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GENERALIZED HYERS-ULAM STABILITY OF MIXED TYPE ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN RANDOM NORMED SPACES

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Abstract. In this paper, using the direct and fixed point methods, we have established the generalized Hyers-Ulam stability of the following additive-quadratic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)] - [f(y)+f(-y)],$$

in random normed spaces.

1. Introduction

Random theory is a setting in which uncertainty arising from problems in various fields of science, can be modeled. It is a practical tool for handling situations where classical theories fail to explain. In fact, there are many cases in which the norm of a vector is impossible to be determined exactly. In these cases the idea of random norm seems to be useful.

Random theory has many application in several fields, for example, population dynamics, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, and so forth. The notion of random normed space goes back to Šherstnev in [26] and extended by Alsina, Schweizer and Sklar in [1]. One of the most important issues in the theory of functional equations concerning the famous Ulam stability problem is as follows: when is it true that a mapping satisfying a functional equation approximately, must be close to an exact solution of the given functional equation?

Ulam [31] in 1940 who was the first person speaking about the stability, proposed a stability problem between a group and a metric group. Hyers [12] was the first mathematician to present an affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. Gavruta [10] obtained generalized result of Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function.

The stability problems of a wide class of functional equations have been investigated by a number of authors, and there are many interesting results concerning

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this problem (see, e.g., [3, 11, 13, 14, 21, 27, 29, 30, 32]). Also by using fixed point method, the stability problems of several functional equations have been extensively investigated by number of authors (see, e.g., [5, 6, 7, 18, 22]).

The generalized Hyers-Ulam stability of different mixed type functional equations in random normed spaces, fuzzy normed spaces and non-Archimedean random normed spaces has been studied by many authors. For example, Park et al. [20] proved the Hyers-Ulam stability of the following additive-quadratic-cubic-quatric functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y)-6f(x) \\ &\quad + f(2y)+f(-2y)-4f(y)-4f(-y), \end{aligned} \tag{1.1}$$

in random normed spaces. Sheng, Saadati, and Sadeghi [28] proved the Hyers-Ulam stability of the following quadratic and additive functional equation

$$f(x+y)+f(x+z)+f(y+z)=f(x)+f(y)+f(z)+f(x+y+z), \tag{1.2}$$

in non-Archimedean random normed spaes.

In 2011 Mohamadi et al. [19] was proved and investigated the generalized Hyers-Ulam stability of the following additive-quadratic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 2f(x+y)+2f(-x-y)+2f(x-y) \\ &\quad + 2f(y-x)-4f(-x)-2f(x) \\ &\quad + f(2y)+f(-2y)-4f(y)-4f(-y) \end{aligned} \tag{1.3}$$

in random normed spaces via fixed point method. In this paper we present the generalized Hyers-Ulam stability of the following mixed type additive and quadratic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)]-[f(y)+f(-y)] \tag{1.4}$$

under arbitrary t-norms by direct method and under min t-norm by fixed point method in random normed spaces and provide an example. Our research is a generalization of the Ravi and Suresh work [24] to random normed spaces.

2. Preliminaries

Before giving the main result, we present some basic facts related to random normed spaces and some preliminary results. We say $f:\mathbb{R}\longrightarrow[0,1]$ is a distribution function if and only if it is a monotone, nondecreasing, left continuos, $\inf_{x\in\mathbb{R}} f(x)=0$ and $\sup_{x\in\mathbb{R}} f(x)=1$. By $\triangle^+$ we denote a collection of all distribution functions and D^+ is a subset of $\triangle^+$ consisting of all functions $f\in\triangle^+$ for which $\mathcal{L}^-f(+\infty)=1$, where $\mathcal{L}^-f(x)$ denotes the left limit of the function f at the point x, that is, $\mathcal{L}^-f(x)=\lim_{t\longrightarrow x^-} f(t)$. The space $\triangle^+$ is partially ordered by the usual point wise ordering of functions , i.e., $F\leq G$ if and only if $F(t)\leq G(t)$ for all $t\in\mathbb{R}$. The maximal element for $\triangle^+$ in this order is the distribution function H_0 given by

$$H_0(t):=\begin{cases}0 & \text{if } t\leq 0\\ 1 & \text{if } t>0\end{cases}.$$

It is obvious that $H_0\geq f$ for all $f\in D^+$.

Definition 2.1. [25, 8] *A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions:*

(1) *T is commutative and associative;*
(2) *T is continuous;*
(3) *$T(a, 1) = a$ for all $a \in [0, 1]$;*
(4) *$T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$.*

Typical examples of continuous t-norms are $T_p(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz t-norm).
Recall (see [11], [8]) that if T is a t-norm and x_n is a given sequene of numbers in $[0, 1]$, $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.
It is known [11] that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 2.2. [26] *A random normed space (briefly RN-space) is a triple (X, μ, T) where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:*

(1) *$\mu_x(t) = H_0(t)$ for all $t > 0$ iff $x = 0$;*
(2) *$\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;*
(3) *$\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.*

Definition 2.3. [17] *Let (X, μ, T) be a RN-space.*

(1) *A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.*
(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.*
(3) *An RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 2.4. [25] If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequnce such that $x_n \longrightarrow x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost every where.

Definition 2.5. [15] *Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if d satisfies*

(1) *$d(x, y) = 0$ if and only if $x = y$;*
(2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(3) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

We recall a fundamental result in fixed point theory.

Theorem 2.6. [9, 4] *Let (X, d) be a complet generalized metric spaces and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

3. Hyers-Ulam Stability of the additive-quadratic functioal equation (1.4) by direct method

One can easily show that an even mapping $f : X \longrightarrow Y$ satisfies equation (1.4) if and only if the even mapping $f : X \longrightarrow Y$ is a quadratic mapping, that is,

$$f(2x + y) + f(2x - y) = 2[f(x + y) + f(x - y)] + 4f(x) - 2f(y).$$

Also, one can easily show that an odd mapping $f : X \longrightarrow Y$ satisfies equation (1.4) if and only if the odd mapping $f : X \longrightarrow Y$ is an additive mapping, that is,

$$f(2x + y) + f(2x - y) = 2[f(x + y) + f(x - y)].$$

For a given mapping $f : X \longrightarrow Y$, we define

$$\begin{aligned} D_s f(x, y) :&= f(2x + y) + f(2x - y) - 2[f(x + y) + f(x - y)] \\ &- 2[f(x) + f(-x)] + [f(y) + f(-y)], \end{aligned}$$

for all $x, y \in X$ and $t > 0$.

In this section, using the direct method, we prove the generalized Hyers-Ulam stability of the additive -quadratic functional equation (1.4) in complete RN-spaces. Also, we present an illustrative example under the min t-norm.

Theorem 3.1. *Let X be a real linear space and (Y, μ, T) be a complete RN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{3.1}$$

for all $x, y \in X$ and $t > 0$, if

$$\lim_{j \to \infty} T_{i=1}^{\infty}(\phi_{2^{i+j-1}x,0}(2^{i+2j+1}t)) = 1, \tag{3.2}$$

and

$$\lim_{m \to \infty} \phi_{2^m x, 2^m y}(2^{2m}t) = 1, \tag{3.3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfies equation (1.4) and the inequality

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{2^{i-1}x,0}(2^{i+1}t), \tag{3.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (3.1) we get

$$\mu_{2f(2x)-8f(x)}(t) \geq \phi_{x,0}(t), \tag{3.5}$$

for all $x \in X$. Then we get

$$\mu_{\frac{f(2x)}{4} - f(x)}(t) \geq \phi_{x,0}(8t), \tag{3.6}$$

therefore,

$$\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}}(t) \geq \phi_{2^k x,0}(2^{2k+3}t), \tag{3.7}$$

that is

$$\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}}\left(\frac{t}{2^{k+1}}\right) \geq \phi_{2^k x,0}(2^{k+2}t), \tag{3.8}$$

for every $k \in N$, $t > 0$. As

$$1 > \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k},$$

by the triangle inequality it follows:

$$\begin{aligned}
\mu_{\frac{f(2^n x)}{2^{2n}} - f(x)}(t) &\geq \mu_{\frac{f(2^n x)}{2^{2n}} - f(x)}(\sum_{k=0}^{n-1} \frac{1}{2^{k+1}} t) \\
&\geq T_{k=0}^{n-1} \left(\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}}(\frac{1}{2^{k+1}} t) \right) \\
&\geqslant T_{k=0}^{n-1} (\phi_{2^k x, 0}(2^{k+2} t) \\
&= T_{i=1}^{n} \left(\phi_{2^{i-1} x, 0}(2^{i+1} t) \right), \qquad (3.9)
\end{aligned}$$

$x \in X$, $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$, we replace x with $2^j x$ and multiplying the left hand side of (3.9) by $\frac{2^{2j}}{2^{2j}}$,

$$\mu_{\frac{f(2^{n+j}x)}{2^{2(n+j)}} - \frac{f(2^j x)}{2^{2j}}}(t) \geq T_{i=1}^{n} \left(\phi_{2^{j+i-1} x, 0}(2^{i+2j+1} t) \right). \qquad (3.10)$$

Since the right hand side of the inequality (3.10) tends to 1 as i and j tend to infinity, the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(2^j x)}{2^{2j}},$$

for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $S : X \longrightarrow Y$ is an even mapping. Replacing x, y with $2^m x$ and $2^m y$, respectiveiy, in (3.1) then multiplying the right hand side by $\frac{2^{2m}}{2^{2m}}$, it follows that:

$$\mu_{\frac{1}{2^{2m}} D_s f(2^m x, 2^m y)}(t) \geq \phi_{2^m x, 2^m y}(2^{2m} t),$$

for all $x, y \in X$. Taking the limit as $m \to \infty$ we find that S satisfies (1.4), that is, S is a quadratic map. To prove (3.4) take the limit as $n \to \infty$ in (3.9).
Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a quadratic function r which satisfies (3.4) and equation (1.4). Therefore

$$\begin{aligned}
\mu_{r(x)-s(x)}(t) =& \mu_{r(x) - \frac{f(2^j x)}{2^{2j}} + \frac{f(2^j x)}{2^{2j}} - s(x)}(t) \\
&\geq T(\mu_{r(x) - \frac{f(2^j x)}{2^{2j}}}(\frac{t}{2}), \mu_{\frac{f(2^j x)}{2^{2j}} - s(x)}(\frac{t}{2})).
\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\mu_{r(x)-s(x)}(t) = 1$. Therefore $r = s$. $\square$

In Theorem 3.1 if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 3.2. *Let X be a real linear space and (Y, μ, T) be a complete RN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \qquad (3.11)$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{j \to \infty} T_{i=1}^{\infty} (\phi_{2^{i+j-1} x, 0}(2^{j+1} t)) = 1, \qquad (3.12)$$

and

$$\lim_{m\to\infty} \phi_{2^m x, 2^m y}(2^m t) = 1, \tag{3.13}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfies equation (1.4) and the inequality

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{2^{i-1}x,0}(2t), \tag{3.14}$$

for all $x \in X$ and $t > 0$.

Corollary 3.3. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{3.15}$$

for all $x \in X$, $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(1 - \frac{\|x\|}{4t + \|x\|}),$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.4. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, and $t > 0$ and $\varepsilon > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{2^{i+1}t}{2^{i+1}t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.5. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and let $L \geq 0$ and p be a real number with $p < 1$ and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\frac{2^{i+1}t}{2^{i+1}t + L2^{(i-1)p}\|x\|^p}),$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

In Corollary 3.5 if

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)L},$$

then the result is similar.

Example 3.6. *Let $(X, \|.\|)$ be a Banach algebra and*

$$\mu_x(t) = \begin{cases} 1 - \frac{\|x\|}{t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases},$$

for all $x, y \in X$ and $t > 0$. Let

$$\varphi_{x,y}(t) = \begin{cases} 1 - \frac{12(\|x\|+\|y\|)}{t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases}.$$

We note that $\varphi_{x,y}(t)$ is a distribution function and $\lim_{j\to\infty} \varphi_{2^j x, 2^j y}(2^{2j}t) = 1$ for all $x, y \in X$ and $t > 0$.
It is easy to show that (X, μ, T_M) is a RN-space. Indeed, $\mu_x(t) = 1$ $\forall t > 0 \Longrightarrow \frac{\|x\|}{t} = 0$ and hence $x = 0$ for all $x \in X$ and $t > 0$. Obviously, $\mu_{\lambda x}(t) = \mu_x(\frac{t}{\lambda})$ for all $x \in X$ and $t > 0$. Now let

$$1 - \frac{\|x\|}{t} \leq 1 - \frac{\|y\|}{s},$$

for all $x, y \in X$.

if $x = y$, we have $s \geq t$. Thus, otherwise, we have

$$\frac{\|x+y\|}{t+s} \leq \frac{\|x\|}{t+s} + \frac{\|y\|}{t+s} \leq 2\frac{\|x\|}{t+s} \leq \frac{\|x\|}{t}.$$

Then

$$1 - \frac{\|x+y\|}{t+s} \geq 1 - \frac{\|x\|}{t}$$

and so

$$\mu_{x+y}(t+s) \geq T_M(1 - \frac{\|x\|}{t}, 1 - \frac{\|y\|}{s}) = T_M(\mu_x(t), \mu_y(s)).$$

It is easy to see that (X, μ, T_M) is complete, for

$$\mu_{x-y}(t) = 1 - \frac{\|x-y\|}{t} \quad \forall x, y \in X$$

and $t > 0$ and $(X, \|.\|)$ is complete. Define a mapping $f : X \longrightarrow X$ by $f(x) = x^2 + \|x\|x_0$ for all $x \in X$, where x_0 is a unite vector in X. A simple computation shows that

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y)\| = \\ &\quad |\|2x+y\| + \|2x-y\| - 2\|x+y\| - 2\|x-y\| - 4\|x\| + 2\|y\|| \\ &\quad \leq 12(\|x\| + \|y\|), \end{aligned}$$

for all $x, y \in X$. Hence $\mu_{D_s f(x,y)}(t) \geq \phi_{x,y}(t)$ for all $x, y \in X$ and $t > 0$. Fix $x \in X$ and $t > 0$, then it follows that,

$$(T_M)_{i=1}^{\infty}\left(\phi_{2^{i+j-1}x,0}(2^{2j+i+1)}t)\right) = 1 - \frac{12\|x\|}{2^{j+2}t},$$

for all $x \in X$, $n \in \mathbb{N}$ and $t > 0$. Hence

$$\lim_{j\to\infty}(T_M)_{i=1}^{\infty}\left(\varphi_{2^{i+j-1}x,0}(2^{1+2j+i)}t)\right) = 1,$$

for all $x \in X$ and $t > 0$. Thus, all the conditions of Theorem 3.1 hold. Since

$$(T_M)_{i=1}^{\infty}\left(\phi_{2^{i-1}x,0}(2^{1+i}t)\right) = 1 - \frac{12.2^{i-1}\|x\|}{2^{i+1}t} = 1 - \frac{3\|x\|}{t},$$

for all $x \in X$ and $t > 0$. We can deduce that $S(x) = x^2$ is the unique quadratic mapping $S : X \longrightarrow X$ such that

$$\mu_{f(x)-s(x)}(t) \geq 1 - \frac{3\|x\|}{t},$$

for all $x \in X$ and $t > 0$.

Using the idea of Theorem 3.2, the following corollaries can be proved.

Corollary 3.7. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, and $t > 0$ and $\varepsilon > 0$. Then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{2t}{2t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.2. □

Corollary 3.8. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and let $L \geq 0$ and p be a real number with $p \leq 0$ and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\frac{2t}{2t + L2^{(i-1)p} \parallel x \parallel^p}),$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.2. □

In Corollary 3.8 if

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)L},$$

then the result is similar.

4. Hyers-Ulam stability of the additive-quadratic functional equation (1.4) by fixed point method

In this section, using the fixed point method, we prove the generalized Hyers–Ulam stability of the additive-quadratic functional equation (1.4) in complete RN-spaces.

Theorem 4.1. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 4,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{4.1}$$

for all $x, y \in X$, and $t > 0$. Then there exists a unique quadratic mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(4-\alpha)t), \tag{4.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. Let $y = 0$ in (4.1); we get

$$\mu_{2f(2x)-8f(x)}(t) \geq \phi_{x,0}(t), \tag{4.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \phi_{x,0}(8t). \tag{4.4}$$

Consider the set

$$E := \{g : X \to Y : \ g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \mathtt{inf}\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(8t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [16, Lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(2x)}{4}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{4}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g, h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(8t)$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{4}) &= \mu_{\frac{g(2x)}{4}-\frac{h(2x)}{4}}(\frac{\epsilon\alpha t}{4}) \\ &= \mu_{g(2x)-h(2x)}(\alpha\varepsilon t) \\ &\geq \phi_{2x,0}(\alpha 8t),\end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 4,$$

we have

$$\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{4}) \geq \phi_{x,0}(8t),$$

that is,

$$d_G(g,h) < \epsilon \Longrightarrow d_G(Jg,Jh) < \frac{\alpha}{4}\epsilon.$$

This means that

$$d_G(Jg,Jh) < \frac{\alpha}{4} d_G(g,h),$$

for all $g, h \in E$. Next, from

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \phi_{x,0}(8t),$$

it follows that $d_G(f,Jf) \leq 1$. Using Theorem 2.6, we show the existence of a fixed point of J, that is, the existence of a mapping $g : X \longrightarrow Y$ such that $g(2x) = 4g(x)$ for all $x \in X$. For all $x \in X$ and $t > 0$,

$$d_G(u,v) < \epsilon \Longrightarrow \mu_{u(x)-v(x)}(t) \geq \phi_{x,0}(\frac{8t}{\epsilon}).$$

Since $d_G(J^n f, g) \longrightarrow 0$, then $\lim_{m\longrightarrow\infty} \frac{f(2^n x)}{4^n} = g(x)$ for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $g : X \longrightarrow Y$ is an even mapping.

Also from

$$d_G(f,g) \leq \frac{1}{1-L} d(f,Jf),$$

for all $g, h \in E$, we have $d_G(f,g) \leq \frac{1}{1-\frac{\alpha}{4}}$, and it immediately follows that

$$\mu_{g(x)-f(x)}(\frac{4}{4-\alpha}t) \geqslant \phi_{x,0}(8t),$$

for all $x \in X$ and $t > 0$. This means that

$$\mu_{g(x)-f(x)}(t) \geqslant \phi_{x,0}(2(4-\alpha)t),$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0,\infty)$ satisfying

$$\mu_{g(x)-f(x)}(Ct) \geqslant \phi_{x,0}(8t),$$

for all $x \in X$ and $t > 0$. This completes the proof. □

In Theorem 4.1 if f is an odd mapping, then the following Theorem can be proved similarly.

Theorem 4.2. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 2,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{4.5}$$

for all $x, y \in X$, and $t > 0$. Then there exists a unique an additive mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(2-\alpha)t), \tag{4.6}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Corollary 4.3. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{4.7}$$

for all $x \in X$, $t > 0$. Then there exists a unique quadratic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(4-\alpha)t + \|x\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $2 \leq \alpha < 4$ and so we get the desired result. □

Corollary 4.4. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique quadratic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(4-\alpha)t}{2(4-\alpha)t + \varepsilon\|x_0\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x \in X$, and $t > 0$ in Theorem 4.1. Then we can choose $1 \leq \alpha < 4$ and so we get the desired result. □

Corollary 4.5. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $p \leq 1$. Then there exists a unique quadrtic mapping $s : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(4-\alpha)t}{2(4-\alpha)t + \theta\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $2^p \leq \alpha < 4$ and so we get the desired result. □

Corollary 4.6. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and let $z_0 \geq 0$ and p be a real number with $p < 1$ and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)z_0},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $s : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(4-\alpha)t}{2(4-\alpha)t + z_0\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)z_0},$$

for all $x, y \in X$ and $t > 0$, in theorem 4.1. Then we can choose $2^{2p} \leq \alpha < 4$ and so we get the desired result. □

Using the idea of Theorem 4.2, the following corollaries can be proved.

Corollary 4.7. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{4.8}$$

for all $x \in X$, $t > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(2-\alpha)t + \|x\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x \in X$ and $t > 0$ in Theorem 4.2. Then we can choose $\alpha = 2$ and so we get the desired result. □

Corollary 4.8. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(2-\alpha)t}{2(2-\alpha)t + \varepsilon\|x_0\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x \in X$, and $t > 0$ in Theorem 4.2. Then we can choose $1 \leq \alpha < 2$ and so we get the desired result. □

Corollary 4.9. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $p < 1$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(2-\alpha)t}{2(2-\alpha)t + \theta\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$ in Theorem 4.2. Then we can choose $2^p \leq \alpha < 2$ and so we get the desired result. □

Corollary 4.10. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and let $z_0 \geq 0$ and p be a real number with $p \leq 0$ and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p \|y\|^p) z_0},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(2-\alpha)t}{2(2-\alpha)t + z_0\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p \|y\|^p) z_0},$$

for all $x, y \in X$ and $t > 0$, in Theorem 4.2. Then we can choose $2^{2p} \leq \alpha < 2$ and so we get the desired result. □

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