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R-Annihilator Small Submodules

A research

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By

Raghda Jallel Al-assadi

Supervisor

Dr. Tha'ar Younis Ghawi

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CERTIFICATION

I certify that this paper was prepared under my supervision at the university of AL-Qadisiyah, college of Education, Dep. of Mathematics, as a partial fulfillment for the degree of B.C. of science in Mathematics.

Signature:

Supervisor: **Dr. Tha'ar Younis Ghawi**

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In view of the available recommendations, I forward this paper for debate by the examining committee.

Signature:

Chairman of Dep. **Dr. Mazin Umran Kareem**

Date: /   / 2018

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

أمن هو قانتٌ آناءَ الليلِ ساجداً وقائماً يحذرُ الاخرةَ ويرجو رحمةَ

ربهِ قُل هل يستوي الذين يَعلمون وَالذين لا يعلمون

انما يتذكرُ اولُو الألبابِ ۞

صدقَ اللهُ العليُ العظم

سورة الزمر (٩)

DEDICATION

I dedicate this humble to cry resounding silence in the sky to the martyrs of Iraq wounded. Also, I dedicate my father treasured, also I dedicate to my supervisor **Dr. Tha'ar Younis Ghawi**. Finally, to everyone who seek knowledge, I dedicate this humble work.

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And later the first to thank in this regard is the Almighty God for the blessing that countless, including the writing of this research modest. Then thanks and gratitude and deepest gratitude to my teacher **Dr. Tha'ar Younis Ghawi**, which prefer to oversee the research and guidance in spite of concern, and not to ask him reordering Almighty to help him I also wish to express my thanks to the staff of the department of mathematics.

CONTENTS

INTRODUCTION

In this work, all rings have identity elements and all modules are right unitary. In [1], Nicholson and Zhou defined annihilator -small right (left) ideals as follows :a left ideal A of a ring R is called annihilator -small if A+T=R, where T is a left ideal , implies that r(T)=0 , where r(T) indicates the right annihilator. Kalati and Keskin consider this problem for modules in [2]as follows :let M be an R-module and S=End(M) . A submodule K of M is called annihilator –small if K+T=M , T a submodule of M, implies that $r_s(T) = 0$, where r_s indicates the right annihilator of T over S=End(M) , where $r_s(T) = \{f \in S | f(T) = 0 \quad \forall t \in T\}$.

These observation lead us to introduce the following concept. A sub-module N of an R-module M is called R-annihilator small if N+T=M , T a submodule of M , implies that $ann_R(T) = 0$, where $ann_R(T) = \{r \in R | r.T = 0\}$. In fact , the set K_M of all elements K such that RK is semisubmodule and annihilator-small and contains both the Jacobson radical and the singular submodule when M is finitely generated and faithful . The submodule A_M generated by K_M is a submodule of M analogue of the Jacobson radical that contains every R-annihilator-small submodules . in this work we give some basic properties of R-annihilator small submodules and various.

CHAPTER ONE

Background of Modules

Definition 1.1 [2] A submodule N of a module M is called small in M (denoted by $N \ll M$) if $\forall K \leq M$ with N+K=M implies that K=M .

Example 1.2 For every module M, we have $0 \ll M$.

Theorem 1.3 [3] $A \ll M \Leftrightarrow \forall \cup \leq_{\neq} M(A + \cup \leq_{\neq} M)$.

Proof. $\Longrightarrow$ Let $A \ll M$ [we will proof by using contradiction] and since $A \leq M$ then U=M. Suppose $\exists U \leq M \ni A + U = M$,and since $A \leq M$ then U=M and this is contradiction (U=M). So $\forall U \leq M(A + U \leq M)$.

$\Longleftarrow$ Suppose $A\ M$, then $\exists U \leq_{\neq} M \ni A + U = M$. And this is a contradiction. Then U=M ,so $A \leq M$.

Theorem 1.4 [1] $M \neq 0\, , A \ll M \Longrightarrow A \neq M$.

Proof. Let $M \neq 0\ \wedge A \ll M$ [We will proof it by using contradiction]

Suppose A=M , then A+0=M , but $A \ll M$ so M=0 and that is contradiction →so$A \neq M$.

Definition 1.5 [4] A module M is said to be semi simple if $\forall N \leq M \exists K \leq M \ni N \oplus K = M$.

Theorem 1.6 If M is a semi simple module then 0 is the only small submodule in M .

Proof. Let $N \ll M\ so\ N \oplus M$ so (since M is semi simple), $\exists K \leq M$

with $N \oplus K = M$, i.e. $N \cap K = 0$ and N+K=M .

$\Longrightarrow$K=M but $N \cap K = 0$ so $N \cap M = 0 \Longrightarrow N = 0.$

Definition 1.7 Let M be an R module A subset X of M is called basis of M iff :

1) X is generated M , i.e. $M = \langle X \rangle$.

2) X is linearly independent , that is for every finite subset$\langle x_1, x_2, \dots, x_n \rangle$ of X with $\sum_{i=1}^{n} X_i \propto_i = 0 , \forall \propto_i \in R \; then \; \propto_i = 0 , \forall 1 \leq i \leq n.$

Definition 1.8 An R-module M is said to be free if satisfy the following condition :

1) M has basis.

2) $M = \bigoplus_{\forall i \in I} A_i \wedge \forall i \in I \; [A_i \equiv R_R].$

Example 1.9 Z as Z-module is a free module.

Example 1.10 Z as Z-module is free since ⟨1⟩=Z

$\langle 1 \rangle = \{1.a | a \in Z\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

And $\forall \propto \in Z, \propto .1 = 0 \Longrightarrow \propto = 0.$

Theorem 2.1.11 In a free Z-module (0) is only submodule.

Proof. Let $F = \bigoplus_{i \in I} x_i$ Z be a free Z-module with basis $\{X_i | i \in I\}$

$A \leq F , a \in A$ and let $\propto = x_{i1} z_1 + \cdots + x_{im} z_m , z_i \in Z , \; with \quad z_1 \neq 0$ let $n \in Z$ with g.c.d$(z_1, n) = 1$ and $n < 1$

Put $U = \bigoplus x_i Z + x_{in} Z$, then aZ+U=F , hence A+U=F with $U \neq F.$

Zoren's lemma 2.1.12 If A is non-empty partial order set such that every chain in A has an upper bound in A , then A has maximal element .

Proposition 1.13 [5] If finitely many arbitrary elements are omitted from an arbitrary generating set X of Q_z, then the set with out these elements omitted is again generating .

Theorem 1.14 [2] Every finitely generating submodule of Q_z is small in Q_z.

Proof. Let $N \leq Q_z$ be a finitely generating submodule , so $\exists\{q_1, q_2, \dots, q_n\} \subseteq Q$ such that $N = \langle q_1, q_2, \dots, q_n\rangle$

Let $K \leq Q_z$ with $Q_z = \langle\langle q_1, q_2, \dots, q_n\rangle \cup K\rangle$, so by the proposition ⟹Q=Z⟹N is small.

Modular law 1.15 [3] If $A, B, C \leq M \wedge B \leq C$, $then$ $(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B$.

Lemma 1.16 If $A \leq B \leq M \leq N$ and $B \ll M \Longrightarrow A \ll N$.

Proof. Let $U \leq N$, let A+U=N [we must proof that U=N]

Since $A \leq B$ then $B + U = N \Longrightarrow (B + U) \cap M = N \cap M \Longrightarrow B + (U \cap M) = M$ (by modular law)

Hence $U \cap M = M(since\ B \ll M)$, and so $M \leq U$ and since assub $B \ll M$, so $A \leq U$ and since A+U=N then $U = N \longrightarrow A \ll N$.

Theorem 1.17

$$A_i \ll M, \qquad i = 1,2, \dots, n \Longrightarrow \sum_{i=1}^{n} A_i \ll M$$

Proof. Let $A_i \ll M$, i=1,2,…,n

If i=1 , $A_1 + U = M \Longrightarrow U = M$ (by hypothesis)⟶ $[A_i \ll M]$

If i=2 ,$A_1 + A_2 + U = M \longrightarrow A_1 + (A_2 + U) = M$

Since $A_1 \ll M$ $then\ A_2 + U = M$

Since $A_2 \ll M \Longrightarrow U = M$ $, so\ \sum_{i=1}^{2} A_i \ll M$

Let it be true at n-1, and we will proof it at n

Let $A = A_1 + A_2 + \cdots + A_{n-1} \leq M \leq\longrightarrow A + A_n + U = M$

Then $A_n + U = M$ $[since\ A \ll M]$, then U=M [since $A_n \ll M]$ $so\ \sum_{i=1}^{n} A_i \ll M.$

Definition 1.18 A homomorphism $\propto: A \longrightarrow B$ is called small $\Leftrightarrow kar\ \propto \ll A.$

If $\propto: M \longrightarrow N$ modular homomorphism on R-ring then if $B \leq N\ then\ \propto \left(\propto^{-1} (B)\right) = B \cap Im(\propto).$

Theorem 1.19 [1] If $A \ll M\ and\ \emptyset \in Hom(M,N) \Longrightarrow \emptyset(A) \ll N.$

Proof. Let $\emptyset(A) + U = N\ \ and\ \ U \leq N\ , so\ \emptyset(m) \in N\ \forall m \in$

$\emptyset(m) = \emptyset(a) + u \qquad with\ \ a \in A\ , u \in U \rightarrow \emptyset(m) - \emptyset(a) = u$

$\rightarrow \emptyset(m-a) = u \rightarrow \emptyset^{-1}\big(\emptyset(m-a)\big) = \emptyset^{-1}(u)$

$\rightarrow m - a \in \emptyset^{-1}(U) \rightarrow m \in A + \emptyset^{-1}(U)$

$\rightarrow A + \emptyset^{-1}(U) = M\ \ but\ \ A \ll M\ , hence\ M = \emptyset^{-1}(U)$

$\rightarrow \emptyset(M) = \emptyset\big(\emptyset^{-1}(U)\big) = U \cap Im(\emptyset)$ [by theorem 2.1.19]

$\rightarrow \emptyset(A) \leq \emptyset(M) \leq U,\ \ hence\ U = \emptyset(A) = N.$

Theorem 1.20 If $\propto: M \rightarrow N$, $\beta N \rightarrow K$ modular homomorphism on R-ring then $\ker(\beta \propto) = \propto^{-1} (\ker(\beta)).$

Proof. Let $x \in \ker(\beta \propto) \rightarrow \beta \propto (x) = 0' \rightarrow \big(\propto (x)\big) = 0' \rightarrow \propto (x) \in \ker(\beta) \rightarrow x \in \propto^{-1} (\ker(\beta))$. So $\ker(\beta \propto) \subseteq \propto^{-1} (\ker(\beta))$ … (1)

Let $x \in \propto^{-1} (\ker(\beta)) \to \propto (x) \in \ker(\beta) \to \big(\propto (x)\big) = 0' \to \beta \propto (x) = 0' \to x \in \ker(\beta \propto)$. So $\propto^{-1} (\ker(\beta)) \subseteq \ker(\beta \propto)$ … (2)

Form (1),(2)→ $\ker(\beta \propto) = \propto^{-1} (\ker(\beta))$.

Theorem 1.21 [2] If $\propto: M \to N$, $\beta: N \to K$ modular homomorphism on R-ring then if $A \leq M$ $then$ $\propto^{-1} \big(\propto (A)\big) = A + \ker(\propto)$.

Proof. Let $x \in \propto^{-1} (\propto (A)) \to \propto (x) \in \propto (A)$.

Then $\exists b \in A \ \ni \propto (x) = \propto (b)$

$\to \propto (x - b) = 0' \to x - b \in \ker(\propto)$, $then\ \exists k \in \ker(\propto) \ni x - b = k$

$\to x = b + k \to x \in A + \ker(\propto)$ $[since\ k \in \ker(\propto), b \in A]$

So $\propto^{-1} \big(\propto (A)\big) \subseteq A + \ker(\propto)$ ... (1)

Let $x \in A + \ker(\propto)$, $then\ \exists b \in B$, $k \in \ker(\propto) \ni x = b + k$

$\to \propto (x) = \propto (b + k) \to \propto (x) = \propto (b) + \propto (k)$

$\to \propto (x) = \propto (b) [since\ k \in \ker(\propto)] \to x \in \propto^{-1} (\propto (A))$

So $A + \ker(\propto) \subseteq \propto^{-1} \big(\propto (A)\big)$ ... (2)

So from (1) ,(2) we get $\propto^{-1} \big(\propto (A)\big) = A + \ker(\propto)$.

Theorem 1.22 [3] If $\propto: A \to B$, $\beta: B \to C$ are small epimorphism then $\beta \propto: A \to C$ also small epimorphism.

Proof. By theorem $\beta \propto$ is also epimorphism

Now we must proof $\ker(\beta \propto) \ll A$

Let $U \leq A$ with $\ker(\beta \propto) + U = A$, then $\propto (\ker(\beta \propto) + U) = \propto (A) \Longrightarrow \propto (\ker(\beta \propto)) + \propto (U) = B \Longrightarrow \propto (\propto^{-1} \ker(\beta)) + \propto (U) = B$ (by theorem2.1.21).

$\Longrightarrow \ker(\beta) + \propto (U) = B$, but $\ker(\beta) \ll B \Longrightarrow \propto (U) = B \Longrightarrow \propto (U) = \propto (A) \Longrightarrow \propto^{-1} \left(\propto (U)\right) = \propto^{-1} \left(\propto (A)\right) \Longrightarrow U + \ker(\propto) = A$ (by theorem 2.1.22). But $\ker(\propto) \leq A \Longrightarrow U = A$.

Definition 1.23 [2] Let $A \leq M$ then $A - B \leq M$ is called addition complement of A in M (briefly adco) iff :

1)A+B=M

2)$B \leq M$ minimal in A+B=M , i.e $\forall B \leq M$ with A+B=M , i.e $\forall U \leq M$ with A+U=M and $U \leq B$ imply U=B

$B - D \leq M$ is called intersection complement of A in M (beieflyinco) iff

1)$A \cap D = 0$

2)D is a maximal in $A \cap D = 0$

i.e. $\forall C \leq M$ with $A \cap C = 0 \ \wedge D \leq C$ implies C=D.

Corollary 1.24 Let $A \leq M \ and \ B \leq M \ \ then \ A \oplus B = M \Leftrightarrow$ B is adco and inco of A in M.

Proof. $\Longrightarrow$ Suppose that B is adco and inco of A

Then A+B=M resp. $A \cap B = 0 \Longrightarrow M = A \oplus B$

$\Longleftarrow$ Suppose that $A \oplus B = M$, hence A+B=M and $A \cap B = 0$

Let $C \leq M$ with A+C=M and $C \leq B, (A + C) \cap B = M \cap B \Longrightarrow (A + C) \cap B = B \rightarrow (A \cap B) = C = B \Longrightarrow C = B[A \cap B = 0]$

So B is adco of A in M

Let $C \leq M$ with $A \cap C = 0 \ and \ B \leq C$

Since A+B=M$\Longrightarrow$A+C=M [since $A + B \subseteq A + C$]

$\rightarrow A \oplus C = M \Longrightarrow A \oplus C = A \oplus B$ [$A \oplus B = M$by assumption]

$\frac{A \oplus C}{A} = \frac{A \oplus B}{A} \Longrightarrow C = B \longrightarrow$ so B is inco of A in M.

Lemma 1.25 [3] Let M=A+B , then we have B is adco of A in M $\Leftrightarrow$ $A \cap B \ll B$.

Proof. $\Longrightarrow let\ U \leq B\ (A \cap B) + U = B$

Then $M = A + (A \cap B) + U \Longrightarrow A + U = M\ [since\ A \cap B \subseteq A]$

But B is so $A \cap B \ll B$

$\Longleftarrow$ We have by assumption M=A+B , let $U \leq M$ with A+U=M and $U \leq B$

$\rightarrow (A + U) \cap B = M \cap B \rightarrow (A + U) \cap B = B\ [B \leq M] \rightarrow (A + B) \cap U = B$ [by modular law]

But $A \cap B \ll B$, hence U=B , thus B is adco to A in M.

CHAPTER TWO

1. R-annihilator-small submodules

Definition 2.1.1 [3] A submodule N of a module M is called R- a-small, if $N + X = M, X$ a submodule of M implies that $ann_R\mathrm{X} = 0$, we write $N \ll^a M$ in this case .

[3] Examples 2.1.2

(*i*) In Z as Z module every proper submodule is Z-a-small. Let nZ be a proper submodule in Z $\ni n \neq \mp 1$, and let mZ be a submodule of Z such that nZ+mZ=Z. We have $ann_Z mZ = \{r \in Z | ra = 0 \quad \forall a \in mZ\}$ where a=m.b ∃b∈Z. So $ann_Z mZ = \{0\}$, and hence nZ is Z-a-small submodule . In particular {0} is a-small submodule in Z as Z-module.

(*ii*) $\{0\}$ is a small in Z_4 as Z-module, but we have $0 + Z_4 = Z_4$ with ann$Z_4 = 4Z \neq 0$ so {0} is not a-small submodule in Z_4.

Proposition 2.1.3 [4] Let A and B be submodule of M such that $A \leq B$, if $A \ll^a B$ then $A \ll^a M$.

Proof. Let M=A+X , where $X \leq M$, by modular law, we have $M \cap B = (A + X) \cap B$,B=A+(X∩B). Since $A \ll^a B$,then ann(X∩B)=0 but $X \cap B \subseteq X$, $annX \subseteq annX \cap B = 0$ then $annX = 0$, thus $A \ll^a M$.

Proposition 2.1.4 [2] Let A and B be submodules of M such that $A \leq B$,if $B \ll^a M$ then $A \ll^a M$.

Proof. Let M=A+X , where $X \leq M$. Then M=B+X. Since $A \subseteq B$, then $A + X \subseteq B + X$ so $M \subseteq B + X$ but $B + X \subseteq M$, thus M=B+X . Since $B \ll^a M$, then annX=0 .

Corollary 2.1.5 Let $\{A_i\}_{i\in I}$, $I = \{1,2,3,\dots,n\}$ be a family of submodules of a module M. If $A_t \ll^a M$ then $\bigcap_{i=1}^{n} A_i \ll^a M$ for some $t \in I$.

Proof. Since $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \dots \cap A_t \cap \dots \cap A_n \leq A_t$, and $A_t \ll^a M$

So by Proposition 2.1.4, we get $\bigcap_{i=1}^{n} A_i \ll^a M$.

Proposition 2.1.6 [3] Let M and N be two R-modules and f:M→N be an epimorphism if $H \ll^a N$ then $f^{-1}(H) \ll^a M$.

Proof. Let $M = f^{-1}(H) + X$, since f is an epimorphism, N=H+f(X). But $H \ll^a N$ therefore ann f(X)=0. To prove ann X ⊆ ann f(X). Let r∈annX, then rX=0, for all x∈X, so f(rX)=f(0)=0 , So f(rX)=0 , but f(rX)=rf(X), then rf(X)=0 for all x∈X, then r∈annf(X). Hence annX=0 .

Notes 2.1.7

(*i*) Let f:M⟶N be an epimorphism, the image of R-a-small submodule of M need not be R-a-small in N as the following example shows : Consider the natural epimorphism $\pi: Z \to Z_4$. Since {0}is a-small in Z as Z-module but $\pi(0) = \bar{0}$ is not a-small in Z_4 as Z-module, since $\bar{0} + Z_{4=}Z_4$ but $annZ_4 = 4Z \neq 0$.

(*ii*) The sum of two R-a-small submodules of a module M need not be R-a-small submodule for example . In Z as Z-module , 2Z and 3Z are a-small submodules but 2Z+3Z=Z is not a-small in itself .

Theorem 2.1.8 [3] Let M be a faithful module if $N \ll M$, then $N \ll^a M$.

Proof. Let $X \leq M$ such that N+X=M. Since $N \ll M$ implies that X=M, hence annX=annM. But M is faithful, annM=0 , thus annX=0, therefore $N \ll^a M$.

Proposition 2.1.9 Let M be a module and $A \leq B$ be submodules of M, then $ann_R B \subseteq ann_R A$.

Proof . Let $r \in ann_R B$, then r.a=0 , for all a∈ B , but we have $A \leq B$, hence $r.a = 0$ for all $a \in A$, therefore $r \in ann_R A$.

The following example show the application of previous proposition.

In Z_{12} as Z-module , $\langle 4\rangle = \{\bar{0},\bar{4},\bar{8}\}$, $\langle 2\rangle = \{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$

$ann_{Z_{12}}\langle 4\rangle = \{\bar{0},\bar{3},\bar{6},\dots\} = 3Z$

$ann_{Z_{12}}\langle 2\rangle = \{\bar{0},\bar{6},\dots\} = 6Z$ but $6Z \subseteq 3Z$

This means $ann_{Z_{12}}\langle 2\rangle \subseteq ann_{Z_{12}}\langle 4\rangle$.

Proposition 2.1.10 [3] Let M_1 , M_2 be modules, if $K_1 \ll^a M_1$ and $K_2 \ll^a M_2$, then $K_1 \oplus K_2 \ll^a M_1 \oplus M_2$.

Proof. Let $P_i: M_1 \oplus M_2 \longrightarrow M_i$, $i = 1,2$ be the projection maps. Since $K_1 \ll^a M_1$, $K_2 \ll^a M_2$, (by proposition 2.1.6) $K_1 \oplus M_2 = {P_1}^{-1}(K_1) \ll^a M_1 \oplus M_2$ and $M_1 \oplus K_2 = {P_2}^{-1}(K_2) \ll^a M_1 \oplus M_2$ (by proposition2. 1.7) $(K_1 \oplus M_2) \cap (M_1 \oplus K_2) = K_1 \oplus K_2 \ll^a M_1 \oplus M_2$ (by corollary 2.1.5) .

Let M be a module over an integral domain R. Define the set T(M) = $\{m \in M | rm = 0$ for some $(r \neq 0) \in R\}$. If T(M)=M, then M is called torsion, if T(M)=0 then M is called torsion free .

Remark 2.1.11[5] Let R be an integral domain and let M be a torsion free module then every proper submodule of M is R-a-small in M .

Proposition 2.1.12 Let N and K be two submodules of a module M then $ann(N + K) = annN \cap annK$.

Proof. Since $N \subseteq N + K,\ K \subseteq N + K, ann(N + K) \subseteq annN,$ and $ann(N + K) \subseteq annK$. Let $r \in ann\,N$ and $r \in ann\,K$ $ra = 0$ for all $a \in N$ and $r.b = 0$. For all b$\in K$ r(a+b)=0 for all a$\in N$ and b$\in K$, then r∈ann(N+K), so ann N∩annK $\subseteq ann(N + K)$ ...(2) from 1 and 2 we get ann(N+K)=annN∩ $annK$.

Proposition 2.1.13 [1] Let M be a faithful R-module, N be a submodule of M such that $annN \leq^e R$, then $N \ll^a$M .

Proof. Let M=N+K ,then 0=ann M=ann (N+K) ann (N+K)=ann N ∩ ann K, by proposition 2.1.12, then 0=ann N ∩ ann K, but ann N$\leq^{e}$R therefor annK=0 thus N$\ll^{a}$M.

Proposition 2.1.14 [1] Let R be an integral domain, let M be a faithful R-module, then every submodule N of M with annN≠0 is R-a-small.

Proof. Assume that M=N+K, then 0=annM=ann(N+K)=annN∩annK, since annN≠0 and R is an integral domain, then annN$\leq^{e}$R. Therefor annK=0. Thus N is R-a-small .

Proposition 2.1.15 Let R be an integral domain and M be a faithful and torsion module, every finitely generated submodule N of M is R-a-small.

Proof. Let N=R$x_1 + Rx_2 + \cdots + Rx_n$ be a submodule of M and M=N+K. Then 0=annM=ann(N+K)=annN ∩ annK = (ann(R$x_1 + Rx_2 + \cdots + Rx_n$)) $\cap\, annK = (\bigcap_{i=1}^{n} ann\, Rx_i) \cap annK$. Since M is torsion, then $annRx_i \neq 0\ forall\ \ i = 1,2, \dots, n$. But R is an integral domain, there for $annRx_i$ is essential in R, for all i, hence $\bigcup_{i=1}^{n} ann\, Rx_i$= annK=0. Thus $N \ll^{a}$M .

2. Characterizations of R-a-small submodules

Proposition 2.2.1[1] Let M be a finitely generated module and $K \ll^a M$, then $K + Rad\ (M) + Z(M) \ll^a M$.

Proof. Let $M = Rm_1 + Rm_2 + \cdots + Rm_n$, $m_i \in M$,$\forall i = 1,2, \dots, n$

and M=K+Rad(M)+Z(M)+X . Since M is finitely generated. Then $Rad(M) \ll M$, and hence M=K+Z(M)+X . So $m_i = k_i + z_i + x_i$, $k_i \in K$, $x_i \in X$, $z_i \in Z(M)$, $\forall i = 1,2, \dots, n$

$M = K + Rz_1 + Rz_2 + \cdots + Rz_n + X$,$but\ K \ll^a M$, therefore

$ann(Rz_1 + Rz_2 + \cdots + Rz_n + X) = 0$. Hence $(\bigcap_{i=1}^{n} ann(Rz_i)) \cap annX = 0$ $since\ z_i \in Z(M)$,$\forall i = 1,2, \dots, n$. Then $ann(z_i) \leq^e R$, $\forall i = 1,2, \dots, n$ $and\ hence$ $\bigcap_{i=1}^{n} ann(z_i) \leq^e R$. So annX=0

Thus $K + Rad(M) + Z(M) \ll^a M$.

Proposition 2.2.2 [3] Let M be a module and $K \ll^a M$. If $Rad\ (M) \ll M$ and Z(M) is finitely generated, then $K + Rad\ (M) + Z(M) \ll^a M$.

Theorem 2.2.3 [2] Let $M = \sum_{\propto \in \wedge} RX_\propto$ be a module and $K \in M$, then the following statements are equivalent :

(*i*) $Rk \ll^a$M.

(*ii*) $\bigcap_{\propto \in \wedge} ann(x_\propto - r_\propto k) = 0\ \ \forall r_\propto \in R$.

Proof. (*i*)→(*ii*) Let $r_\propto \in R$ for each $\propto \in \wedge$ $then\ x_\propto = x_\propto - r_\propto k + r_\propto k$, $\forall \propto \in \wedge$, then $M = \sum_{\propto \in \wedge} R(x_\propto - r_\propto k) + Rk$ $since\ Rk \ll$ $\ll^a M$ $then$ $0 = ann(\sum_{\propto \in \wedge} R(x_\propto - r_\propto k)) = \bigcap_{\propto \in \wedge} annR(x_\propto - r_\propto k)$

(*ii*)→ (*i*) Let M=RK+B . then for each $\propto \in \wedge$. $x_\propto = r_\propto k + b_\propto$ $r_\propto \in R\ and\ b_\propto \in B$. Now let $tx_\propto = tr_\propto k + tb_\propto$, $since\ \ tb_\propto = 0$ $then\ \ t(x_\propto - r_\propto k) = 0$, $\forall \propto \in \wedge$ $so\ \ t \in ann(x_\propto - r_\propto k) = 0$,$\forall \propto \in \wedge$. Hence $t \in \bigcap_{\propto \in \wedge} ann(x_\propto - r_\propto k) = 0$.

Theorem 2.2.4 [5] Let R be a commutative ring, and $M = \sum_{\propto\in\Lambda} Rx_{\propto}$ be a module and $k \in M$ then the following statements are equivalent" :

(i) $Rk \ll^{a}$M

(ii) $\bigcap_{\propto\in\Lambda} ann(x_{\propto} - r_{\propto}k) = 0 \quad \forall r_{\propto} \in R$

(iii) there exists ∝∈Λ $\; such\ that \quad bx_{\propto} \notin Rbk \quad \forall 0 \neq b \in R.$

Proof. (***i***)→(***ii***) By Theorem 2.2.3.

(***ii***)→(***iii***) "Let $0 \neq b \in R$, assume that $bx_{\propto} \in Rbk, \quad \forall_{\propto} \in \Lambda$ then $bx_{\propto}\ let\ 0 \neq b \in R.$ There for $b \in ann(x_{\propto} - x_{\propto}k), \quad \forall_{\propto} \in \Lambda$ $and\ hence\ \ 0 \neq b \in \bigcap_{\propto\in\Lambda} ann(x_{\propto} - r_{\propto}k) = 0$ which is a contradiction.

(***iii***)→(***ii***) let $b \in \bigcap_{\propto\in\Lambda} ann\ (x_{\propto} - r_{\propto}k)\ \ and\ hence\ b \in ann(x_{\propto} - x_{\propto}k) \quad \forall \propto \in \Lambda\ .\ there\ for\ \ bx_{\propto} = r_{\propto}bk\ , \forall \propto \in \Lambda\ .\ \ so\ bx_{\propto} \in Rbk \quad \forall_{\propto} \in \Lambda$ By our assumption , b=0 .

Theorem 2.2.5 [2] Let R be a commutative ring, and let $M = \sum_{\propto\in\Lambda} Rx_{\propto}$ be a module and $K \leq M$ then the following statements are equivalent:

(***i***) $K \ll^{a} M$

(***ii***)$\bigcap_{\propto\in\Lambda} annR\ (x_{\propto} - k_{\propto}) = 0\ , \ \forall k_{\propto} \in K$

Proof. (***i***)→(***ii***) let $k_{\propto} \in K \ \ \forall \propto \in \Lambda \ \ then\ \ x_{\propto} = x_{\propto} - k_{\propto} + \ k_{\propto}\ , \ \forall \propto \in \Lambda$ $and\ hence \quad M = \sum_{\propto\in\Lambda} R(x_{\propto} - k_{\propto}) + k.$ But $k \ll^{a} M,$ therefor

0=ann($\sum_{\propto\in\Lambda} R(x_{\propto} - k_{\propto})$) =$\bigcap_{\propto\in\Lambda} annR(x_{\propto} - k_{\propto})$

(***ii***)→(***i***) let M=K+A. then for each ∝∈Λ, $\ x_{\propto} = k_{\propto} + a_{\propto}, \quad a_{\propto} \in A, k_{\propto} \in$

Hence $a_{\propto} = x_{\propto} - k_{\propto}$, for each ∝∈Λ , $\quad so\ M = \sum_{\propto\in\Lambda} R(x_{\propto} - k_{\propto}) + k$

Now let $t \in annA\ \ there\ for\ t(x_{\propto} - k_{\propto}) = 0\ , \ \forall \propto \in \Lambda \quad so\ t \in \bigcap_{\propto\in\Lambda} annR(x_{\propto} - k_{\propto}) = 0 \ \ thus\ \ annA = 0\ \ and \ \ k \ll^{a} M\ .$

Definition 2.2.6 [3] Let M be an R-module and $k \in M,$ we say that k is R-a-small in M if $Rk \ll^{a} M.$ Let $k_{m} = \{k \in M | Rk \ll^{a} M\}.$

Example 2.2.7 In Z as Z-module we know that every proper submodule is R-a-small, this implies the set all R-a-small elements are $Z|\{-1,1\}$.

Notes 2.2.8 [5]

(*i*) That $Z(M) \subseteq k_M$ and $rad\ (M) \subseteq k_M, when$ M is finitely generated and faithful .

(*ii*) k_M is not closed under addition in general. For example consider Z as Z- module the sum of R-a-small need not be R-a-small. clearly that $3Z \ll^a$Z and $2Z \ll^a Z$ but Z=3Z+2Z is not R-a-small in Z.

Remark 2.2.9 Let M be a module and $k \in k_M$, then $Rk \subseteq k_m$.

Proof. Let $r \in R$ $clearly\ that\ Rrk \subseteq Rk \ll^a M$ by proposition (2.1.4) $Rrk \ll^a M$ and hence $rk \in K_M$ $thus$ $Rk \subseteq K_M$.

Remark 2.2.10 [4] Let M be a module and $A \ll^a M$ $then$ $A \subseteq K_M$. Let $x \in A,$ $then\ Rx \subseteq K \ll^a M\ and\ hence\ Rx \ll^a M$ by proposition (2.1.4) Thus $x \in K_M$ as we have seen , the sum of R-a-small submodules need not be R-a-small (consider 3Z+2Z in Z).

Definition 2.2.11 [1] Let M be a module and let R-a-small submodule A_M of M be the sum of R-a-small submodule of M . If M has no R-a-small submodule , we write $A_M = M$. It is clear that $K_M \subseteq A_M$ in every module , but this may not be equality (consider Z as Z-module) .

Proposition 2.2.12 [2] Let M be a module such that $K_M \neq \emptyset$ then :
(*i*) A_M is a submodule of M , A_M contains all R-a-small submodule of M .
(*ii*) $A_M = \{k_1 + k_2 + \cdots + k_n | k_i \in K_M$ $for\ each\ i\ , n \geq 1\}$.
(*iii*) A_M is generated by K_M.
(*iv*) If M finitely generated , then $Rad\ (M) \subseteq A_M$ and $Z(M) \subseteq A_M$.

Proposition 2.2.13 [5] Let M be a module such that $K_M \neq \emptyset$ then the following are equivalent:

(***i***) K_M is closed under addition , that is a finite sum of R-a-small elements is R-a-small.
(***ii***) $A_M = K_M$.

Proof. (***i***)→(***ii***) let $k_1 + k_2 + \cdots + k_n \in A_M, k_i \in K_i \quad and\ K_i \ll^a M\ , \quad \forall i = 1,2,\ldots,n$
Then $RK_i \ll^a M \quad by\ proposition\ (2.1.4)\ \ hence\ K_i \in K_M\ \ \forall i = 1,2,\ldots,n$. By our assumption , $k_1 + k_2 + \cdots + k_n \in K_M$, thus $A_M = K_M$.
(***ii***)→(***i***) assume that $A_M = K_M \quad and\ let\ x, y \in K_M \quad since\ K_M \subseteq A_M,\ \ then\ x, y\ \in A_M$. But A_M is a submodule of M by proposition (2.2.11). Therefor $x + y \in A_M = K_M\ thus\ K_M$ is closed under addition.

Proposition 2.2.14 [3] Let M be a module such that $K_M \neq \emptyset$ consider the following statements :
(***i***) $A_M \ll^a M$.
(***ii***) If $K \ll^a M \quad and\ \ L \ll^a M \quad then\ K + L \ll^a M$.
(***iii***) K_M is closed under addition , that is the sum of R-a-small elements is R-a-small.
(***iv***)$A_M = K_M$.

Then (i)→ $(ii) \leftrightarrow (iii) \leftrightarrow (iv)$, if M is finitely generated (i)⟷(ii).

Proof. (***i***)→ $(\boldsymbol{ii})$ Assume that $A_M \ll^a M \quad and$ let K and L be R-a-small submodule of M, then $K + L \subseteq A_M$. But $A_M \ll^a M\ \ therefor\ \ K + L \ll^a M$ by proposition (2.1.4)
(***iii***)→(***iv***) by proposition (2.2.12)
To show that (***ii***)→(***i***) Let $M = R_{m_1} + R_{m_2} + \cdots + R_{m_n} \quad and\ let\ M = A_M + X \quad then\ \ m_i = a_i + x_i\ \ , \quad a_i \in A_M \quad and\ x_i \in X\ ,\ \ \forall i = 1,2,\ldots,n \quad there\ for\ M = \sum_{i=1}^{n} Ra_i + X \quad since\ \ a_i \in A_M,\ \forall i = 1,2,\ldots,n\ , then\ \ a_1 + a_2 + \cdots + a_n \in A_M \quad hence \quad Ra_i \ll^a M\ ,\ \ \forall i = 1,2,\ldots,n\ \ (by\ our\ assumption)\ \sum_{i=1}^{n} Ra_i \ll^a M$. So annX=0 .

Proposition 2.2.15 [1] Let M be a finitely generated module such that $A_M \ll^a M$ then:

(***i***) A_M is the unique largest R-a- small submodule of M
(***ii***) $A_M = \cap \{W | W\ \ maximal\ submodule\ of\ M\ with\ A_M \subseteq W\}$

Proof.(*ii*) Let$a \in \cap \{W|W \ \ maximal\ submodule\ of\ M\ \ with\ A_M \subseteq W\}$
Claim that $Ra \ll^a M$ assume not , then M=Ra+X ,
$X \leq M \ \ and\ \ annX \neq 0$. Since $A_M \ll^a M\ ,\ then\ M \neq A_M + X$. but M is finitely generated then there exist a maximal submodule such that $A_M + X \subseteq B$. Now , if $a \in B$ we get B=M which is a contradiction so $a \notin B$. But $a \in \cap \{W|W\ maximal\ submodule\ of\ M\ with\ A_M \subseteq W\}$ which is a contradiction .Thus $Ra \ll^a M\ \ and\ hence\ \ a \in A_M$.

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