Republic of Iraq
Ministry Of higher education
and Scientific research
University of AL-Qadisiyah
College of Education

T-small Submodules

A research

Submitted to the department of education Al-Qadisiyah University as a partial fulfillment requirement for the degree of Bachelor of science in mathematics

By

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2018

Certification

I certify that this paper was prepared under my supervision at the university of AL-Qadisiyah, college of Education, Dep. of Mathematics, as a partial fulfillment for the degree of B.C. of science in Mathematics.

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ (١) خَلَقَ الْإِنْسَانَ مِنْ عَلَقٍ (٢) اقْرَأْ وَرَبُّكَ
الْأَكْرَمُ (٣) الَّذِي عَلَّمَ بِالْقَلَمِ (٤) عَلَّمَ الْإِنْسَانَ مَا لَمْ يَعْلَمْ (٥) كَلَّا إِنَّ الْإِنْسَانَ
لَيَطْغَى (٦) أَنْ رَآهُ اسْتَغْنَى (٧) إِنَّ إِلَى رَبِّكَ الرُّجْعَى

صدقَ اللهُ العليُ العظيم

سورة العلق

Dedication

I dedicate this humble to cry resounding silence in the sky to the martyrs of Iraq wounded. Also, I dedicate my father treasured, also I dedicate to my supervisor **Dr. Tha'ar Younis Ghawi**. Finally, to everyone who seek knowledge, I dedicate this humble work.

Acknowledgment

And later the first to thank in this regard is the Almighty God for the blessing that countless, including the writing of this research modest. Then thanks and gratitude and deepest gratitude to my teacher **Dr. Tha'ar Younis Ghawi**, which prefer to oversee the research and guidance in spite of concern, and not to ask him reordering Almighty to help him I also wish to express my thanks to the staff of the department of mathematics.

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INTRODUCTION

In this work, all rings have identity elements and all modules are right unitary. We use the notations "$\subseteq$ " and "$\leq$ " to denote inclusion and submodule, respectively. For two integers n and m, we denote n/m in ease n divides m and gcd (n, m) denotes the greatest common divisor of n and m. Let R be an ring and M be an R-module. Recall that a submodule X of M is small, denoted by $N \ll M$, if for any submodule X of M, $X + N = M$ implies that X = M. More details about small submodules can be found in [2, 3, 4] . The concept of small submodule has been extended by some researchers, for this see [1, 6]. In [5], the authors extended the concept of essential submodule with respect to an arbitrary submodule. This motivates us to define a new generalization of small submodules. Let T be an arbitrary submodule of M . We say that a submodule N of M is an T-small submodule of M provided for each submodule $X \leq M$, $T \subseteq X + N$ implies that $T \subseteq X$. Note that the notions of smallness and T-smallness coincide if $T = M$. We investigate the basic properties of T-small submodules.

CHAPTER ONE

Some Properties of small submodules

Definition 1.1 [3] A submodule N of a module M is called small in M (denoted by N$\ll$ M) if $\forall\ K \leq M$ with N + K =M implies K = M.

Example 1.2 For every module M we have 0 $\ll$ M.

Theorem 1.3 [3] $A \ll M \Leftrightarrow \forall\ U \leq M(A + U \leq M)$.

Proof $\Longrightarrow Let\ A \ll M$ [WE will proof by using contradiction]

And since A $\leq$ M then U = M

Suppose $\exists$ U $\leq$ M $\ni$ A + U = M, And since A $\leq$ M then U = M

And this is contradiction $\longrightarrow$ Then U = M, So A $\ll$ M

Definition 1.4 [6] An R module M is said to be semisimple if $\forall N \leq M \exists K \leq M \ni N \oplus K = M$.

Theorem 1.5 [6] If M is a semisimple module then 0 is the only small submodule in M .

Proof Let $N \leq M\ so\ N \leq \oplus M$ so (since M is semisimple)

$\exists K \leq M\ with\ N \oplus K = M\ i.e\ N \cap K = 0\ and\ N + K = M \Longrightarrow K = M\ but\ N \cap K = 0\ so\ N \cap M = 0 \Rightarrow N = 0$.

Definition 1.6 [6] Let M be an R-module A subset X of M is called basis of M if **:**

(i) X is generated M, i.e. M = $\prec$ X $\succ$

(ii) X is linearly independent, that is for every finite subset

$\prec X_1, X_2, \dots., X_2 \succ$

of X with $\sum_{i=1}^{n} Xi\ \propto i = 0, \forall \propto i \in R$ then $\propto i = 0, \forall 1 \leq i \leq n$.

Definition 1.6 [2] An R-module M is said to be free if satisfy the following conditions:

(*i*) M has basis

(*ii*) M = $\bigoplus_{\forall i \in I} A_i \ \wedge \forall i \in I[A_i \equiv R_R]$.

Example 1.7 [2] Z as Z-module is a free module.

Example 1.8 Z as Z-module is free since $< 1 >= Z$

$< 1 >= \{1. a \setminus a \in Z\} = \{\ldots - 3, -2, -1,0,1,2,3, \ldots\}$

and $\forall \propto \in Z, \propto .1 = 0 \setminus \Rightarrow \propto = 0$.

Theorem 1.9 [1] In a free Z-module (0) is only small submodule.

Proof Let $F = \bigoplus_{i \in IXi} Z$ be a free Z-module with basis $\{X_i \ / \ i \in I\}$ $A \leq F, a \in A$ and let $\propto = X_{i1Z1} + .. + X_{im} Z_m, Z_i \in Z$, with $Z_i \neq I\}$Let $n \in Z$ with $g.c.d(Z_1, n) = 1$ and $n \prec 1$.

Put $U = \bigoplus_{X_i} Z + X_i nZ$, then $aZ + U = F$, hence $A + U = F$ with $U \neq F$.

Zoren's lemma 1.10 [2] If A is non-empty partial order set such that every chain in A has an upper bound in A, then A has maximal element.

Proposition 1.11[3] If finitely many arbitrary elements are omitted from an arbitrary generating set X of Q_z, then the set with out these elements omitted is again generating.

Theorem 1.12 Every finitely generating submodule of Q_z is small in Q_z.

Proof Let $N \leq Q_z$ be a finitely generating sub module , So $\exists \{q_1, q_2, \ldots, q_n\} \subseteq Q$ Such that $N =< q_1, q_2, \ldots, q_n >$ Let $K \leq Q_z$ with $Q_z =<< q_1, q_2, \ldots, q_n > \cup K >$, by the proposition $Q = Z \Rightarrow N$ is small.

Modular law 1.13 [3] If A, B, C $\leq$ M and B $\leq$ C, then $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Lemma 1.14 [3] If $A \leq B \leq M \leq N$ and $B \ll M \Rightarrow A \ll N$

Proof Let $U \leq N$, Let $A + U = N$ [We must proof that U = N]

Since A≤ B then B+U=N⇒ (B+U) ∩ M=N ∩ M

⇒ B+(U∩ M)=M (by modular law)

Hence U∩ M= (since B≪ M, So A ≤ U and since A + U=N

Then U=N ⟶ A ≪ N.

Theorem 1.15 [3] If $A_i \ll M$, i = 1,2,……, n⇒ $\sum_{i=1}^{n} A_i \ll M$

Proof Let $A_i \ll M$, i = 1,2,……, n

If i=1, $A_1 + U = M \Rightarrow U = M$ (by hypothesis) ⟶ $[A_i \ll M]$

If n=2 ⟶ $A_1 + A_2$ + U=M ⟶ A_1+$(A_2 + U) = M$

Since $A_1 \ll M$ then $A_2 + U = M$

Since $A_2 \ll M$ ⇒ U=M, So $\sum_{i=1}^{2} A_i \ll M$

Let it be true at n-1, And we will proof it at n

Let A= $A_1 + A_2$…+$A_{n-1} \leq M$ ⟶ A+A_n+U=M

Then A_n+U=M [since A ≪ M], Then U=M [since $A_2 \ll M$]

So $\sum_{i=1}^{n} A_i \ll M$.

Definition 1.16 A homomorphism $\propto: A \longrightarrow B$ is called small ⇔ ker∝≤ A.

Theorem 1.17 [6] If $\propto: M \longrightarrow N$ modular homomorphism on R-ring. If B≤ N then $\propto\left(\propto^{-1} (B)\right) = B \cap IM(\propto)$.

Theorem 1.18 [6] If A≪ M and $\varphi \in Hom(M, N) \Rightarrow \varphi(A) \ll N$

Proof Let $\varphi(A) + U = N$ and $U \leq N$, so $\varphi(m) \in N \forall m \in M$

$\varphi(m) = \varphi(a) + u$ with $a \in A, u \in U \longrightarrow \varphi(m) - \varphi(a) = u$

$\longrightarrow \varphi(m - a) = u \longrightarrow \varphi^{-1}\left(\varphi(m - a)\right) = \varphi^{-1} = (u)$

$\longrightarrow m - a \in \varphi^{-1}(U) \longrightarrow m \in A + \varphi^{-1} = (U)$

$\longrightarrow A + \varphi^{-1} = (U) = M$ but $A \leq M$, hence $M = \varphi^{-1} = (U)$

$\longrightarrow \varphi(m) = \varphi\big(\varphi^{-1} = (U)\big) = U \cap Im(\varphi)$[by theorem (2.1.19)]

$\longrightarrow \varphi(A) \le \varphi(m) \le U$, hence $U = \varphi(A) = N$

Theorem 1.19 [2] If $\propto: M \longrightarrow N, B: N \longrightarrow K$ modular homomorphism on R-ring then Ker $(B\propto) = \propto^{-1}$ (Ker(B)).

Proof Let $X \in Ker(B \propto) \longrightarrow B \propto (X) = 0 \longrightarrow \big(\propto (X)\big) = 0 \longrightarrow \propto (X) \in Ker(B) \longrightarrow X \in \propto^{-1}$ (Ker(B))

So Ker $(B \propto) \subseteq \propto^{-1} \big(Ker(B)\big) \ldots (1)$

Let $X \in \propto^{-1} \big(Ker(B)\big) \longrightarrow \propto (X) \in (B) \longrightarrow \big(\propto (X)\big) = 0 \longrightarrow B \propto (X) = 0 \longrightarrow X \in Ker(B \propto)$ So$\propto^{-1} \big(Ker(\mathcal{B})\big) \subseteq Ker\ (\mathcal{B} \propto) \ldots. (2)$ From (1), (2) $\longrightarrow$ Ker $(\mathcal{B} \propto) = \propto^{-1} \big(Ker(\mathcal{B})\big)$.

Theorem 1.20 [2] If $\propto: M \longrightarrow N, \mathcal{B}: N \longrightarrow K$ modular homomorphism on R-ring. If $A \le M$ then $\propto^{-1} \big(\propto (A)\big) = A + Ker(\propto)$.

Proof Let $X \in \propto^{-1} \big(\propto (A)\big) \longrightarrow \propto (X) \in \propto (A)$, Then $\exists\, b \in A \ni \propto (X) = \propto (b) \longrightarrow \propto (X - b) = 0 \longrightarrow X - b \in Ker(\propto)$, then $\exists\, K \in Ker(\propto) \ni X - b = K \longrightarrow X = b + K \longrightarrow X \in A + Ker(\propto)$[since $K \in Ker(\propto), b \in A$]

So $\propto^{-1} \big(\propto (A)\big) \subseteq A + Ker(\propto) \ldots.. (1)$ Let $X \in A + Ker(\propto)$, $Then\ \exists\, b \in B, K \in Ker(\propto) \ni X = b + K \longrightarrow \propto (x) = \propto (b + K) \longrightarrow \propto (X) = \propto (b) + \propto (K) \longrightarrow \propto (x) = \propto (b)$[$since\ K \in Ker(\propto)$] $\longrightarrow X \in \propto^{-1} \big(\propto (A)\big)$ So $A + Ker(\propto) \subseteq \propto^{-1} \big(\propto (A)\big) \ldots (2)$

So from (1), (2) we get $\propto^{-1} \big(\propto (A)\big) = A + Ker(\propto)$.

Theorem 1.21 [2] If $\propto: A \longrightarrow B, \mathcal{B}: B \longrightarrow C$ are small epimorphisms, then $\mathcal{B} \propto: A \longrightarrow C$ also small epimorphism.

Proof By theorem $\mathcal{B} \propto$ is also epimorphism Now we must proof $Ker(\mathcal{B} \propto) \ll A$ Let $U \le A$ with $Ker(\mathcal{B} \propto) + U = A$, Then $\propto (Ker(\mathcal{B} \propto) + U) = \propto (A)$

$\Rightarrow \propto \left(\mathrm{Ker}(\mathcal{B} \propto)\right) + \propto (\mathrm{U}) = \mathrm{B} \Rightarrow \propto (\propto^{-1} \left(\mathrm{Ker}(\mathcal{B} \propto)\right) + \propto (\mathrm{U}) =$
B (by theorem(2.1.21)) $\Rightarrow Ker(\mathcal{B}) + \propto (\mathrm{U}) = \mathrm{B}$, But $\mathrm{Ker}(\mathcal{B}) \ll \mathrm{B} \Rightarrow \propto$
$(U) = B \Rightarrow \propto (U) = \propto (A)$

$\Rightarrow \propto^{-1} \left(\propto (U)\right) = \propto^{-1} \left(\propto (A)\right) \Rightarrow \mathrm{U} + \mathrm{Ker}(\propto) =$
A $\left(\text{by theorem } (2.1.22)\right) But\ Ker(\propto) \ll A \Rightarrow \mathrm{U} = \mathrm{A}$

Definition 1.22 [4] Let A≤ M then A-B ≤ M is called addition complement of A in M (briefly adico) iff :

(*i*) A + B = M

(*ii*) B ≤ M minimal in A+B=M, i.e. $\forall\, B \leq M$ with A+B=M, i.e $\forall\, U \leq M$ with A+U=M and U≪ B imply U = B

B- D ≤ M is called intersection complement of A in M (briefly inco) iff :

(*i*) A∩D=0

(*ii*) D is a maximal in A∩D=0

i.e. $\forall\ \mathrm{C} \leq \mathrm{M}$ with $\mathrm{A} \cap \mathrm{C} = 0\ \wedge \mathrm{D} \leq \mathrm{C}$ implies $\mathrm{C} = \mathrm{D}$.

Corollary 1.23 [4] Let A≤ M and B ≤ M, then A⊕B=M ⟺ B is adco and inco of A in M.

Proof ⟹ suppose that B is adco and inco of A

Then A+B=M resp. A∩B=0 ⟹ M= A⊕B

⟸ suppose that A⊕B=M, hence A+B=M and A∩B=0

Let C ≤ M with A+C=M and C≤ B, (A+C) ∩B=M∩B⟹

(A+C)∩B=B $\longrightarrow (\mathrm{A} \cap \mathrm{B}) + \mathrm{C} = \mathrm{B} \Rightarrow \mathrm{C} + \mathrm{B}[\mathrm{A} \cap \mathrm{B} = 0]$

So B is adco of A in M Let C≤ M with $\mathrm{A} \cap \mathrm{C} = 0$ and B≤C

Since A+B=M⟹ A+C=M [since A+B$\subseteq \mathrm{A} + \mathrm{C}$]

$\longrightarrow \mathrm{A} \oplus \mathrm{C} = \mathrm{M} \Rightarrow \mathrm{A} \oplus \mathrm{C} = \mathrm{A} \oplus \mathrm{B}$ [$\mathrm{A} \oplus \mathrm{B} = \mathrm{M}$ by assumption]

$\frac{\mathrm{A} \oplus \mathrm{C}}{\mathrm{A}} = \frac{\mathrm{A} \oplus \mathrm{B}}{\mathrm{A}} \Rightarrow \mathrm{C} = \mathrm{B} \longrightarrow$ so B is inco of A in M .

Lemma 1.24 [4] Let M=A+B, then B is adco of A in M $\Leftrightarrow A\cap B \leq B$.

Proof $\Longrightarrow$ let $U\leq B$ and $(A\cap B)+U=B$

Then $M=A+(A\cap B)+U \Longrightarrow A+U=M$ [since $A\cap B\subseteq A$]

But B is so $A\cap B\leq B$

$\Longleftarrow$ we have by assumption M=A+B, Let $U\leq B$ with A+U=M and $U\leq B$

$\longrightarrow (A+U)\cap B = M\cap B \longrightarrow (A+U)\cap B = B$ $[B\leq M]$ $\longrightarrow (A+B)\cap U = B$ [by modular law]

But $A\cap B \leq B$, hence U=B, thus B is adco to A in M.

CHAPTER TWO

T-small submodules

Let M be an R-module. A submodule N of M is called T-small in M provided for each submodule X of M, T ⊆ X + N implies that T⊆ X. We study this mentioned notion which is a generalization of the small submodules and we obtain some related results .

Definition 2.1. [1] Let R be a ring and T be a submodule of an R-module M. A submodule N of M is called T-small (in M), denoted, by N$\ll_T M$, in case for any submodule X ≤M, T ⊆ X + N implies that T ⊆ X . Under the notations of the above definition, if T=0, then every submodule of M is T-small in M. Also if T ≠ 0. Then N $\ll_T M$ implies that T ⊈ N, for if not, T ⊆ N +(0) and hence T ⊆ (0), a contradiction. If T = M, then N$\ll_T M$ if and only if N ≪ M .

Remarks 2.2.

(*i*) If T = 0, then every submodule of M is T-small in M.

Proof. Let N ≤ M and let X ≤ M, where T = 0. Since ∌ 0 ≤ X + N it clear that 0 ≤ X then N $\ll_T M$.

(*ii*) If T ≠ o, N$\ll_T M$ implies T ⊈ N.

Proof. If T ≤ N and let $x = o$, $x \leq N$ then $T \subseteq N + (0)$ but N$\ll_T M$, T ≤ o which is a contradiction, T ⊈ N .

(*iii*) If T = M, then N$\ll_T M$ if and only if $N \ll M$.

Proof. Let N$\ll_T M$ where T = M to prove $N \ll M$.

Suppose $K \ll M$ such that N + K = M implies N + K = T, so $T \subseteq N + K$ but N$\ll_T M$ implies $T \subseteq K$, $M \subseteq K$ implies $M = K$ and hence $N \ll M$.

Let $N \ll M$ to prove $N \ll_T M$ suppose $K \leq M$ such that $T \subseteq K + N$. So, $M \subseteq K + N$. But $K + N \leq M$ (clear). Then $M = K + N$ but $N \ll M$, hence $M = K$, so $T = K$, thus $T \subseteq K$. Therefore $N \ll_T M$.

Example 2.3.

(i) Let $\mathbb{Z}$ be the ring of integers. It is easy to see that (0) is the only small submodule of $\mathbb{Z}$ and also for any nonzero integer m, the submodule (0) is the only $m\mathbb{Z}$ -small submodule of $\mathbb{Z}$.

Solution. Suppose $n\,\mathbb{Z} \leq \mathbb{Z}$ where $n \neq \pm 1$ such that $m\,\mathbb{Z} \leq n\,\mathbb{Z} + o$

Then $m\mathbb{Z} \leq n\mathbb{Z}$. Therefore $o \ll_{mZ} \mathbb{Z}$.

(ii) In $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module. Let $N_1 =< 4 >$, $N_2 =< 6 >$,

$T_1 =< 2 >$ and $T_2 =< 3 >$ be submodules of $\mathbb{Z}_{12}$.

Is N_1 is a $T_1 -$ small submodule ?

Is N_2 is a $T_2 -$ small submodule ?

Solution. All submodules of $\mathbb{Z}_{12}$ are :

$X_1 = \{0\}$

$X_2 =< 2 >= \{0, 2, 4, 6, 8, 10\}$

$X_3 =< 3 > = \{0, 3, 6, 9\}$

$X_4 =< 4 > = \{0, 4, 8\}$

$X_5 =< 6 > = \{0,\ 6\}$

$X_6 = \mathbb{Z}_{12}$

When $N_1 =< 4 >$ and $T_1 =< 2 >$. $T \subseteq X + N$ implies $T \subseteq X$

$< 2 > \subseteq < 2 > + < 4 > = < 2 >$ implies $< 2 > \subseteq < 2 >$

$< 2 > \subseteq < 3 > + < 4 > = \mathbb{Z}_{12}$ implies $< 2 > \nsubseteq < 3 >$

Then N_1 is not $T_2 -$ small submodule.

when $N_2 =< 6 >$ and $T_1 =< 3 >$

$< 3 > \subseteq < 3 > + < 6 > = < 3 >$ implies $< 3 > \subseteq < 3 >$

$< 3 > \subseteq \mathbb{Z}_{12} + < 6 > = \mathbb{Z}_{12}$ implies $< 3 > \subseteq \mathbb{Z}_{12}$

Then N_2 is $T_1 -$ small submodule.

(*iii*) $4\mathbb{Z}_{24} \ll_{3\mathbb{Z}24} \mathbb{Z}_{24}$ is not small in $\mathbb{Z}_{24}$.

Solution.

$4\mathbb{Z}_{24} = \{0, 4, 8, 12, 16, 20\}$

$3\mathbb{Z}_{24} = \{0, 3, 6, 9, 12, 15, 18, 21\}$

$4\mathbb{Z}_{24} + 3\mathbb{Z}_{24} = \{0, 1, 2, 3, \dots, 23\} = \mathbb{Z}_{24}$ but $3\mathbb{Z}_{24} + 4\mathbb{Z}_{24}$. Hence

$4\mathbb{Z}_{24}$ is not small in $\mathbb{Z}_{24}$. To prove $4\mathbb{Z}_{24}$. To prove $4\mathbb{Z}_{24} \ll_{3\mathbb{Z}24} \mathbb{Z}_{24}$.

$4\mathbb{Z}_{24} \ll \mathbf{3_{Z24}} \ \mathbb{Z}_{24}$

T = $< 3 >$ and N = $< 4 >$

All submodules of $\mathbb{Z}_{24}$ are:

$X_1 = \{0\}$

$X_2 =< 2 > = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$

$X_3 =< 3 > = \{0, 3, 6, 9, 12, 15, 18, 21\}$

$X_4 =< 4 > = \{0, 4, 8, 12, 16, 20\}$

$X_5 =< 6 > = \{0, 6, 12, 18\}$

$X_6 =< 8 > = \{0, 8, 16\}$

$X_7 =< 12 > = \{0, 12\}$

$X_8 = \mathbb{Z}_{24}$

T $\subseteq X + N$ implies T $\subseteq X$

$< 3 > \subseteq < 3 > + < 4 > = \mathbb{Z}_{24}$ implies $< 3 > \subseteq < 3 >$

$< 3 > \subseteq \mathbb{Z}_{24} + < 4 > = \mathbb{Z}_{24}$ implies $< 3 > \subseteq \mathbb{Z}_{24}$

Proposition 2.4. Let M be an R-module, L $\leq$ T $\leq$ M and K $\leq$ M .

(*i*) If K $\ll_T M$, then K $\cap T \ll M$.

(*ii*) L $\ll_T M$ if and only if L $\ll$ T.

Proof. (i) Suppose that $(K \cap T) + X = M$ for some $X \leq M$. Then $T < (K \cap T) + X$ and hence $T \leq M$ then $T \subseteq K + X$ but K $\ll_T M$, we have $T \subseteq X$. Thus $K \cap T \subseteq T \subseteq X$ implies $K \cap T \subseteq X$ and hence $X = (K \cap T) + X = M$.

(ii) Suppose that L $\ll_T M$ and $T \subseteq L + X$ for some $X \subseteq T$ then $T \subseteq L + X$ and so $T \subseteq X$. Thus $X = T$. Conversely, suppose that L ≪ T and $T \subseteq L + X$ for some $X \leq M$ then $T = (L + X) \cap T$ by module Law $T = L + (X \cap T)$ but L ≪ T and hence $X \cap T = T$ Thus $T \subseteq X$.

Proposition 2.5. [4] Let M be an *R-module* with submodules N ≤ K ≤ M and T ≤ K. If N $\ll_T K$, then N $\ll_T M$.

Proof. Suppose that T⊆ N +X for some $X \subseteq M$. Then $T \subseteq (N + X) \cap K$ for some $T \subseteq K$ and $T \subseteq N + X$ and by modular Law $T \subseteq (N + X) \cap K = N + (X \cap K)$, since N $\ll_T K$, we have $T \subseteq X \cap K \subseteq X$ implies $T \subseteq X$.

Proposition 2.6. [4] Let M be an R-module with submodules N_1, N_2 and T. Then $N_1 \ll_T M$ and $N_2 \ll_T M$ if and only if $N_1 + N_2 \ll_T M$.

Proof. Suppose that $N_1 \ll_T M$ and $N_2 \ll_T M$ to prove $N_1 + N_2 \ll_T M$. Let $X \leq M$ such that T ⊆ $(N_1, +N_2) + x$ implies T ⊆ $N_1 + (N_2 + X)$ T ⊆ $N_2 + X$, since $N_1 \ll_T M$. T ⊆ X , since $N_2 \ll_T M$.

Proposition 2.7.[4] Let M be an R-module with submodules K ≤ N ≤ M and K ≤ T . Then N $\ll_T M$ if and only if K $\ll_T M$ and N/K $\ll_{T/K}$ M/K.

Proof. Suppose $N \ll_T M$,to prove T ⊆ $K + X$ for some $X \leq M$. Then T ⊆ $N + X$ and by hypothesis, T ⊆ X. Thus $K \ll_T M$. Now assume that T/K ⊆ N/K + $X/K = (N + X)/K$ for some $K \leq X \leq M$. Then $T \leq N + X$ and so $T \leq X$. Thus T/K ⊆ X/K . Conversely, suppose that K $\ll_T M$ and N/K ≪ M/K and also T⊆ $K + X$ for some $X \leq M$. Then T/K ⊆

$(N+X)/K = N/K + (X+K)/K$. Since N/K $\ll_{T/K}$ M/K, T/K $=$ $(X+K)/K$ and so T $\subseteq K+X$. Since K $\ll_T M$, we have T $\subseteq X$, as desired .

Proposition 2.8.[3] Let M be an R-module with submodules $K_1 \leq M_1$ $\leq$ M and $K_2 \leq M_2 \leq$ M such that $T \subseteq M_1 \cap M_2$. Then $K_1 \ll_T M_1$ and K_2 $\ll_T M_2$ if and only if $K_1 + K_2 \ll_T M_1 + M_2$.

Proof. First assume that $K_1 \ll_T M_1$ and $K_2 \ll_T M_2$. By proposition (2.5), $K_1 \ll_T M_1 \leq M_1 + M_2$ implies $K_1 \subseteq M_1 + M_2$ and $K_2 \ll_T M_2 \subseteq$ $M_1 + M_2$ implies $K_2 \subseteq M_1 + M_2$, Also by proposition (2.6) $K_1 + K_2$ $\ll_T M_1 + M_2$**.** Suppose that $K_1 + K_2 \ll_T M_1$ to prove $K_1 \ll_T M_1$ and K_2 $\ll_T M_2$. $T \subseteq K_1 + X$ $\forall X_1 \subseteq M_1$ **.** $T \subseteq K_1 + X \subseteq K_1 + K_2 + X$
$T \subseteq K_1 + K_2$+X. Since $X \leq M_1$ implies $M_1 \subseteq M_1 + M_2$, $X \subseteq M_1 + M_2$ and since $K_1 + K_2 \ll_T M_1 + M_2$ Then $T \subseteq X$.

Theorem 2.9. [4] Let $\{T_i\}_{i \in I}$ be an indexed set of submodules of an R-module M and K be a submodule of M. If for each $i \in I$, K $\ll_{Ti} M$, then K $\ll_{\sum i \in I\, Ti} M$.

Proof. Suppose $\sum_{i \in I} T_i \subseteq K + X$ for some $X \leq M$. Since $T_i \subseteq \sum T_i \subseteq$ $K + X$ Then $T_i \subseteq K + X$, since K $\ll_{Ti} M$ this implies that $T_i \subseteq X$. Then $\sum T_i \subseteq X$.

Corollary 2.10. [3] Let K_1 and K_2 be submodules of an R-module M such that $K_1 \ll_{K_2} M$ and $K_2 \ll_{K1} M$. Then $K_1 \cap K_2 \ll_{K_1+K_2} M$.

Proof. Since $K_1 \ll_{K_2} M$ and $K_2 \ll_{K_1} M$, by theorem 2.6 , $K_1 \cap K_2 \subseteq$ $K_2 \leq M$ and $K_1 \cap K_2 \subseteq K_1 \leq M$. Since $K_1 \ll_{K_2} M$ and $K_2 \ll_{K_1} M$. Then by theorem 2.9, $K_1 \cap K_2 \ll_{K_1} M$ and $K_1 \cap K_2 \ll_{K_2} M$ implies $K_1 \cap K_2$ $\ll_{K_1+K_2} M$.

Proposition 2.11. [3] Let K and $0 \neq T$ be two submodules of a right R-module M. The following statements are equivalent:

(i) K $\ll_T M$;

(ii) The natural map $\pi : M \longrightarrow M/K$ is T-small;

(iii) For every right R-module N and R-homomorphism $h : N \longrightarrow M,\ T \subseteq K + Imh$ implies that $T \subseteq Imh$.

Proof. $(i) \Leftrightarrow (ii)$ Suppose that K $\ll_T M$ to prove $\pi : M \longrightarrow M/K$ is T-small. By definition (1) A monomorphism $f : M \longrightarrow \grave{M}$ is called T-small if Im f $\ll_{\mathrm{T}} \grave{M}$.

(ii) An epimorphism $f : M \longrightarrow \grave{M}$ is called T-small if ker f $\ll_T M$

We must find ker π and prove is T-small.

$\pi : M \longrightarrow M/K$ and π (x) = x + K

$\leftrightarrow$ $\ker \pi = \{x \in M : f(x) = 0\} = \{x \in M : x + K = 0_{\frac{M}{K}}\}$

$\leftrightarrow$ $= \{ x \in M : x + K = 0 + K\}$

$\leftrightarrow$ $= \{ x \in M : x + K = K\} = \{x \in M : x \in K\}$ =K

$\leftrightarrow$ Hence $\ker \pi = K$

$\leftrightarrow$ then $\pi : M \longrightarrow M/K$ is T-small.

$(i) \Rightarrow (iii)$ Clear.

$(iii) \Rightarrow (i)$ Suppose that $T \subseteq K + X$ for some $X \subseteq M$. Let $i : X \longrightarrow M$ be the inclusion map. Since $i(x) = x$ *and* $\mathrm{Im}(X) = X$ Then $T \subseteq K + Imi = K + X$ and $f(T) = 0$ then by (iii) $f(x) \ll M$ and $f(T) \neq 0$ then $T \subseteq X$.

Lemma 2.12. [4] Let M and N be right R-modules and $f : N \longrightarrow M$ be an R-homomorphism. If K and T are submodules of M such that K $\ll_T M$, then $f(K) \ll_{f(T)} N$. In particular, if K $\ll_T M \leq N$, then K $\ll_T N$.

Proof. We may assume that $f(T) \neq 0$. Let $f(T) \subseteq f(K) + X$, for some $X \leq N$. We claim that $T \subseteq K + f^{-1}(x)$. Let $t \in T$. Then $f(t) \in f(T)$ implies $f(t) \in f(K) + X$, then $f(t) = f(k) + x$ for some $x \in X$ and $k \in K$ Thus $f(t) - f(K) = x$, $f(t - K) = x \in X$ and so $f(t - K) \in X$. $t - k \in f^{-1}(x)$. This implies that $t \in k + f^{-1}(x)$ implies $t \subseteq k + f^{-1}(x)$. Since $K \ll M$, We have $t \subseteq f^{-1}(x)$ and hence $f(T) \subseteq X$.

Corollary 2.13. [4] Let M and N be right R-modules and $f : M \longrightarrow N$ be an R-homomrphism, then $f(K) \ll_{f(T)} N$.

Note. Let M and N be R-modules and $f : M \longrightarrow N$ is a homomorphism. If consider $f: \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{20}$ with $f(\bar{x}) = 2\bar{x}$. Then $10\mathbb{Z}_{20} \ll \mathbb{Z}_{20}$ but $f^{-1}(10\mathbb{Z}_{20}) = 5\mathbb{Z}_{10}$ is not small in $\mathbb{Z}_{10}$.

Example 2.14. Let f: $\mathbb{Z} \longrightarrow \mathbb{Z}_4$ be a homomrphism defined by $f(x) = \bar{x}$ for all $x \in Z$. Put K = $2Z_4$ and T = $\{\bar{0}\}$. We note that $2Z_4 \ll_{\{\bar{0}\}} \mathbb{Z}_4$. Now,
$f^{-1}(K) = f^{-1}(2Z_4) = f^{-1}\{\bar{0}, \bar{2}\} = \{\ldots., -4, -2, 0, 2, 4, \ldots\} = 2Z$
$f^{-1}(T) = f^{-1}(\{\bar{0}\}) = \{\ldots\ldots, -8, -4, 0, 4, 8, \ldots.\} = 4Z.$
So $f^{-1}(T), f^{-1}(K)$ are submodules of $\mathbb{Z}$. Now, let X = 3$\mathbb{Z} \leq \mathbb{Z}$. Hence $f^{-1}(T) \subseteq X + f^{-1}(K)$ $(i.e \quad 4\mathbb{Z} \subseteq 3\mathbb{Z} + 2\mathbb{Z} = \mathbb{Z})$, but $4\mathbb{Z} \nsubseteq 3\mathbb{Z}$ $(i.e \quad f^{-1}(T) \nsubseteq X)$. Therefore $f^{-1}(K)$ is not $f^{-1}(T)$-small in $\mathbb{Z}$.

Definition 2.15. Let M be an R-module and N ⊆ M. If N' ≤ M is minimal with respect to N + N' = M, then N' is called a supplement of N in M.

Proposition 2.16. [3] Let N and T be submodules of an R-module M and N' be a supplement of N in M. If N $\ll_T M$, then T⊆ N'. Moreover, if N $\ll_T M$ and N + T = M, then N' = T .

Proof: Since N' is supplement of N in M and N' is minimal with respect to N + N' = M and T ⊆ M then T ⊆ N + N' but N $\ll_T$ M then T ⊆ N' and moreover, since N + T = M and since N' is supplement of N in M then N' is minimal with respect to N + N' = M but N + T = M implies N' ⊆ T but T ⊆ N' then T = N .

Theorem 2.17. [3] Let K be a submodule of an R-module M and K' is a supplement of K in M. The following are equivalent:

(i) K $\ll_{K'}$ M;

(ii) For each submodule N of M, the relation K + N = M implies K' ⊆ N.

Proof: $(i) \Longrightarrow (ii)$ suppose that K + N = M and since K' ⊆ K + N by definition K' ⊆ N

$(ii) \Longrightarrow (i)$ Suppose that K' ⊆ K + X some $X \subseteq M$. Since M = K + K' ⊆ K + X hence M ⊆ K + X and since X ⊆ M and K ≤ M then K + $X \subseteq M$ we have M = K + X and by hypothesis K' ⊆ X .

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