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# Stability Of Numerical Methods For Solve Ordinary Differential Equations 

A reasarch<br>Submitted to the department of mathematics college of education University AL-Qadisiya as a partial fulfillment of the requirement for the degree of bachelor of science in mathematics

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## Background

## 1. Definition : [ 1 ]

Many ordinary differential equations encountered do not have easily obtain able closed from solutions, and we must seek other methods by which solutions can be constructed . Numerical methods provide an alternative way of constructing solutions to these sometimes difficult problems . In this chapter we present an introduction to some numerical methods which can be applied to a wide variety of ordinary differential equations. These methods can be programmed into a digital computer or even programmed into some hand - held calculators. Many of the numerical techniques introduced in this chapter are readily available in the from of subroutine packages available from the internet .

An equation that consists of derivatives is called a differential equation.

Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations.

So , it is important for engineers and scientists to know how to set up differential equations and solve them .

Differential equations are of two types ordinary differential equation " ODE " and partial differential equations "PDE" .

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable . Examples of ordinary differential equation include . [ 2 ]

1. $\frac{d^{2} y}{d x^{2}}+2 \frac{d y}{d x}+y=0 \quad, \quad \frac{d y}{d x}(0)=2 \quad, \quad y(0)=4$.
$\begin{aligned} 2-\frac{d^{3} y}{d x^{3}}+3 \frac{d^{2} y}{d x^{2}}+5 \frac{d y}{d x}+y=\sin x \quad & \frac{d^{2} y}{d x^{2}}(0)=12\end{aligned} \quad \begin{array}{ll}\frac{d y}{d x}(0)=2, & y(0)=4\end{array}$
Note : In this first part, we will see how to solve ODE of the form .

$$
\frac{d y}{d x}=f(x, y) \quad, \quad y(0)=y_{0}
$$

In another section, we will discuss how to solve higher ordinary differential equations or coupled "simultaneous" differential equations .

But first, How to write a first order differentia equation in the form?

## Example " 1 "

$$
\frac{d y}{d x}+2 y=1.3 e^{-x} \quad, \quad y(0)=5
$$

Is rewritten as :

$$
\frac{d y}{d x}=1.3 e^{-x}-2 y \quad, \quad y(0)=5
$$

Is this case : $\mathrm{f}(\mathrm{x}, \mathrm{y})=1.3 \mathrm{e}^{\mathrm{x}}-2 \mathrm{y}$
Example (2): $e^{y} \frac{d y}{d x}+x^{2} y=2 \sin (3 x) \quad, \quad y(0)=5$
We consider the problem of developing numerical methods to solve a first order initial value problem of the form [3].

$$
\begin{equation*}
\frac{\mathrm{dy}}{\mathrm{dx}}=\mathrm{f}(\mathrm{x}, \mathrm{y}) \quad, \quad \mathrm{y}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0} \tag{8.1}
\end{equation*}
$$

and then consider how to generalize these methods to solve systems of ordinary differential equations having the form .

$$
\begin{array}{ll}
\frac{\mathrm{dy}_{1}}{\mathrm{dx}}=\mathrm{f}_{1}\left(\mathrm{x}, \mathrm{y}_{1}, \mathrm{y}_{2}, \ldots \ldots, \mathrm{y}_{\mathrm{m}}\right) \quad, & \mathrm{y}_{1}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{10} \\
\frac{\mathrm{dy}_{2}}{\mathrm{dx}}=\mathrm{f}_{2}\left(\mathrm{x}, \mathrm{y}_{1}, \mathrm{y}_{2}, \ldots \ldots, \mathrm{y}_{\mathrm{m}}\right) \quad, & \mathrm{y}_{2}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{20}
\end{array}
$$

:

$$
\frac{\mathrm{dy}_{\mathrm{m}}}{\mathrm{dx}}=\mathrm{f}_{\mathrm{m}}\left(\mathrm{x}, \mathrm{y}_{1}, \mathrm{y}_{2}, \ldots \ldots ., \mathrm{y}_{\mathrm{m}}\right) \quad, \quad \mathrm{y}_{\mathrm{m}}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{\mathrm{m} 0}
$$

Coupled systems of ordinary differential equations are sometimes written in the vector from
$\frac{d \vec{y}}{d x}=\vec{f}(x, \vec{y}) \quad \vec{y}\left(x_{0}\right)=\vec{y}_{0}$
Where $\vec{y}, \vec{y}\left(x_{0}\right)$ and $\vec{f}(\mathrm{x}, \overrightarrow{\mathrm{y}})$ are column vectors given by $\overrightarrow{\mathrm{y}}=\operatorname{col}\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \ldots \ldots, \mathrm{y}_{\mathrm{m}}\right) \quad, \overrightarrow{\mathrm{y}}\left(\mathrm{x}_{0}\right)=\operatorname{col}\left(\mathrm{y}_{10}, \mathrm{y}_{20}, \ldots \ldots \mathrm{y}_{\mathrm{m} 0}\right)$
and $\vec{f}(x, \vec{y})=\operatorname{col}\left(f_{1}, f_{2}, \ldots . . . f_{m}\right)$
We start with developing numerical methods for obtaining solutions to the first order initial value problem (8.1) over an interval $\mathrm{x}_{0} \leq \mathrm{x} \leq \mathrm{x}_{\mathrm{n}}$ many of the techniques developed for this first order equation can with modifications, also be applied to solve a first order system of differential equations.

## 1-2 Higher order equations

By defining new variables , higher order differential equations can be reduced to a first order system of differential equations. As an example , consider the problem of converting order liner homogeneous differential equation .

$$
\begin{equation*}
\frac{\mathrm{d}^{\mathrm{n}} \mathrm{y}}{\mathrm{dx}^{\mathrm{n}}}+a_{1} \frac{d^{n-1} \mathrm{y}}{\mathrm{dx}^{\mathrm{n}-1}}+a_{2} \frac{\mathrm{~d}^{\mathrm{n}-\mathrm{y}} \mathrm{y}}{\mathrm{dx}^{\mathrm{n}-2}}+\ldots \ldots \ldots .+a_{n-1} \frac{d y}{d x}+a n y=0 \tag{8.4}
\end{equation*}
$$

To a vector representation. To convert this equation to vector from we define new variables. Define the vector quantities .
$\overrightarrow{\mathrm{y}}=\operatorname{col}\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{y}_{3}, \ldots \ldots \ldots, \mathrm{y}_{\mathrm{m}}\right)=\operatorname{col}\left(\mathrm{y}, \frac{\mathrm{dy}}{\mathrm{dx}}, \frac{\mathrm{d}^{2} y}{\mathrm{dx}}, \ldots \ldots, \frac{\mathrm{d}^{\mathrm{n}-1} \mathrm{y}}{\mathrm{dx}^{\mathrm{n}-1}}\right)$
$\vec{f}(x, y)=A \vec{y}$,
Where $\mathrm{A}=\left[\begin{array}{ccccccc}0 & 1 & 0 & 0 & \ldots . . . & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots \ldots . & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots \ldots . & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots \ldots . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots . . & 0 & 1 \\ -a n & -a n-1 & -a n-2 & -a n-3 & \ldots \ldots . . & -a_{2} & -a_{1}\end{array}\right]$
Observe that the linear the order differential equation (8.4) can now represented in the from of equation (8.3) . In this way higher order linear ordinary differential equations can be represented as a first order vector system of differential equations.

## 2- Numerical Solution: [ 4 ]

In our study of the scalar initial value problem (8.1) it is assumed that $\mathrm{f}(\mathrm{x}, \mathrm{y})$ and its partial derivative $\mathrm{f}_{\mathrm{y}}$ both exist and are continuous in a rectangular region about a point ( $\mathrm{x}_{0}, \mathrm{y}_{0}$ ). If these conditions are satisfied, then theoretically there exists unique solution of the initial value problem (8.1) .

Which is a continuous curve $\mathrm{y}=\mathrm{y}(\mathrm{x})$, which passes through the point ( $\mathrm{x}_{0}, \mathrm{y}_{0}$ ) and satisfies the differential equation.

In contrast to the solution being represented by continuous function $\mathrm{y}=\mathrm{y}(\mathrm{x})$, the numerical solution to the initial value problem (8.1) is represented by a set of data points ( $\mathrm{x}_{\mathrm{i}}, \mathrm{y}_{\mathrm{i}}$ ) for i $=0,1,2, \ldots \ldots, \mathrm{n}$ where $\mathrm{y}_{\mathrm{i}}$ is an approximation to the true solution $\mathrm{y}\left(\mathrm{x}_{\mathrm{i}}\right)$. we shall investigate various methods for constructing the data points $\left(\mathrm{x}_{\mathrm{i}}, \mathrm{y}_{\mathrm{i}}\right)$, for $\mathrm{i}=1,2, \ldots, \mathrm{n}$ which approximate the true solution to the given initial value problem. The given rule or technique used to obtain the numerical solution is called a numerical method or algorithm . There are many numerical methods for solving ordinary differential equations. In this chapter we will consider only a select few of the more popular methods. The numerical methods considered can be classified as either single - step methods or multi-step methods . We begin our introduction to numerical methods for ordinary differential equations by considering single step methods .

## 3- Initial - Value Problems For Ordinary Differential

 Equations[2]Many problems in engineering and science can be formulated in terms of differential equations .

A differential equation is an equation involving a relation between an un known function and one or more of its derivatives . Equations involving derivatives of only one independent variable are called ordinary differential equations and may be classified as either initial - value problems " IVP " or boundary value problems "BVP" . Examples of the two types are:
IVP : y" =-yx

$$
\begin{equation*}
Y(0)=2 \quad, \quad y^{\prime}(0)=1 \tag{1.1a}
\end{equation*}
$$

BVP: $y^{\prime \prime}=-y x$

$$
\begin{equation*}
Y(0)=2, \quad y(1)=1 \tag{1.2a}
\end{equation*}
$$

Where the prime denotes differentiation with respect to x . The distinction between the two classifications lies in the location where the extra conditions [Eqs . (1.1b) and (1.2b) ] are specified. For an IVP, the conditions are given at the same value of, where as in the case of the BVP, They are prescribed at two different values of x .

Since there are relatively few differential equations arising from practical problems for which anal arising from practical problems for which analytical solutions are known , one must resort to numerical methods. In this situation it turns out that the numerical methods for each type of problem, IVP or BVP, are quite different and require separate treatment. In this chapter we discuss. IVPs, leaving BVPS to chapter 2 and 3.

Consider the problem of solving the mth-order differential equation.
$\mathrm{y}^{(m)}=f\left(x, y^{\prime}, y^{\prime \prime}, \ldots, y^{(m-1)}\right) \quad$ with initial conditions

$$
\begin{aligned}
y\left(x_{0}\right) & =y_{0} \\
y^{\prime}\left(x_{0}\right) & =y_{0}^{\prime}
\end{aligned}
$$

$$
y^{(m-1)}\left(x_{0}\right)=y_{0}^{(m-1)}
$$

Where f is known function and $\mathrm{y}_{0}, \mathrm{y}_{0}^{\prime}, \ldots \ldots . . ., \mathrm{y}_{0}^{(\mathrm{m}-1)}$ are constants . It is customary to rewrite (1.3) as an equivalent system of m first - order equations . To do so, we define a new set of dependent variables $\mathrm{y}(\mathrm{x}), \mathrm{y}_{2}(\mathrm{x}), \ldots \ldots, \mathrm{y}_{\mathrm{m}}(\mathrm{x})$ by :

$$
\begin{gathered}
\mathrm{y}_{1}=y \\
\mathrm{y}_{2}=\mathrm{y}^{\prime} \\
\mathrm{y}_{3}=\mathrm{y}^{\prime \prime} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{y}_{\mathrm{m}}=\mathrm{y}^{(\mathrm{m}-1)}
\end{gathered}
$$

And transform (1.3) into :
$y_{1}^{\prime}=y_{2} \quad=f_{1}\left(x, y_{1}, y_{2}, \ldots \ldots \ldots ., y_{m}\right)$
$y_{2}^{\prime}=y_{3} \quad=f_{2}\left(x, y_{1}, y_{2}, \ldots \ldots \ldots, y_{m}\right)$
$y_{m}^{\prime}=f\left(x, y_{1}, y_{2}, \ldots \ldots \ldots, y_{m}\right)=f_{m}\left(x, y_{1}, y_{2}, \ldots \ldots ., y_{m}\right)$

$$
\begin{aligned}
& \mathrm{y}_{1}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0} \\
& \mathrm{y}_{2}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0}^{\prime}
\end{aligned}
$$

$$
\mathrm{y}_{\mathrm{m}}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0}^{\mathrm{m}-1}
$$

In vector notation (1.5) because :

$$
\begin{align*}
& \mathrm{y}^{\prime}(\mathrm{x})=\mathrm{f}(\mathrm{x}, \mathrm{y}) \\
& \mathrm{y}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0} \tag{1.6}
\end{align*}
$$

Where

$$
\mathrm{y}(\mathrm{x})=\left[\begin{array}{c}
\mathrm{y}_{1}(\mathrm{x}) \\
\mathrm{y}_{2}(\mathrm{x}) \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{y}_{\mathrm{m}}(\mathrm{x})
\end{array}\right] \quad, \quad \mathrm{f}(\mathrm{x}, \mathrm{y})=\left[\begin{array}{c}
\mathrm{f}_{1}(\mathrm{x}, \mathrm{y}) \\
\mathrm{f}_{2}(\mathrm{x}, \mathrm{y}) \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{f}_{\mathrm{m}}(\mathrm{x}, \mathrm{y})
\end{array}\right] \quad, \quad \mathrm{y}_{0}=\left[\begin{array}{l}
\mathrm{y}_{0} \\
\mathrm{y}_{0}^{\prime} \\
\cdot \\
\cdot \\
\cdot \\
\mathrm{y}_{0}^{(\mathrm{m}-1)}
\end{array}\right]
$$

It is easy to see that (1.6) can represent either an mthorder different equation . A system of equations of mixed order but with total order of m , or a system of, first order equations.

In general, subroutines for solving IVPS as same that the problem is in the form (1.6). In order to simplify the analysis , we begin by examining a single first - order IVP , after which we extend the discussion to include systems of the form (1.6). Consider the initial - value problem .

$$
\begin{gathered}
\mathrm{y}^{\prime}=\mathrm{f}(\mathrm{x}, \mathrm{y}) \quad \mathrm{y}\left(\mathrm{x}_{0}\right)=\mathrm{y}_{0} \\
\mathrm{x}_{0} \leq \mathrm{x} \leq \mathrm{x}_{n}
\end{gathered}
$$

We assume that $\partial \mathrm{f} / \partial \mathrm{y}$ is continuous on the strip $\mathrm{x}_{0} \leq \mathrm{x} \leq \mathrm{x}_{n}$, thus guaranteeing that (1.7) possesses unique solution If $y(x)$ is the exact solution to (1.7) its graph is a curve in the xy -plane passing through the point ( $\mathrm{x}_{0}, \mathrm{y}_{0}$ ) .

A discrete numerical solution of (1.7) is defined to be a set point $\left(\mathrm{x}_{\mathrm{i}}, \mathrm{u}_{\mathrm{i}}\right)_{i=0}^{n}$, Where $\mathrm{u}_{0}=\mathrm{y}_{0}$ and each point $\left(\mathrm{x}_{\mathrm{i}}, \mathrm{y}\left(\mathrm{x}_{\mathrm{i}}\right)\right)$ on the solution curve. Note that the numerical solution is only a set of points, and nothing is said about values between the points . In the remainder of this chapter we describe various methods for obtaining a numerical solution $\left[\left(\mathrm{x}_{\mathrm{i}}, \mathrm{u}_{\mathrm{i}}\right)\right]_{i=0}^{N}[2]$.

## 2- Stability of Numerical Methods

## 2-1 Numerical Solution of Linear Multi step Method [2]

Consider $\mathrm{y}^{\prime}(\mathrm{x})=\mathrm{f}(\mathrm{x}, \mathrm{y}) \quad, \mathrm{y}(\mathrm{a})=\mathrm{y}_{0} \quad, \mathrm{a} \leq \mathrm{x} \leq \mathrm{b} \quad$ (I.V.P)
Def 1: A linear k-step method of solving $y^{\prime}=f(x, y), y(a)=y_{0}$ is a linear diff.eq. of the kk.th order of the form .

$$
\alpha_{\mathrm{k}} \mathrm{y}_{\mathrm{n}+\mathrm{k}}+\alpha_{k-1} \mathrm{y}_{\mathrm{n}+\mathrm{k}-1}+\ldots . .+\alpha_{0} \mathrm{y}_{\mathrm{n}}=h\left[\beta_{k} f_{n+k}+\ldots . .+\beta_{0} f_{n}\right]
$$

Where $y_{n+j}$ is an approximate to $y\left(x_{n+j}\right)$ and $x_{n+j}=a+(n+j) h$, $\mathrm{j}=0, \ldots, \mathrm{k}$ and $x_{n}=a+n h \quad, \mathrm{n}=0,1, \ldots$ and $f_{n+j}=f\left(x_{n+j}, y_{n+j}\right)$, $j=0, \ldots, k$.
$\sum_{j=0}^{k} \alpha_{j} y_{n+j}=h \sum_{j=0}^{k} \beta_{j} f_{n+j}$
k.th order numerical method .

Example Euler's method $y_{n+1}-y_{n}=h f_{n}\left(x_{n}, y_{n}\right)+o\left(h^{2}\right)$ and Trapezeum method $y_{n+1}-y_{n}=\frac{h}{2}\left(f_{n}+f_{n+1}\right)+o\left(h^{3}\right)$
Def 2:The Method M is said to be of order P if $\mathrm{c}_{\mathrm{o}}, \mathrm{c} \square, \ldots, \mathrm{c} \square \square 0$ but $c \square_{+1} \neq 0$ and $c \square_{+1}$ is called the (error constant)

## 2-2 Consistency and Zero Stability [7]

Def 3: The method $M$ is said to be consistency iff its order $p$ satisfies: $p \geq 1$

To find value of :
$c_{0}=\sum_{j=0}^{k} \alpha_{j}$
$c_{1}=\sum_{j=0}^{k} j \alpha_{j}-\sum_{\mathrm{j}=0}^{\mathrm{k}} \beta_{j}$
$c_{p}=\frac{1}{p!}\left(\alpha_{1}+2^{p} \alpha_{2}+\ldots+k^{p} \alpha_{k}\right)-\frac{1}{(p-1)!}\left(\beta_{1}+2^{p-1} \beta_{2}+\ldots+k^{p-1} \beta_{k}\right)$

Theorem1: M is convergent $\Rightarrow \mathrm{M}$ is consistency .
Let M be convergent $\Rightarrow \operatorname{Lim} \mathrm{y}_{\mathrm{n}}=\mathrm{y}\left(\mathrm{x}_{\mathrm{n}}\right)$ as $\mathrm{h} \rightarrow 0, \mathrm{n} \rightarrow \infty\left(\mathrm{x}_{\mathrm{n}}\right.$ fixed).

Also, $y_{n+j}=y\left(x_{n}\right)+\Theta_{j}(h)$ Where $\Theta_{j}(h) \rightarrow 0$ as $h \rightarrow 0$.
$\sum_{j=0}^{k} \alpha_{j} y_{n+j}=\sum_{j=0}^{k} \alpha_{\mathrm{j}} y\left(x_{n}\right)+\sum_{j=0}^{k} \alpha_{j} \Theta_{j}(h)$
$\mathrm{h} \sum \beta_{\mathrm{j}} f_{n+j}=y\left(x_{n}\right) \sum \alpha_{j}+\sum \alpha_{j} \Theta_{j}(h)$
$\Rightarrow y\left(x_{n}\right) \sum \alpha_{j} \rightarrow 0 \quad$ as $\mathrm{h} \rightarrow 0$
$y^{\prime}\left(x_{n}\right)=\operatorname{Lim}_{h \rightarrow 0} \frac{y\left(x_{n}+j h\right)-y\left(x_{n}\right)}{j h} \quad$, then $\frac{\mathrm{y}_{\mathrm{n}+\mathrm{j}}-y_{n}}{j h} \rightarrow \mathrm{y}^{\prime}\left(\mathrm{x}_{\mathrm{n}}\right)$
$\therefore \frac{\mathrm{y}_{\mathrm{n}+\mathrm{j}}-y_{n}}{j h}=\mathrm{y}^{\prime}\left(\mathrm{x}_{\mathrm{n}}\right)+Q j(h), Q j(h) \rightarrow 0$ as $\mathrm{h} \rightarrow 0 \forall \mathrm{j}=0, \ldots \mathrm{k}$
$\therefore \sum \alpha_{\mathrm{j}} \mathrm{y}_{\mathrm{n}+\mathrm{j}}=\sum \alpha_{\mathrm{j}} \mathrm{y}_{\mathrm{n}}=\mathrm{h} \sum \mathrm{j} \alpha_{\mathrm{j}} \mathrm{y}^{\prime}\left(\mathrm{x}_{\mathrm{n}}\right)+h \sum j \alpha_{j} Q_{j}(h)$
$\mathrm{h} \sum \beta_{\mathrm{j}} f_{n+j}=h y^{\prime}\left(x_{n}\right) \sum j \alpha_{j}+h \sum j \alpha_{j} \mathrm{Q}_{\mathrm{j}}(h)$
$\mathrm{f}_{\mathrm{n}+\mathrm{j}}=f\left(x_{n+j_{n}}, y_{n+j}\right) \rightarrow y^{\prime}\left(x_{n}\right)$
$\mathrm{y}^{\prime}\left(\mathrm{x}_{\mathrm{n}}\right) \sum \beta_{j}=y^{\prime}\left(x_{n}\right) \sum j \alpha_{j}+\sum j \alpha_{j}+\sum j \alpha_{j} Q j(h)$
$\Rightarrow \sum \beta_{j}=\sum j \alpha_{j}$ when $\mathrm{h} \rightarrow 0$
$\mathrm{M} \Rightarrow \sum \alpha_{j} \mathrm{y}_{\mathrm{n}+\mathrm{j}}=\mathrm{h} \sum \beta_{\mathrm{j}} \mathrm{f}_{\mathrm{n}+\mathrm{j}} \quad$ "

Def 4: The first charachecteristic polynomial of M is the poly
$\mathrm{P}(\mathrm{r})=\sum_{\mathrm{j}=0}^{\mathrm{k}} \alpha_{\mathrm{j}} \mathrm{j}^{\mathrm{j}}=\alpha_{0}+\alpha_{1} r+\alpha_{2} r^{2}+\ldots \ldots .+\alpha_{k} r^{k}$
The second charachecteristic polynomial M is
$\delta(\mathrm{r})=\sum_{j=0}^{k} \beta_{j} r^{j}=\beta_{0}+\beta_{1} r+\ldots .+\beta_{k} r^{k}$

Def 5: The method $M$ is said to be zero stable if all roots $r_{j}$ of $\mathrm{p}(\mathrm{r})=0$ satisfy $\left|r_{j}\right| \leq 1 \forall \mathrm{j}=1, \ldots, \mathrm{k}$ and if $\mathrm{r}_{\mathrm{m}}$ is a multiple root of $\mathrm{p}(\mathrm{r})=0$, then $\left|r_{m}\right|<1$

Example for consistency and zero stable

## 1-Euler's Method

$$
\begin{aligned}
\mathrm{y}_{\mathrm{n}+1}-\mathrm{y}_{\mathrm{n}} & =\mathrm{hf}_{\mathrm{n}}+\mathrm{o}\left(\mathrm{~h}^{2}\right) \\
\mathrm{sol}:-\mathrm{c}_{0} & =\sum_{j=0}^{1} \alpha_{j}=\alpha_{0}+\alpha_{1}=-1+1=0 \\
\mathrm{c}_{1} & =\sum_{j=0}^{1} \mathrm{j} \alpha_{\mathrm{j}}-\sum_{\mathrm{j}=0}^{1} \beta_{j}=0 \alpha_{0}+1 \alpha_{1}-\left(\beta_{0}+\beta_{1}\right)=1-1=0 \\
\mathrm{c}_{2} & =\frac{1}{2!}\left(\alpha_{1}+2^{2} \alpha_{2}\right)-\frac{1}{1!}\left(\beta_{1}+2 \beta_{2}\right) \\
& =\frac{1}{2}(1+0)-(0+0)=\frac{1}{2} \neq 0 \\
\therefore \mathrm{p} & =1 \Rightarrow \therefore \text { Euler's method is consistency }
\end{aligned}
$$

To show the method is zero stable then satisfy $\left|r_{j}\right| \leq 1$

$$
\begin{aligned}
& p(r)=\sum_{j=0}^{k} \alpha_{\mathrm{j}} \mathrm{r}^{\mathrm{j}}=0 \\
& p(r)=r-1=0 \\
& \mathrm{r}=1 \\
& \text { since }\left|r_{j}\right| \leq 1
\end{aligned}
$$

$\therefore$ Euler's method is zero stable
$\therefore$ Euler's method is consistency and zero stable

## 2-Midpoint method

$$
\mathrm{y}_{\mathrm{n}+2}-y_{n}=2 h f_{n+1}+o\left(h^{3}\right)
$$

Sol:

$$
\begin{aligned}
& \mathrm{c}_{0}=\sum_{j=0}^{2} \alpha_{j}=\alpha_{0}+\alpha_{1}+\alpha_{2}=-1+0+1=0 \\
& \mathrm{c}_{1}=\sum_{j=0}^{2} j \alpha_{j}-\sum_{j=0}^{2} \beta_{j}=0 \alpha_{0}+1 \alpha_{1}+2 \alpha_{2}-\beta_{0}-\beta_{1}-\beta_{2} \\
&=0+0+2(1)-0-2-0=0 \\
& \mathrm{c}_{2}= \frac{1}{2!}\left(\alpha_{1}+2^{2} \alpha_{2}\right)-\frac{1}{1!}\left(\beta_{1}+2 \beta_{2}\right) \\
&= \frac{1}{2}(0+4(1))-(2+2(0))=\frac{1}{2}(4)-2=2-2=0 \\
& \mathrm{c}_{3}= \frac{1}{3!}\left(\alpha_{1}+2^{3} \alpha_{2}\right)-\frac{1}{2!}\left(\beta_{1}+2^{2} \beta_{2}\right)=\frac{1}{6}(0+8(1))-\frac{1}{2}(2-4(0)) \\
&=\frac{1}{6}(8)-2\left(\frac{1}{2}\right)=\frac{4}{3}-1=\frac{1}{3} \neq 0
\end{aligned}
$$

$\therefore \mathrm{p}=2 \Rightarrow \mathrm{p} \geq 1$
$\therefore$ mid point method is consistency
To show the method is zero stable then satisfy $\left|r_{j}\right| \leq 1$

$$
\begin{gathered}
p(r)=\sum_{j=0}^{k} \alpha_{\mathrm{j}} \mathrm{r}^{\mathrm{j}}=0 \\
\mathrm{p}(\mathrm{r})=0 \Rightarrow \mathrm{r}^{2}-1=0 \Rightarrow r^{2}=1 \Rightarrow r=\mu 1 \\
\quad \therefore\left|\mathrm{r}_{\mathrm{j}}\right| \leq 1
\end{gathered}
$$

$\therefore$ Midpoint method is zero stable.
$\therefore$ Midpoint method is consistency and zero stable.

3-TrapezeomeMethod

$$
\begin{aligned}
& \mathrm{y}_{\mathrm{n}+1}-y_{n}=\frac{h}{2}\left(f_{n}+f_{n+1}\right)+o\left(h^{3}\right) \\
& \text { sol:- } \mathrm{c}_{0}=\sum_{j=0}^{1} \alpha_{j}=\alpha_{0}+\alpha_{1}=-1+1=0 \\
& \mathrm{c}_{1}=\sum_{j=0}^{1} \mathrm{j} \alpha_{\mathrm{j}}-\sum_{j=0}^{1} \beta_{\mathrm{j}}=0 \alpha_{0}+\alpha_{1}-\beta_{0}-\beta_{1} \\
&=0+(1)-\frac{1}{2}-\frac{1}{2}=1-1=0 \\
& \mathrm{c}_{2}=\frac{1}{2!}\left(\alpha_{1}+2^{2} \alpha_{2}\right)-\frac{1}{1!}\left(\beta_{1}+2^{1} \beta_{2}\right) \\
&=\frac{1}{2}(1+0)-\left(\frac{1}{2}+0\right)=\frac{1}{2}-\frac{1}{2}=0 \\
& \mathrm{c}_{3}=\frac{1}{3!}\left(\alpha_{1}+2^{3} \alpha_{2}\right)-\frac{1}{2!}\left(\beta_{1}+2^{2} \beta_{2}\right) \\
&=\frac{1}{6}(1+0)-\frac{1}{2}\left(\frac{1}{2}+0\right)=\frac{1}{6}-\frac{1}{4}=\frac{4-6}{24}=\frac{-2}{24}=\frac{-1}{12} \neq 0 \\
& \therefore \mathrm{p}=2 \Rightarrow \mathrm{p} \geq 1 \\
& \therefore \text { Trapezeomis consistency }
\end{aligned}
$$

To show the method is zero stable then satisfy $\left|r_{j}\right| \leq 1$

$$
p(r)=\sum_{j=0}^{k} \alpha_{\mathrm{j}} \mathrm{r}^{\mathrm{j}}=0
$$

$\mathrm{p}(\mathrm{r})=0 \Rightarrow \mathrm{r}-1=0 \Rightarrow \mathrm{r}=1$
$\therefore\left|\mathrm{r}_{\mathrm{j}}\right| \leq 1$
$\therefore$ Trapezeomis Zerostabloe.
$\therefore$ Trapezeomis consistency and Zerostable.

## Theorem 2:

If M is convergent to the solution of P as $\mathrm{h} \rightarrow 0, \mathrm{n} \rightarrow \infty$ $\mathrm{x}_{\mathrm{n}}=\mathrm{a}+\mathrm{nh}$ fixed then M is zero stable .[6]

## Theorem 3:

A necessary and sufficient condition for the convergence of $M$ to exact solution of P as $\mathrm{h} \rightarrow 0, \mathrm{n} \rightarrow \infty$ are that M should be to consistency and zero stable .

Note: From above theorem can be easy to show the methods is converge.
Def 6: The equation $\pi(r, \bar{h})=p(r)-\bar{h} \delta(r)$ is said to be stability poly.

Def 7: The method M is said to be absolutely stable for some value of $\bar{h}$ if the roots $\mathrm{r}_{\mathrm{j}}$ of $\pi(r, \bar{h})=0$ satisfy $\left|r_{j}\right|<1 \quad \forall \mathrm{j}=1, \ldots, \mathrm{k}$ An interval $\left(\alpha^{*}, \beta^{*}\right)$ is an interval of absolutely stable of M if M is absolutely stable $\forall \overline{\mathrm{h}} \in\left(\alpha^{*}, \beta^{*}\right)$. [5] .

Def 8:The method M is said to be relatively stable for value $\bar{h}$ if $\left|r_{j}\right|<\left|r_{1}\right| \forall \mathrm{j}=2, \ldots ., \mathrm{k},\left(\alpha^{*}, \beta^{*}\right)$ is an interval of relatively stable of M. [4]

Example stability poly

## 1- Euler method :

$$
\begin{aligned}
& \mathrm{y}_{\mathrm{n}+1}-y_{n}=h f_{n}+o\left(h^{2}\right) \\
& \pi(r, \bar{h})=r-1-\bar{h}>0 \Rightarrow r=1+\bar{h} \\
& |\mathrm{r}|<1 \Rightarrow|1+\bar{h}|<1 \\
& \\
& \quad-1<1+\overline{\mathrm{h}}<1 \Rightarrow-2<\bar{h}<0
\end{aligned}
$$

$(-2,0)$ is the interval of absolutely stable $\overline{\mathrm{h}} \in(-2,0)$

## 2- Trapezam Method :

$$
\begin{aligned}
& \left.\mathrm{y}_{\mathrm{n}+1}-y_{n}=\frac{h}{2} f_{n+1}+f_{n}\right)+o\left(h^{3}\right) \\
& \pi(r, \bar{h})=r-1-\frac{\bar{h}}{2}(r+1)=0
\end{aligned}
$$

$$
\left(1-\frac{\overline{\mathrm{h}}}{2}\right) r-1-\frac{\bar{h}}{2}=0
$$

$$
\mathrm{r}=\frac{1+\overline{\mathrm{h}} l 2}{1-\overline{h l 2}}
$$

$$
|r|<1 \Rightarrow\left|1+\frac{\bar{h}}{2}\right|<\left|1-\frac{\bar{h}}{2}\right|
$$

but $\overline{\mathrm{h}}<0 \Rightarrow\left|1+\frac{\bar{h}}{2}\right|<1-\frac{\bar{h}}{2}$

$$
-1+\frac{\overline{\mathrm{h}}}{2}<1+\frac{\bar{h}}{2}<1-\frac{\bar{h}}{2} \Rightarrow-\infty<\bar{h}<0 \Rightarrow \bar{h} \in(-\infty, 0)
$$

## 2-4 A. stable [4]

Def: A numerical method is said to be A-stable if its region of absolutely stable contains the left half of $\mathrm{h} \kappa$-plane .
i.e $\{\mathrm{h} K: \operatorname{Re}(\mathrm{h} K)<0\}$ (No restriction on h )

## Remark :

1- No explicit linear multistep method is A-stable .
2 - The order of any implicit A stable method $\mathrm{p} \leq 2$.
3- The trap. Method $\mathrm{p}=2$ is A stable with the smallest error constant.

For example implicit Euler method :

$$
\begin{aligned}
& y_{n+1}-y_{n}=h f_{n+1} \\
& \pi(r, \bar{h})=p(r)-\bar{h} \delta(r) \\
& =\mathrm{r}-1-\bar{h} r=0 \\
& =\mathrm{r}(1-\bar{h})-1=0 \\
& \text { since }|\mathrm{r}|<1 \Rightarrow\left|\frac{1}{1-\bar{h}}\right|<1 \\
& \therefore|1-\bar{h}|>1 \\
& \text { if } 1-\bar{h}>1 \Rightarrow-\bar{h}>0 \Rightarrow \bar{h}<0 \\
& \therefore \bar{h} \in(-\infty, 0) \\
& \text { or } 1-\bar{h}<1 \Rightarrow-\bar{h}<0 \Rightarrow \bar{h} \text { (nodefine) }
\end{aligned}
$$

Then the interval of absolutely stable $(-\infty, 0)$ thus the method is A -stable .

Def:- A one step implicit method is said to be L-stable if :
1 - If is A stable .
2- When applied to the scalar equation $\mathrm{y}^{\prime}=\mathrm{dy}(\operatorname{Re}(\mathrm{h} \kappa)<0)$ preduces $\mathrm{y}_{\mathrm{n}+1}=Q(h \lambda) y_{n}$
and $\operatorname{Lim}|\mathrm{Q}(\mathrm{h} \lambda)|=0$ as $\operatorname{Re}(\mathrm{h} \lambda) \rightarrow-\infty$
In L-stable method no restriction on $h+f a s t ~ d e c r e a s e ~ o f ~ y_{n}$ to 0 .

Note that : L-stability $\rightarrow$ A-stability but the converse not true .

## References

[1] A.Edalat and D.pattinson , Domain -thoretic solution of differential equations, London 2006 .
[2] B.Tasic , Numerical methods for solving O.D.E flow , ph.D. thesis , Technsch universteil Eind hover, geborate Ni, 2006.
[3] D.Honcqheq ordinary Differential Equation, North western University 2008.
[4] E.S. pav , Numerical methods course notes, University of California, 2007.
[5] R.Bronson , Differential Equations , second Edition , London, 2003.
[6] R.Piche , solving Bups using differentiation matrices , Tampere Unversity of Technology, V012, 2007.
[7] S.Mauch . Introduction to methods of Applied . Mathematics, April , 2002 $\square$

