y]^2 \cup [Z]^2.$$

And therefore, $\bar{\sigma} \xleftrightarrow{[Y]\times[Z]} \bar{\tau}$, (i.e $\bar{\sigma}$, $\bar{\tau}$ are adjacent relations).

Conversely: Suppose that $\bar{\sigma}$ and $\bar{\tau}$ are adjacent relations then $G_0^{\bar{\sigma}}([X]_\sigma) = G_0^{\bar{\tau}}([X]_\tau)$ and in particular $[X]_\sigma = [X]_\tau$. Let $[X] = [X]_\sigma = [X]_\tau$. Since $\bar{\sigma}$ and $\bar{\tau}$ are adjacent relations then there exist $[X] - [Y] \cup [Z]$ such that $\bar{\sigma} \xleftrightarrow{[Y]\times[Z]} \bar{\tau}$ and again from the definition (2) if $Y = \{x \in X: \bar{x} \in [Y]\}$ and $Z = \{x \in X: \bar{x} \in [Z]\}$ then we get the following:

$$\sigma(x,y) = \bar{\sigma}(\bar{x},\bar{y}) = 0 \text{ for all } (x,y) \in Y \times Z,$$

$$\tau(x,y) = \bar{\tau}(\bar{x},\bar{y}) = 1 - \bar{\sigma}(\bar{y},\bar{x}) = 1 - \sigma(y,x) \text{ for all } (x,y) \in Y \times Z,$$

$$\tau(x,y) = \bar{\tau}(\bar{x},\bar{y}) = 0 \text{ for all } (x,y) \in Z \times Y,$$

$$\tau(x,y) = \bar{\tau}(\bar{x},\bar{y}) = \bar{\sigma}(\bar{x},\bar{y}) = \sigma(x,y) \text{ for all } (x,y) \in Y^2 \cup Z^2.$$

And therefore, $\sigma \xleftrightarrow{Y\times Z} \tau$ (i.e σ and τ are adjacent relations).

Proposition 2.4.6 For any $\sigma \in V(X)$ Then the connected graphs $G_\sigma(X)$ and $G_0^{\bar{\sigma}}([X]_\sigma)$ are isomorphic.

Proof: It is clear from proposition (4).

Proposition 2.4.7 Let $\sigma \in V(X)$ and $x \in X$. There exist a unique $\tau \in G_\sigma(X)$ such that $\tau(x,y) = \tau(y,x) = \delta_{\bar{x}\bar{y}}$ for all $y \in X$.

Proof: The connected graphs $G_\sigma(X)$ and $G_0^{\bar{\sigma}}([X]_\sigma)$ are isomorphic and from the proposition in [1] there exist a unique $\bar{\tau} \in G_0^{\bar{\sigma}}([X]_\sigma)$ such that $\bar{\tau}(\bar{x},\bar{y}) = \bar{\tau}(\bar{y},\bar{x}) = \delta_{\bar{x}\bar{y}}$ for all $\bar{y} \in [X]_\sigma$. Through τ is the generated relation $\bar{\tau}$ from the isomorphism graphs $G_\sigma(X) \to G_0^{\bar{\sigma}}([X]_\sigma)$ then $\tau \in G_\sigma(X)$ and $\tau(x,y) = \bar{\tau}(\bar{x},\bar{y}) = \delta_{\bar{x}\bar{y}}$ and $\tau(y,x) = \bar{\tau}(\bar{y},\bar{x}) = \delta_{\bar{y}\bar{x}}$ for all $y \in X$. ■

And hence if we fixed a reflexive – transitive relation $\sigma \in V(X)$ we can define the mapping $X \to G_\sigma(X)$, where $x \in X$ we get reflexive – transitive relation $\sigma^{[x]}$ and it is clear that if $x \sim y$, then $\sigma^{[x]} = \sigma^{[y]}$,

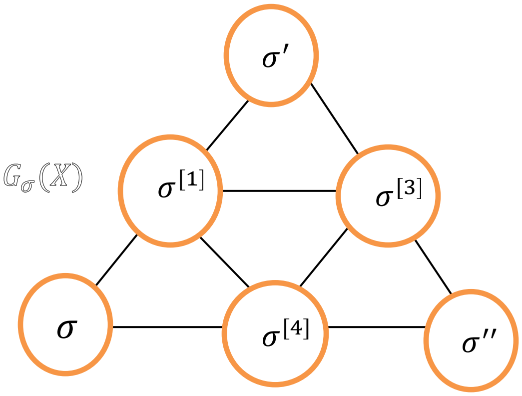
Example 2.4.8 We can show that the graphs $G_\sigma(X)$ and $G_0^{\bar{\sigma}}([X]_\sigma)$ in example 2 are isomorphic and $\sigma^{[1]} = \sigma^{[2]} = \sigma^{[3]} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $X = \{1,2,3,4\}$,

$$\sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Then } [X]_\sigma = \{\{1,2\},\{3\},\{4\}\},\ \sigma^{[1]} = \sigma^{[2]} =$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma^{[3]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \sigma^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$G_\sigma(X)$:



$$\text{Where } \sigma' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \sigma'' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

CHAPTER THREE

GRAPH OF FINITE TOPOLOGY

3.1 Bijection between finite-reflexive transitive relations and finite topologies.

The relation $\sigma \in V(X)$ will be called finite, if the set $[X]_\sigma$ consists of a finite number of equivalence classes, i.e. $[X]_\sigma < \infty$. The collection of all such relations denote by $W(X)$*:* Obviously, $W(X) \subseteq V(X)$. We fix an arbitrary set X*:* The set of all subsets is called a topology on X see[2] . Next, we consider that $T -$ the finite topology on the set X*:* Then for any $x \in X$ there is the smallest open set $S_T(x)$*;* containing the point x*;* where $S_T(x) -$ is the intersection of all open sets containing the point x.

Proposition 3.1.1 Let T $-$ the finite topology defined on the set X, $S \in T$ and $x, y, z \in X$.

1. If $x \in S$, then $S_T(x) \subseteq S$. Then the following holds: $S = \bigcup_{x \in S} S_T(x)$.
2. $y \in S_T(x)$ if and only if $S_T(y) = S_T(x)$.
3. If $y \in S_T(x)$, $z \in S_T(y)$, then $z \in S_T(x)$.

Proof: 1. Let $x \in S$. Since $S_T(x) -$ is the intersection of all open sets containing the point x, then $S_T(x) \subseteq S$. Furthermore, $S = \bigcup_{x \in S}\{x\} \subseteq \bigcup_{x \in S} S_T\,(x) \subseteq S$.

1. If $y \in S_T(x)$, then $S_T(y) \subseteq S_T(x)$. And the conversely it is clear.
2. Since $S_T(y) \subseteq S_T(x)$ and $S_T(z) \subseteq S_T(y)$, then $S_T(z) \subseteq S_T(x)$, therefore, $z \in S_T(x)$. ■

Now we fix on the set X finite topology T and construct the function $\sigma: X^2 \to B$ such that $\sigma(x, y) = 1$, if $y \in S_T(x)$, otherwise $\sigma(x, y) = 0$. We show that σ the characteristic function for some reflexive- transitive relation therefore for any $x \in X$ then $x \in S_T(x)$, therefore $\sigma(x, x) = 1$.

Proved the equality $\sigma(x, y)\sigma(y, z) \leq \sigma(x, z)$. If $\sigma(x, y) = 1, \sigma(y, z) = 1$, then $y \in S_T(x)$, $z \in S_T(y)$, therefore $z \in S_T(x)$, (see proposition 7) hence $\sigma(x, z) = 1$. And the others cases travails. And therefore $\sigma \in V(X)$, and hence $\sigma \in W(X)$. Then for any $x \in X$ we get the following:

$S_T(x) = \{y \in X: \sigma(x, y) = 1\}$ and since from (1) we get $U_\sigma(x) = S_T(x)$ …………………(5)

And since for $S_T(x) \in T$, then for any set $U_\sigma(x)$ generated the topology T, hence the number of these finite . Since $[x]_\sigma = \{y \in X: U_\sigma(y) = U_\sigma(x)\}$, then the number of the equivalence classes $[x]_\sigma$ also finite then $\sigma \in W(X)$.

For any topological spaces (X, T), $card\, T < \infty$, generated the binary relation $\sigma \in W(X)$ in other words if $T(X) -$ is the collection of all topological spaces defined on the set X, the defined the mapping $\Phi: T(X) \to W(X)$, $\quad T \to \sigma$.

Theorem 3.1.2 The mapping $\Phi: T(X) \to W(X)$ is bijective.

Proof: If $T, T' \in T(X)$, $T \neq T'$, then there exist a set $S \subseteq X$ such that $S \in T$ and $S \notin T'$. Then $S = \bigcup_{x \in S} S_T(x)$ and there is $y \in S$

such that $S_T(y) \neq S_{T'}(y)$ and we get a contradiction since $S = \bigcup_{x \in S} S_T(x) = \bigcup_{x \in S} S_{T'}(x) \in T'$ therefore, if $\sigma = \Phi(T)$, $\sigma' = \Phi(T')$, then there exist $z \in X$ such that $\sigma(y, z) \neq \sigma'(y, z)$. Hence $\sigma \neq \sigma'$ then $\Phi: T(X) \to W(X)$ injective mapping.

Now to prove that $\Phi: T(X) \to W(X)$ surjective mapping we fixed $\sigma \in W(X)$. T denoted the family of all subset of X , each set of these is from the form (1) in other word $S \in T$, if there exist $A \subseteq X$, such that $card\ A < \infty$ and $S = \bigcup_{x \in A} U_\sigma(x)$. It is clear that $\emptyset \in T$. And $X = \bigcup_{[x]_\sigma \subseteq X} [x]_\sigma \subseteq \bigcup_{[x]_\sigma \subseteq X} U_\sigma(x) \subseteq X$, we get $X = \bigcup_{[x]_\sigma \subseteq X} U_\sigma(x)$. Since $card\ [X]_\sigma < \infty$, then $X \in T$. If $F, G \in T$, then it is clear that $F \cup G \in T$ then we get the following implication:

$$F = \bigcup_{x \in A} U_\sigma(x), \quad G = \bigcup_{y \in B} U_\sigma(y) \implies F \cap G$$
$$= \bigcup_{x \in A, y \in B} \big(U_\sigma(x) \cap U_\sigma(y)\big).$$

If $S = U_\sigma(x) \cap U_\sigma(y) \neq \emptyset$, then $z \in S$ because $z \in U_\sigma(x)$ and $z \in U_\sigma(y)$, therefore $U_\sigma(z) \subseteq U_\sigma(x)$ and $U_\sigma(z) \subseteq U_\sigma(y)$, therefore $[z]_\sigma \subseteq U_\sigma(z) \subseteq S$ and $S = \bigcup_{z \in S} \{z\} \subseteq \bigcup_{z \in S} U_\sigma(z) \subseteq S$. Hence $S = \bigcup_{z \in S} U_\sigma(z)$. Next let $Q = \bigcup_{[z]_\sigma \subseteq S} U_\sigma(z)$ (since $card\ [X]_\sigma < \infty$, then $Q \in T$.) If $w \in Q$, then there exist $z \in S$ such that $[z]_\sigma \subseteq S$ and $w \in U_\sigma(z)$, therefore $w \in S$ and hence $Q \subseteq S$ now if $w \in S$ then there exist $z \in S$ such that $w \in U_\sigma(z)$ then from the above we proved that the implication $z \in S \Longrightarrow [z]_\sigma \subseteq S$, and hence $w \in Q$, then $S \subseteq Q$. Then $S = Q \in T$, and since $card\ A < \infty$ and $card\ B < \infty$ then $F \cap G \in T$. Therefore, $T \in T(X)$. and from the definition of the family of T show that $U_\sigma(x) \in T$ for all $x \in X$, then there exist $U_\sigma(x) -$ open set in topology T since $x \in U_\sigma(x)$, and $S_T(x) -$ the intersection of all open sets contains the point x, then $S_T(x) \subseteq U_\sigma(x)$. In the other word for the set $S_T(x)$ as element in topology T we get $S_T(x) = \bigcup_{z \in A} U_\sigma(z)$, $card\ A < \infty$, and since $x \in S_T(x)$ then there exist $z \in A$ such that $x \in U_\sigma(z)$, therefore, $U_\sigma(x) \subseteq U_\sigma(z) \subseteq S_T(x)$. Hence $S_T(x) = U_\sigma(x)$ for all $x \in X$. Let $\sigma' = \Phi(T)$. And from (5) we get $U_{\sigma'}(x) = S_T(x)$, therefore, $U_\sigma(x) = U_{\sigma'}(x)$ for all $x \in X$ then $\sigma = \sigma'$ and $\Phi(T) = \sigma$. ■

3.2 The graph of finite topology. Now suppose that $card\ X < \infty$, it clear that for any topology T define on the set X is finite and for any $\sigma \in V(X)$ then $card\ [X]_\sigma < \infty$, therefore, $\sigma \in W(X)$. Hence $W(X) = V(X)$ and $Im\ \Phi = V(X)$. And from the bijection $\Phi^{-1}: V(X) \to T(X)$ we can consider that the vertices of the graph $(V(X), E(X))$ is the finite topology (the elements of the set $T(X)$). And from this we can say that if $T, T' \in T(X)$ are adjacent vertices if $\Phi(T), \Phi(T') \in V(X)$ are adjacent and hence we can said that $\left(T(X), E(X)\right) -$ graph of finite topology.

Example 3.2.1 In the example 3 the topology $\Phi^{-1}(\sigma) = \{\emptyset, \{1,2,3,4\}, \{3,4\}, \{4\}\}$ adjacent with the topologies:

$$\Phi^{-1}\left(\sigma^{[1]}\right) = \left\{\emptyset, \{1,2\}, \{3,4\}, \{4\}, \{1,2,4\}, \{1,2,3,4\}\right\},$$

$$\Phi^{-1}\left(\sigma^{[4]}\right) = \left\{\emptyset, \{1,2,3\}, \{3\}, \{4\}, \{3,4\}, \{1,2,3,4\}\right\}$$

Which these also adjacent.

The family $\{X_1, \dots, X_m\}$, which contains the all subset of the set X, is called the partitions of the set X if $\bigcup_{k=1}^{m} X_k = X$ and $X_i \cap X_j = \emptyset$ where $i \neq j$.It clear that $m \leq n$ and we denoted the collection of all these partition by $\beta(X)$, and we denoted that the family of partition which equle to m component by $\beta_m(X)$ then it is known that $\beta_m(X) = S(n,m)$, where $S(n,m) -$ is Stirling numbers of the second kind (see the example in [3] p. 102) and we known that $S(n,m) = \frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}K^n$ (see [3] p. 121).

It is clear that for any $\sigma \in V(X)$ the family $[X]_\sigma$ represented a partition of the set X.

Remark 3.2.2 Through $V_0(X) -$ denote the collection of all partial order defined on the set X. Then there exist one-to-one correspondence between these collection and the collection of all labeled transitive graphs define on the set X(see example in [4] p.28) and in turn, there is a one-to-one correspondence between this set and the set of all labeled T_0 $-$topology defined on the set

X. (see example in [5] p. 256) and denote the number of these topologies by $T_0(n)$ and $T_0(0) = 1$.

Now we prove that the following theorem

Theorem 3.2.3 Let $n \in \mathbb{N}$, $G(X) = \langle V(X), E(X) \rangle$ the graph of reflexive- transitive relation defined on the set $X = \{1,2, \dots, n\}$. Then $card\, V(X) = \sum_{m=1}^{n} S(n,m) T_0(m)$, and if $[G(X)] -$ is the collection of all component graph $G(X)$ then $card\, [G(X)] = \sum_{m=1}^{n} S(n,m) T_0(m-1)$,

Proof: This formula $card\, V(X) = \sum_{m=1}^{n} S(n,m) T_0(m)$ we can find in [6],[7] and [8] Now to prove that $card\, [G(X)] = \sum_{m=1}^{n} S(n,m) T_0(m-1)$. Suppose that $\Phi: T(X) \to V(X)$. Since from proposition(3) all vertices of connected component $G_\sigma(X)$ of the graph $\langle V(X), E(X) \rangle$ have the same type $[X]_\sigma$, therefore for any $\{X_1, \dots, X_m\} \in \beta(X)$ define the family of connected graphs $G(X_1, \dots, X_m) = \left\{G_\sigma(X): [X]_\sigma = \{X_1, \dots, X_m\}\right\}$. Let $[G(X)]$ $-$the family of all connected component graphs. Then :

$$[G(X)] = \bigcup_{\{X_1,\dots,X_m\} \in \beta(X)} G(X_1, \dots, X_m),$$

$$card[G(X)]$$
$$= \sum_{\{X_1,\dots,X_m\}\in\beta(X)} cardG(X_1,\dots,X_m) =$$
$$= \sum_{m=1}^{n} \sum_{\{X_1,\dots,X_m\}\in\beta(X)} card\ G(\{X_1,\dots,X_m\} \in \beta(X)).$$

We fixed the partition $P = \{\{X_1, \dots, X_m\}$. It is clear that, $P \in \beta_m(X)$. Moreover P generated the family $G(X_1, \dots, X_m)$ and the graph $G_0(P)$, the vertices of these graph are partial orders wich define on the set P .We denoted the family of all connected component graphs $G_0(P)$ by $[G_0(P)]$ and from the theorem (1) in [1] we get that $card\ [G_0(P)] = T_0(m-1)$.

Fix the connected component $G_\sigma(X) \in G((X_1, \dots, X_m)$, and let $\sigma' \in V(X) -$ its representative (without loss of generality, we can assume that $\sigma' = \sigma$). It is clear that, $[X]_\sigma = P$. Since from the remark(6) the relation σ generated partial order $\bar{\sigma} \in V_0([X]_\sigma) = V_0(P)$ and connected components $G_0^{\bar{\sigma}}([X]_\sigma) = G_0^{\bar{\sigma}}(P)$ of the graph $G_0([X]_\sigma) = G_0(P)$. And hence we can define the mapping $\varphi: G_\sigma(X) \to G_0^{\bar{\sigma}}(P)$, acting from $G((X_1, \dots, X_m)$ in $[G_0(P)]$.

<u>Iinjectivity $\boldsymbol{\varphi}$.</u> Suppose that $G_0^{\bar{\sigma}}(P) = G_0^{\bar{\tau}}(P)$ for some $\sigma, \tau \in V(X)$. Then $\bar{\tau} G_0^{\bar{\sigma}}(P)$ without loss of generality, we can assume that $\bar{\sigma}, \bar{\tau} -$ are adjacent partial order and since from proposition (4) σ and τ also adjacent therefore $G_\sigma(X) = G_\tau(X)$.

Surjectivity φ. Let $G_0^{\bar{\tau}}(P) \in [G_0(P)]$ for some partial order $\bar{\tau} \in V_0(P)$. Then generated the function $\sigma: X^2 \to B$ such that $\sigma(x, y) = \bar{\tau}(X_i, X_j)$ for all $(x, y) \in X_i \times X_j$. If $(x, y, z) \in X_i \times X_j \times X_k$, then

$$\sigma(x, x) = \bar{\tau}(X_i, X_j) = 1$$

$$\sigma(x, y)\sigma(y, z) = \bar{\tau}(X_i, X_j)\bar{\tau}(X_j, X_k) \leq \bar{\tau}(X_i, X_k) = \sigma(x, z),$$

Therefore, $\sigma \in V(X)$ and hence define the partition $[X]_\sigma$. We fixed the index i and the element $x \in X_i$. For all $y \in X_i$, $\eta \in X$ we get $\sigma(x, \eta) = \bar{\tau}(X_i, X_j) = \sigma(y, \eta)$ (where j such that $\eta \in X_j$). And therefore, $U_\sigma(x) = U_\sigma(y)$, $x \sim y$, $y \in [x]_\sigma$. Hence $X_i \subseteq [x]_\sigma$. Next suppose $z \in [x]_\sigma$, and j such that $z \in X_j$ since $x \sim z$, then $\sigma(x, \eta) = \sigma(z, \eta)$ for all $\eta \in X$. Therefore:

1) $If\ \eta \in X_j\ , then\ \bar{\tau}(X_i, X_j) = \sigma(x, \eta) = \sigma(z, \eta) = \bar{\tau}(X_j, X_j) = 1,$

2) $If\ \eta \in X_i\ , then\ \bar{\tau}(X_j, X_i) = \sigma(z, \eta) = \sigma(x, \eta) = \bar{\tau}(X_i, X_i) = 1,$

Therefore, $\bar{\tau}(X_i, X_j)\bar{\tau}(X_j, X_i) = 1$, hence $i = j$(since $\bar{\tau} -$ partial order and $z \in X_i$ then $[x]_\sigma \subseteq X_i$, then $[x]_\sigma = X_i$ hence we get the implication $x \in X_i \Longrightarrow [x]_\sigma = X_i$ the conversely of implication it clear and hence $[x]_\sigma = P$ and $G_\sigma(X) \in G(X_1, \dots, X_m)$ and since $(\xi \in X_i \Leftrightarrow \bar{\xi} \in X_i)$, then $\bar{\sigma}(\bar{x}, \bar{y}) = \sigma(x, y) = \bar{\tau}(X_i, X_j) = \bar{\tau}(\bar{x}, \bar{y})$, therefore $\bar{\sigma} = \bar{\tau}$. And hence $\varphi(G_\sigma(X)) = G_0^{\bar{\sigma}}(P) = G_0^{\bar{\tau}}(P)$, and that's mean $Im\, \varphi = [G_0(P)]$. Then φ −bijective then the sets $G(X_1, \dots, X_m)$ and $[G_0(P)]$ are the same cardinality hence card $G(X_1, \dots, X_m) = T_0(m - 1)$ and

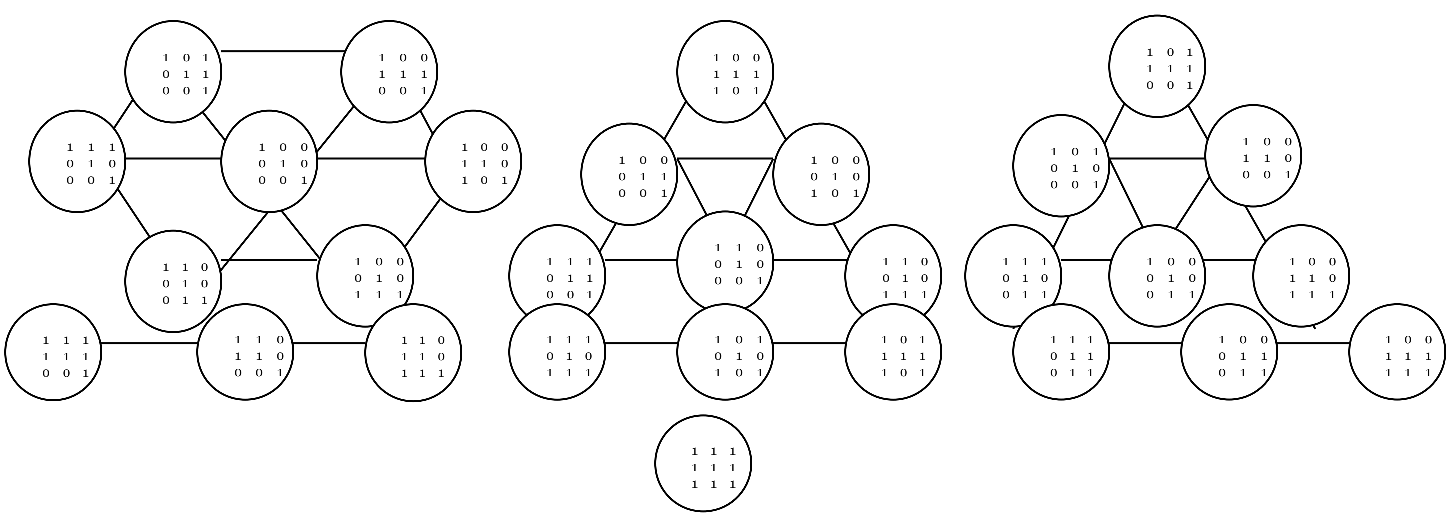
$$card[G(X)]$$

$$= \sum_{m=1}^{n} \sum_{\{X_1,\dots,X_m\}\in\beta_m(X)} T_0(m-1)$$

$$= \sum_{m=1}^{n} card\, \beta_m(X) T_0(m-1) =$$

$$= \sum_{m=1}^{n} S(n,m) T_0(m-1)$$

Example 3.2.4 for small n the number of connectivity of the graph $\langle T(X), E(X)\rangle$ equal to 1,2,7,45,… for example, 45=1.1+7.1+6.3+1.19. in other word 355 finite topologies of the fourth order are contained in forty five connected components of the graph $G(\{1,2,3,4\})$. Below there are 7 connectivity components (7=1.1+3.1+1.3) of the graph $G = (\{1,2,3\})$ contains 29 finite topology (all 29 are reflexive-transitive relations).



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