Nonlinear Analysis of Continuous Composite Beam by Finite Element Method

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Abstract

In this paper a composite beam element has been developed. The composite beam element can be used to model the nonlinear behavior of composite beams. The element is implemented in a nonlinear finite element program (written by the researcher) and its implementation is verified by the analysis of continuous beam tested by others. The good agreement between the computed results and experimental data demonstrates accuracy of the used element. It was found that the increase of steel reinforcement gives an increase in ultimate load by 14.3%, also the increase of flange width at plastic hinge region gives an increase in the ultimate load by 39.3%.

الخـــلاصـــة

في هذا البحث تم تطوير عنصر العتب المركب. ويمكن استخدامه في تمثيل التصرف اللاخطي للعتب المركب. العنصر ينفذ في برنامج عناصر محدده لاخطي (مكتوب من قبل الباحث) وهذا التنفيذ قد دقق بواسطة تحليل عتب مستمر مركب مفحوص من قبل باحثين آخرين. المقارنة الجيدة بين القيم المحسوبة والمعلومات العملية بينت دقه العنصر المستخدم. لقد وجد إن الزيادة في حديد التسليح في مناطق (plastic hinge) يعطي زيادة في قابليه تحمل العتب القصوى بنسبه ١٤.٣ % كذلك الزيادة في عرض الشفة (flange) في مناطق المفصل اللدن (plastic hinge) سوف يعطي زيادة في تحمل العتب القصوى بنسبه ٣٩.٣ %.

1. Introduction

When two elements capable of resisting bending moments are elastically connected together at the interface, interaction partial or complete, between the two elements takes place. If the elastic connection is flexible, the differential axial strains at the common interface exist resulting in slip, and differential deflections may also result giving rise to uplift between the two elements.

Many studies have been conducted concerning the analysis of composite structures in the past. However, generally either full interaction has been assumed [1] or the shear connectors have been treated as rigid or elastic springs [2,3]. Some of studies assumed that the shear connector is continuous along the length, i.e. discrete connectors are assumed to be replaced by a medium of negligible thickness having normal and tangential modulus [4-7]. Yam and Chapman [4,5] developed an approach to incorporate nonlinear material and shear connector behavior, the resulting nonlinear differential equations had been solved iteratively.

2. Finite Element Idealization

The composite beam has two coordinate system, $\bar{X}$, $\bar{Z}$ for the concrete part and X, and Z for the steel part. Each part of the element has its pertaining end nodes 1 and 2 with three degrees of freedom per node as shown in **Fig.(1)**, consequently, there are six degrees of freedom (four transitional and two rotational displacements) for each node of the element.

Assuming that the plane section within each material remains plane, the axial displacement and strain can be expressed in the terms of displacements u and w relative to the local x and z axes. According to **Fig.(2)** the horizontal displacement and strain in each component of the horizontal beam are:

i) Steel Beam

$$u_{st} = u_{ost} - z\frac{dw_{st}}{dx} \quad \text{..........} \quad (1)$$

$$\varepsilon_{st} = \frac{du_{ost}}{dx} - z\frac{d^2w_{st}}{dx^2} = \varepsilon_{ost} - z\frac{d^2w_{st}}{dx^2} \quad \text{..........} \quad (2)$$

ii) Concrete Slab

$$u_c = u_{oc} - \bar{z}\frac{dw_c}{dx} \quad \text{..........} \quad (3)$$

$$\varepsilon_c = \frac{du_{oc}}{dx} - \bar{z}\frac{d^2w_c}{dx^2} = \varepsilon_{oc} - \bar{z}\frac{d^2w_c}{dx^2} \quad \text{..........} \quad (4)$$

iii) Slab Reinforcement

$$\mathbf{u_{sr} = u_{oc} - d_z \frac{dw_c}{dx}} \quad \text{..} \mathbf{(5)}$$

$$\mathbf{\varepsilon_{sr} = \frac{du_{oc}}{dx} - d_z \frac{d^2w_c}{dx^2} = \varepsilon_{oc} - d_z \frac{d^2w_c}{dx^2}} \quad \text{..} \mathbf{(6)}$$

where u_{oc} and u_{ost} are axial displacements in the concrete slab and steel beam, respectively, and $\frac{dw_c}{dx}$ and $\frac{dw_{st}}{dx}$ are the slopes of concrete slab and steel beam in z direction, respectively.

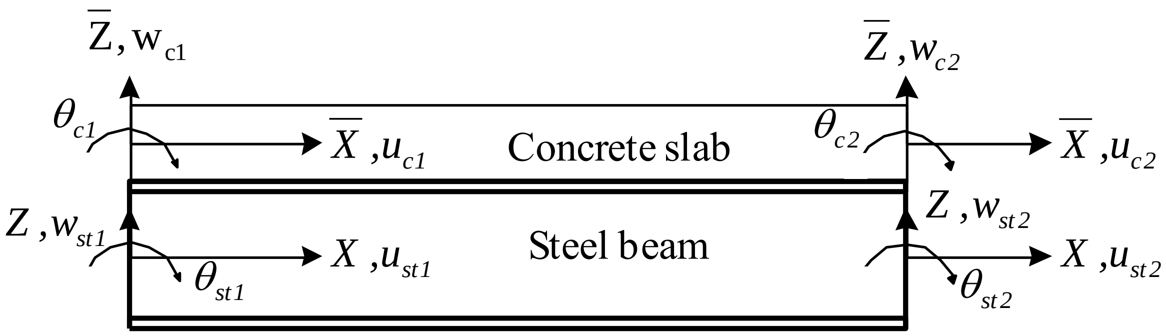

Figure (1) Displacement components of an element of a composite beam

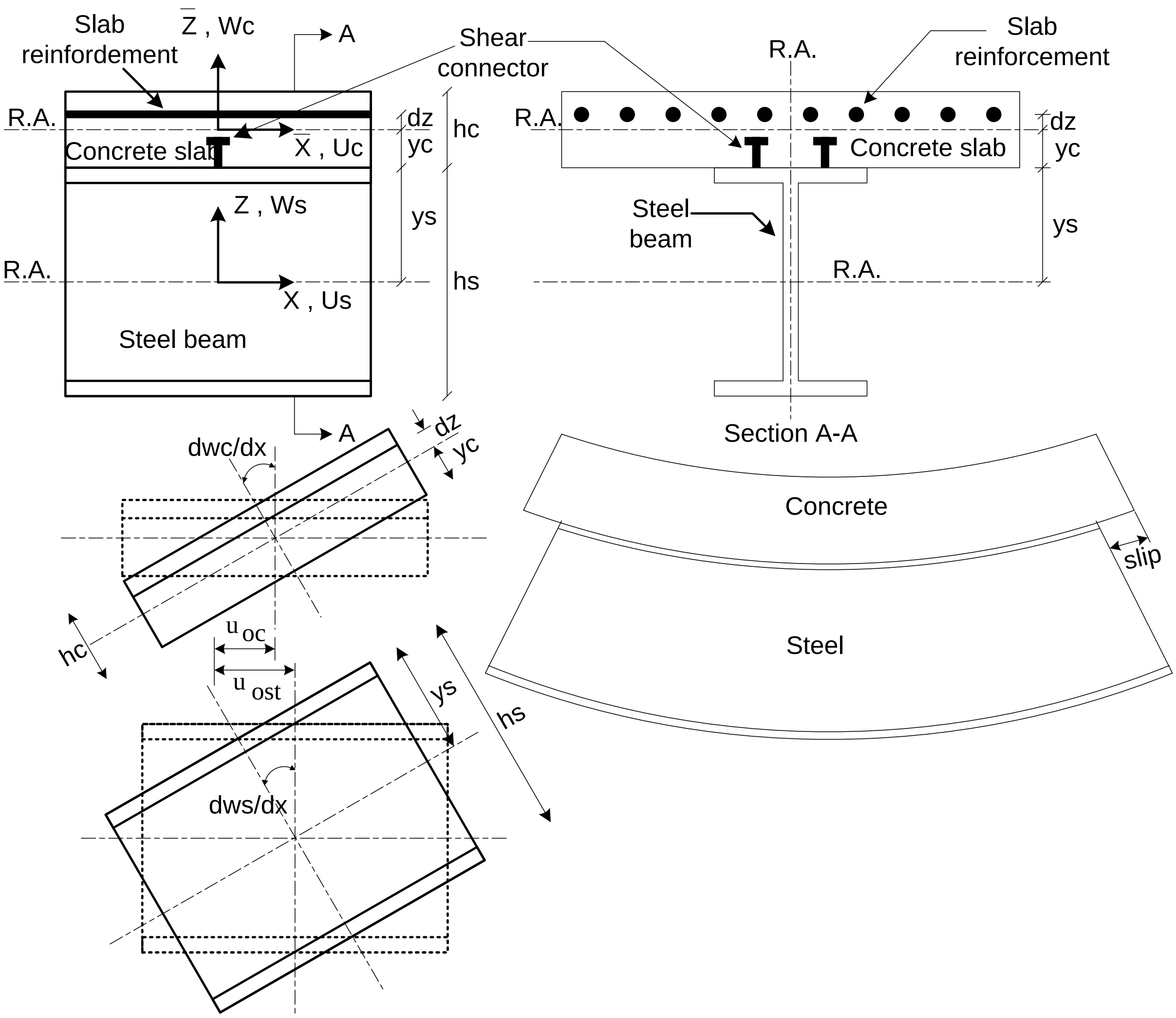

Figure (2) Deformations of composite beam segment

The slip, S, between the concrete slab and steel beam is given as the difference in the displacement between bottom surface of concrete slab and the top surface of steel beam at the centerline of the interface, i.e.

$$S = u_{c(\bar{z}=-y_c)} - u_{st(z=y_s)} \quad \ldots\ldots\ldots\ldots (7)$$

$$S = u_{oc} - u_{ost} + y_s \frac{dw_{st}}{dx} + y_c \frac{dw_c}{dx} \quad \ldots\ldots\ldots\ldots (8)$$

The separation (uplift), f_a, between the concrete slab and steel beam in the vertical direction is the difference in deflection (in z-direction) between the steel beam and concrete slab at the node under consideration. It may be expressed as:

$$f_a = w_{st} - w_c \quad \ldots\ldots\ldots\ldots (9)$$

Vectors represent the axial displacement and the bending displacements are {u} and {v}, respectively,

$$\left.\begin{aligned}\{u\} &= [u_1 \quad u_2]^T \\ \{v\} &= [w_1 \quad \theta_1 \quad w_2 \quad \theta_2]^T\end{aligned}\right\} \quad \ldots\ldots (10)$$

These displacement components can be assembled in one column vector {d} as:

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \ldots\ldots (11)$$

From Equations (2), (4) and (6) it can be concluded that [8], a C_o-continuity shape function (linear) and C_1-continuity shape function (cubic Hermitten) are required for representing the axial and flexural displacements, respectively. Then let $u_o(x)$ and $v_o(x)$ be the axial and bending displacements at any point along x-axis, respectively, then:

$$u_o = Na\{u\} \quad \& \quad v_o = Nb\{v\} \quad \ldots\ldots (12)$$

where, Na is the shape function defining a linear interpolation of $u_o(x)$ between the nodes, and Nb comprises the cubic beam function interpolation polynomial [8]

$$Na = [N1 \quad N2]^T \quad \& \quad Nb = [N3 \quad N4 \quad N5 \quad N6]^T \quad \ldots\ldots (13)$$

where:

$$\left.\begin{aligned} N1 &= 1 - \frac{x}{L} & N2 &= \frac{x}{L} \\ N3 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} & N4 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ N5 &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & N6 &= -\frac{x^2}{L} + \frac{x^3}{L^2} \end{aligned}\right\} \quad \ldots\ldots (14)$$

thus, the displacements field, {d}, is

$$\{d\} = [u_{st1} \quad u_{c1} \quad w_{st1} \quad w_{c1} \quad \theta_{st1} \quad \theta_{c1} \quad u_{st2} \quad u_{c2} \quad w_{st2} \quad w_{c2} \quad \theta_{st2} \quad \theta_{c2}]^T \quad \ldots\ldots (15)$$

As stated before, six degrees of freedom are needed at each node in the finite element discretization. The nodal displacements at each node will be,

$$\{di\} = [u_{sti} \quad u_{ci} \quad w_{sti} \quad w_{ci} \quad \theta_{sti} \quad \theta_{ci}]^T \quad \ldots\ldots (16)$$

For beam element under external load, using the virtual work principles [8]

$$\textbf{External virtual work} = \int_0^L R_i\, U_i\, dx \quad \text{.......................................} \quad (17)$$

where R_i is the applied load and U_i is the virtual displacement. For nodal displacements, {d}, and equivalent external load $\{R_j\}$

$$\textbf{External virtual work} = [R_j]\delta\{d\}^T \quad \text{.......................................} \quad (18)$$

But the internal work = internal work in steel beam + internal work in concrete slab + internal work in shear connector + internal work in slab reinforcement. Then:

$$[R_j]\{d\} = \int_{vol.steel} \delta\varepsilon_{st}\,\sigma_{st}\,dvol + \int_{vol.concrete} \delta\varepsilon_c\,\sigma_c\,dvol + \sum_{i=1}^{n}\int_0^L \delta\varepsilon_{sr}\,\sigma_{sr}\,Asr\,dx$$
$$+ \sum_{m=1}^{ns}[q_x\,\delta S]_{x=xs} + \sum_{m=1}^{ns}[F_a\,\delta f_a]_{x=xs} \quad \text{...................................} \quad (19)$$

where q_x is the shear force (kN) in x-direction, F_a is the normal force (kN), $F_a=f(f_a)$, n is the number of layer reinforcement, ns is the number of shear connectors in each element, and xs is the location of shear connector. The strains in a composite beam component are expressed as:

$$\left.\begin{aligned} \delta\varepsilon_{st} &= [B_{st}]\delta\{d\} \\ \delta\varepsilon_c &= [B_c]\delta\{d\} \\ \delta\varepsilon_{sr} &= [B_{sr}]\delta\{d\} \\ \delta S &= [B_S]\delta\{d\} \\ \delta F_a &= [B_f]\delta\{d\} \end{aligned}\right\} \quad \text{.......................................} \quad (20)$$

where [B]'s are the strain-displacement relationship matrices. Combing of Equations (19) and (20) leads to:

$$\{R_j\} = \int_{steel}[B_{st}]^T\sigma_{st}\,dvol + \int_{concrete}[B_c]^T\sigma_c\,dvol + \sum_{i=1}^{n}\int_0^L[B_{sr}]^T\sigma_{sr}\,A_{sr}\,dx +$$
$$\sum_{m=1}^{ns}\left[[B_S]^T q_x\right]_{x=xs} + \sum_{m=1}^{ns}\left[[B_f]^T F_a\right]_{x=xs} \quad \text{...................................} \quad (21)$$

and the strain vectors may be written in one column vector, {ε}, as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{st} \\ \varepsilon_{c} \\ \varepsilon_{sr} \\ S \\ f_a \end{Bmatrix} = \left[B_{5\times12}\right] \begin{Bmatrix} u_{st1} \\ u_{c1} \\ w_{st1} \\ w_{c1} \\ \theta_{st1} \\ \theta_{c1} \\ u_{st2} \\ u_{c2} \\ w_{st2} \\ w_{c2} \\ \theta_{st2} \\ \theta_{c2} \end{Bmatrix} \quad \text{where} \left[B\right]^T = \begin{bmatrix} N_1' & 0 & 0 & -N_1 & 0 \\ 0 & N_1' & N_1' & N_1 & 0 \\ -zN_3'' & 0 & 0 & y_sN_3' & N_3 \\ 0 & -\bar{z}N_3'' & -dzN_3'' & y_cN_3' & -N_3 \\ -zN_4'' & 0 & 0 & y_sN_4' & N_4 \\ 0 & -\bar{z}N_4'' & -dzN_4'' & y_cN_4' & -N_4 \\ N_2' & 0 & 0 & -N_2 & 0 \\ 0 & N_2' & N_2' & N_2 & 0 \\ -zN_5'' & 0 & 0 & y_sN_5' & N_5 \\ 0 & -\bar{z}N_5'' & -dzN_5'' & y_cN_5' & -N_5 \\ -zN_6'' & 0 & 0 & y_sN_6' & N_6 \\ 0 & -\bar{z}N_5'' & -dzN_5'' & y_cN_6' & -N_6 \end{bmatrix} \quad \ldots (22)$$

Eq. (21) can be written in compact form as:

$$[K]\{d\} = \{R_j\} \quad \ldots\ldots (23)$$

where [K] is the stiffness matrix. The stiffness matrix is generated at the mid-length of composite beam element and assumed to be constant along the element for the non-linear behavior. The stiffness matrix of a composite beam element is given by:

$$[K]^e = \int_{vol} [B]^T [D][B] dvol \quad \ldots\ldots (24)$$

It is composed from the contribution of composite beam components and can be expressed as:

$$[K]^e = [K]^e_{st} + [K]^e_{c} + [K]^e_{sr} + [K]^e_{S} + [K]^e_{f} \quad \ldots\ldots (25)$$

where:

$[K]^e_{st}$: steel beam element stiffness matrix, $[K]^e_{c}$: concrete slab element stiffness matrix, $[K]^e_{sr}$: slab reinforcement element stiffness matrix, $[K]^e_{S}$: shear connector element stiffness matrix in x-direction, and, $[K]^e_{f}$: shear connector element stiffness matrix in z-direction.

3. Non-Linear Analysis: Cross-Section Properties

The modulus of elasticity for each material of composite beam is a function of strain value at the point under consideration. But the strain varies across the depth of the beam. Steel beam and concrete slab section are divided into a number of layers as shown in **Fig.(3)** so that:

$$\mathbf{EA} = \int_A \mathbf{E\,dA} = \sum_{ie=1}^{n} \mathbf{E_{ie}\,A_{ie}} \quad \text{.......} \quad \mathbf{(26)}$$

$$\mathbf{EI} = \int_A \mathbf{E\,z^2\,dA} = \sum_{ie=1}^{n} \mathbf{E_{ie}\,z^2\,A_{ie}} \quad \text{.......} \quad \mathbf{(27)}$$

where, n is the number of layers in the material under consideration. E_{ie} is the modulus of elasticity of element. z is the distance from layer to the reference axis of concrete slab or steel beam. A_{ie} is the cross-sectional area of the layer.

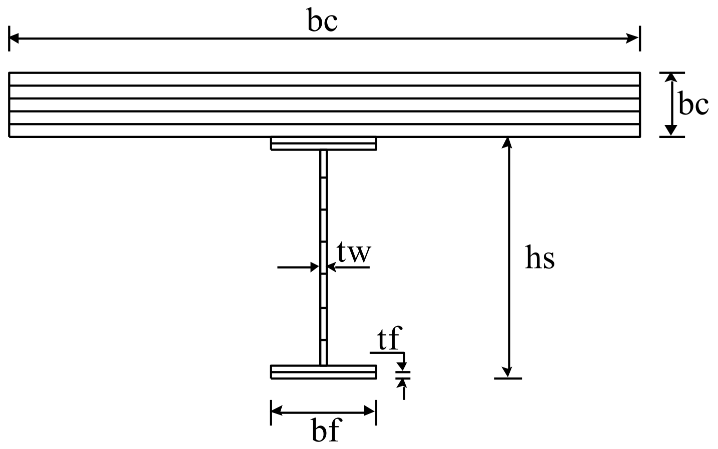

Figure (3) Layered beam section

4. Materials Constitutive Relationships

4-1 Concrete

For concrete in compression the used model for the stress-strain relationship is that proposed in BS 8110 [9] as shown in **Fig. (4-A)**, the ultimate compressive strain, $\boldsymbol{\sigma_{cu}}$ is limited to 0.0035, the curved portion of stress-strain curve is defined by:

$$\boldsymbol{\sigma = 5500\sqrt{\sigma_{cu}}\,\varepsilon - 11.3 * 10^6 \varepsilon^2} \quad \text{.......} \quad \mathbf{(28)}$$

with $\varepsilon_o = 2.44*10^{-4}\sqrt{\sigma_{cu}}$, and the initial modulus of elasticity is:

$$\mathbf{Ei = 5500\sqrt{\sigma_{cu}}} \quad \text{.......} \quad \mathbf{(29)}$$

in which σ_{cu} is the concrete cube strength in MPa.

The tensile strength of concrete is relatively low so that, concrete is assumed incapable to resist any tension.

4-2 Steel Beam and Reinforcement

A bilinear stress-strain curve is adopted for this type of steel as shown in **Fig.(4-B)**. In this stress-strain curve, the yield stress, fy, in tension and compression is equal.

4-3 Shear Connectors

Load-slip curves and information concerning shear connectors can be obtained from push-out tests, although they cannot be assumed to represent what really happens because the distribution of longitudinal stress in the concrete flange of a beam is different from that in the slab in push-out test [10].

Many different load-slip relationships for stud connectors have been proposed, an exponential model is the best of these models. An exponential model for the load-slip relationship of shear connectors was used by Yam and Chapman [4]. This is represented by the following function,

$$\mathbf{Q = a\left(1 - e^{-bS}\right)} \quad \text{.......} \quad \mathbf{(30)}$$

where $a = \left(\frac{Q1}{2Q1 - Q2}\right)$ and $b = \frac{1}{S}\ln\left(\frac{Q1}{Q2 - Q1}\right)$, are two constants chosen to give the best fit with experimental load-slip curve. Alternatively, two points are chosen from the experimental curve, so that S_2=2S_1, this is shown in **Fig.(4-C)**.

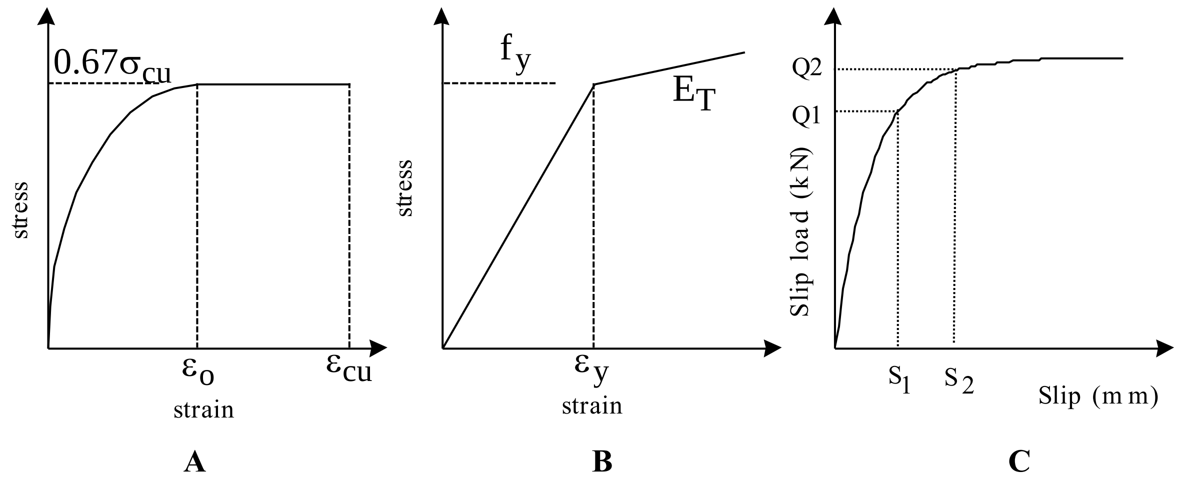

Figure (4) A-Stress-strain curved for concrete; B- Bilinear stress-strain curved for steel; C-Load-slip for shear connector

5. Convergence Criteria

The nonlinear algebraic equations can be solved iteratively, as illustrated in **Fig.(5)** in which R and d denote a representative load and displacement respectively. For the first stage of solution, the material properties are assumed constant and a set of nodel displacements corresponding to a specified applied loading is determined. From these displacements, strains throughout the beam are determined, which are used to define the secant values of material properties for the second stage of the solution. The process is repeated until the calculated displacements have converged.

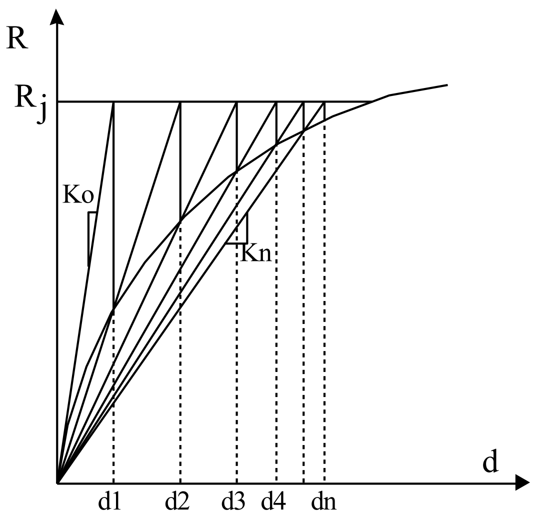

Figure (5) Solution procedure in a nonlinear problem (secant method)

6. Results of Numerical Example

Yam and Chapman [5] tested a series of continuous composite beams, CBI is one of the tested continuous beams, and **Fig.(6)** illustrates the dimensions of this beam. The material properties are listed below.

***Steel beam*:** 152 mm* 76 mm *17.86 kg/m rolled steel joist. Flange 76 mm* 9.6 mm. Web thickness 5.9 mm. Young's Modulus 206700 MPa. Yields stress 301 MPa. Strain–hardening factor 0.005.

Concrete slab: 610 mm* 60.3 mm. cylinder compressive strength 48 MPa. Young's Modulus 27600 MPa.

Shear Connector: 9 mm diameter. 50 mm height. Spacing 146 mm. Number of rows 2. Load-slip relation $Q = 32(1 - e^{-4.725\ S})$.

Longitudinal reinforcement: Young's Modulus 206800 MPa. Yield stress 321 MPa. Area of top bars 445 sq. mm. Area of bottom bars 142 sq. mm.

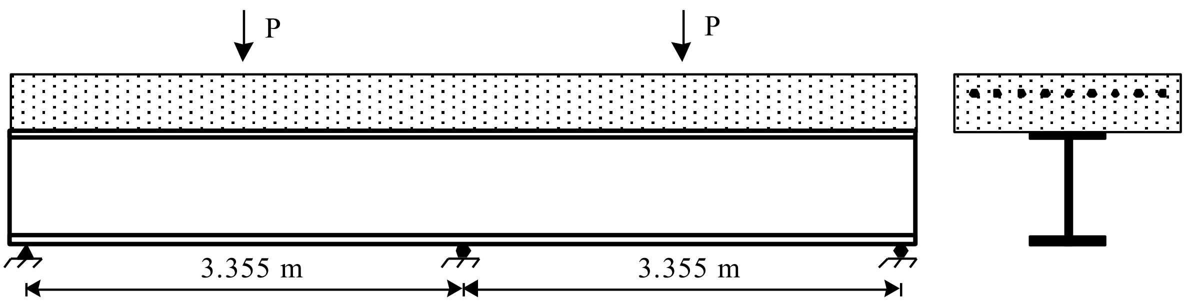

Figure (6) Continuous composite beam CBI [5]

The results in **Fig.(7-A)**, **(7-B)**, and **(7-C)** for beam CBI show similarly good agreement. **Figure (7-A)** shows the deflection shape at P=122 kN, which corresponds to 87% of ultimate load. It should be pointed out that in the experiment the loads on the two spans were not identical. They were 121 kN and 123 kN, but in the analysis for simplicity they were assumed equal. The calculated results are close to the experimental values although the analysis overestimates the measured deflections in the left span and underestimates the same measured deflection in the right span. The slip variation along the span at 87% of the ultimate load is illustrated in **Fig.(7-B)** which shows good agreement. Here again a situation similar to that in **Fig.(7-A)** can be observed. Comparison of the computed strains in the extreme bottom flange of the beam is rather closely with their experimental counterparts, **Fig.(7-C)**.

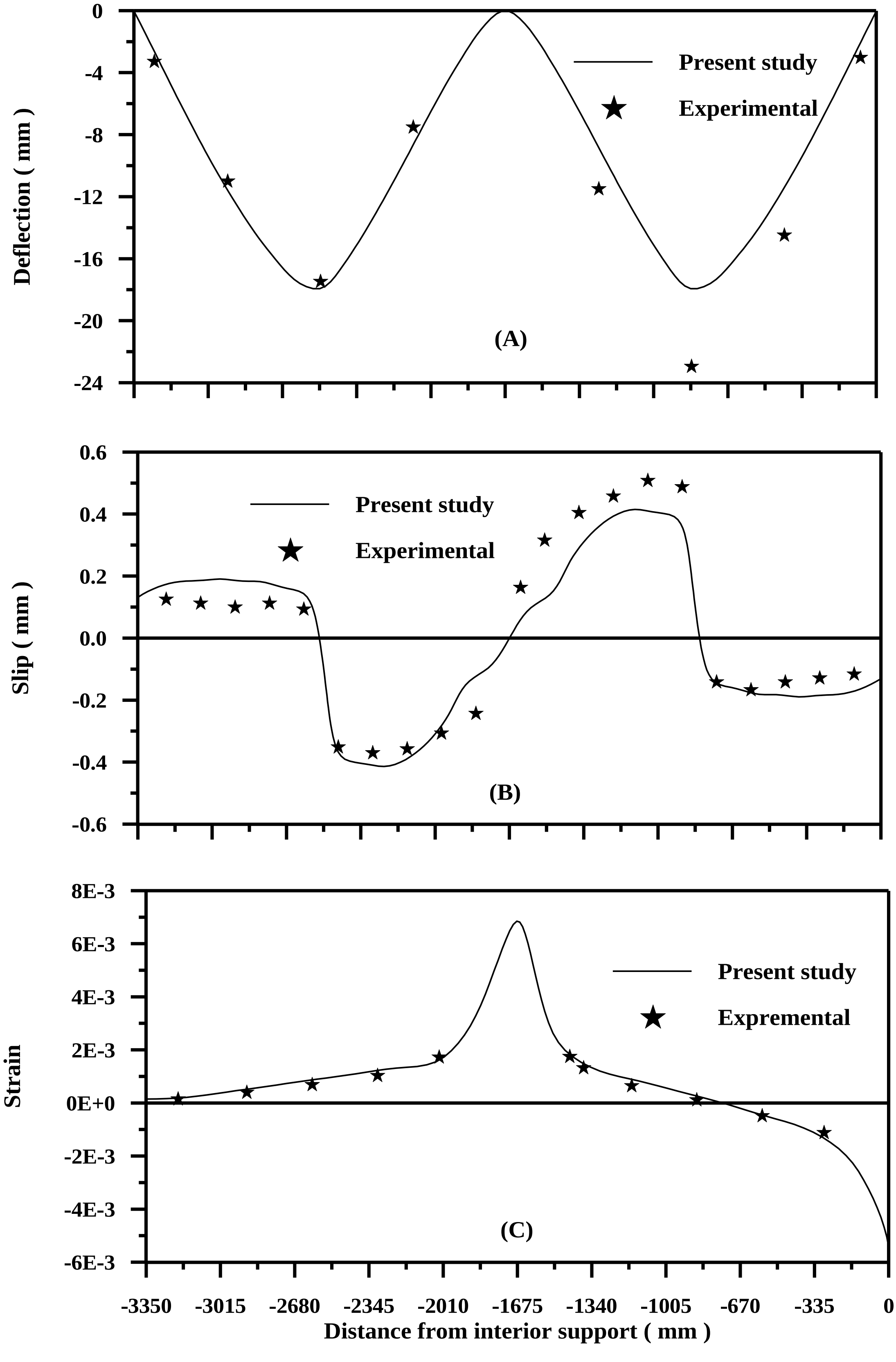

Figure (7) Comparison between computed and reference values [5] at 122 kN, A-Deflected shape, B-Slip distribution, C-Longitudinal strain of bottom flange

From **Figures (8)** and **(9)**, the failure load of the original beam was 140 kN and this equals to reference value, this failure load was simultaneous for beam and shear connector. At this load a plastic hinge was found at mid support and under point loads (beam failure by plastic hinge mechanism). Also the maximum slip at this load level was 1.393 mm and it is much closed to the failure value of stud connector (1.4 mm) [5]. One of the study cases the steel reinforcement especially at mid support (top) and under point loads (bottom) is doubled. **Figure (8)** shows an increase in the ultimate load value by percent 14.3%, and this increase gives low slip at load level of 140 kN, **Fig.(9)** shows the comparison between slip values at 140 kN for original beam, increasing of steel reinforcement, and increasing in flange width. The effect of increasing in steel reinforcement is very clear on slip **(Fig. (9))**.

Finally, by doubling flange width at plastic hinge zone the increase in the ultimate load value is very good (39.3%, **Fig.(8)**), also the slip is very little at 140 kN when compared with original values as shown in **Fig.(9)**.

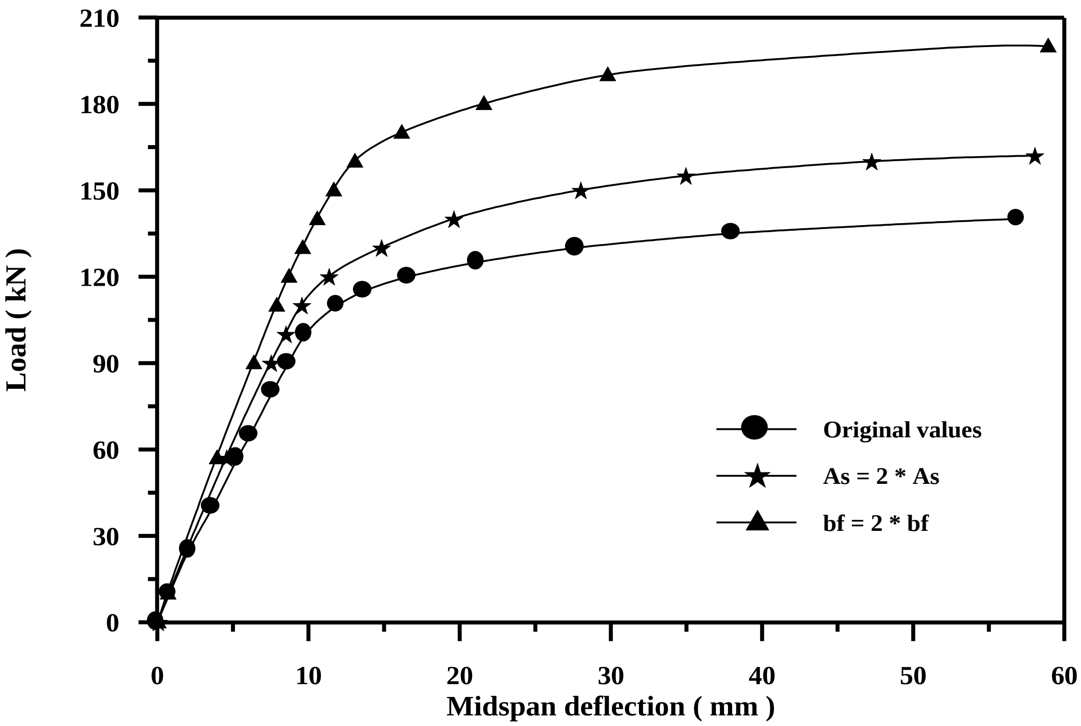

Figure (8) Load-deflection curve for original beam and increasing in steel reinforcement and flange width

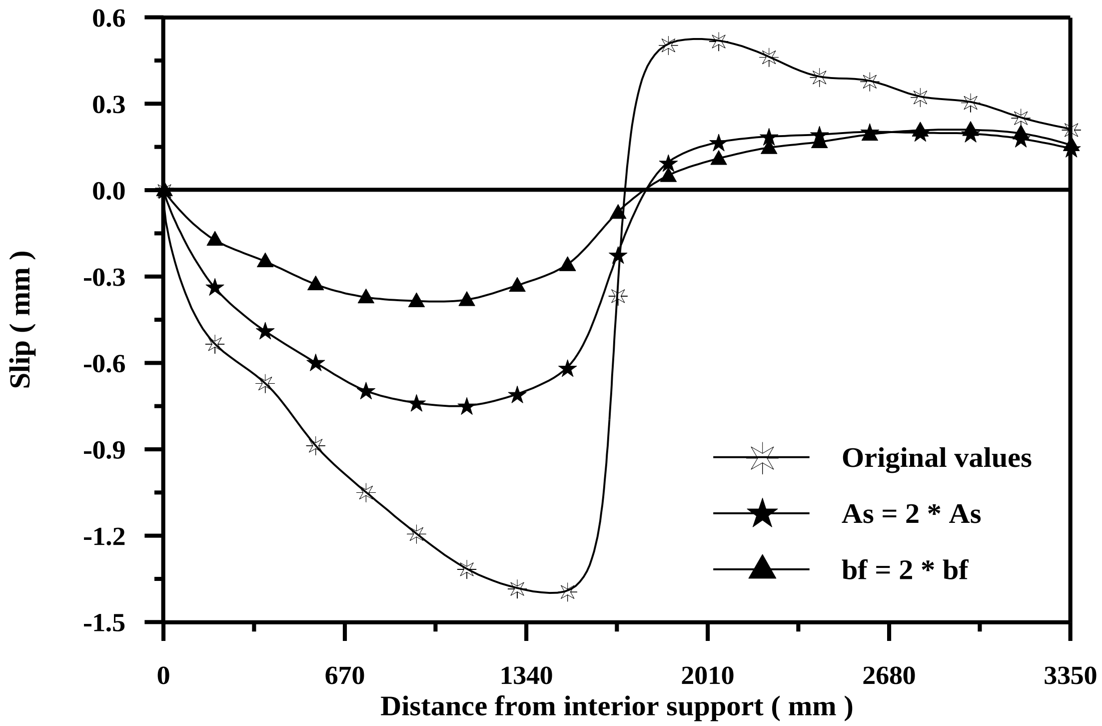

Figure (9) Slip distribution along the beam axis at load P=140 kN for original beam and increasing in steel reinforcement and flange width

7. Conclusions

The following points are the results concluded from the above discussion:-

1. The developed composite beam element gives good results when compared with experimental data.

2. The double increase in steel reinforcement at plastic hinge region will give increase in the ultimate load value by 14.3%.

3. The double increase in flange width of steel beam at plastic hinge region will give increase in the ultimate load value by 39.3%.

4. The increase in steel reinforcement and flange width at plastic hinge region will give decrease in slip value at the same load.

8. References

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Notations

$\varepsilon_{st}, \varepsilon_{c} \& \varepsilon_{sr}$: axial strains in steel beam, concrete slab and steel reinforcement, respectively.

$\varepsilon_{ost} \& \varepsilon_{oc}$: central strains in steel beam and concrete slab, respectively.

$\sigma_{st}, \sigma_{c} \& \sigma_{sr}$: stresses in steel beam, concrete slab and steel reinforcement, respectively.

A_{sr} : area of steel reinforcement.

[B] : strain-displacement matrix.

d_z : distance from steel reinforcement to reference axis of concrete slab.

f_a : uplift:

Fa : normal force.

$[K]^e$: element stiffness matrix.

Na & Nb : shape function.

Q : shear force.

S : slip.

$u_{st}, u_{c} \& u_{sr}$: axial displacements in steel beam, concrete slab and steel reinforcement, respectively.

$u_{ost} \& u_{oc}$: central axial displacements in steel beam and concrete slab, respectively.

$w_{st} \& w_{c}$: deflections of steel beam and concrete slab, respectively.

yc : distance from reference axis of concrete slab to interface.

ys : distance from reference axis of steel beam to interface.