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Generalized stability results on certain functional equations in random normed spaces

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GENERALIZED STABILITY RESULTS ON CERTAIN FUNCTIONAL EQUATIONS IN RANDOM NORMED SPACES

By

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To my beloved country, Iraq

To my mother and father with all love and respect

To all outstanding teachers with all the appreciation and pride

To my husband and my daughter Hawraa with great thanks and gratitude

To all those who helped me, even if simple

Give them this humble work.

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Abstract

In this thesis, we prove stability of a sextic functional equation and additive-quadratic functional equations in random normed spaces, intuitionistic random normed space and non-Archimedean random normed space via direct method under arbitrary t-norms. Also stability for these functional equations will be proved in random normed spaces and intuitionistic random normed spaces via fixed point method.

Keywords: Random normed space, Intuitionistic random normed space, non-Archimedean random normed space, Fixed point, Sextic functional equation, Additive mapping, Quadratic mapping.

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Preface

In the fall of 1940, S.M. Ulam [70] gave a wide-ranging talk before a mathematical colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those he asked a question concering the stability of homomorphisms: given a group G_1, a metric group G_2 with the metric $d(.,.)$, and a positive number ε, dose there exist $\delta > 0$ such that, if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \longrightarrow G_2$ exists with $d(f(x), h(x)) \leq \varepsilon$ for all $x \in G_1$? If the answer is affirmative, we say that the functional equation is stable.

Several mathematicians have dealt with special cases as well as generalizations of Ulam's problem. Hyers [35] provided a partial solution to Ulam's problem for the case of approximately additive mappings in which G_1 and G_2 are Banach spaces with $\varepsilon = \delta$.

Taking this famous result into consideration, the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability on (E_1, E_2) if for every function $f : E_1 \longrightarrow E_2$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \delta,$$

for some $\delta \geq 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \longrightarrow E_2$ such that $f - A$ is bounded on E_1.

In 1968, Forti [26] proved that Hyers' proof remains unchanged if G_1 is an Abelian semigroup. In 1950, Aoki [8] addressed the Hyers' stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$. In 1978, Th.M. Rassias [55] formulated and proved the stability theorem for the linear mapping between Banach spaces E_1 and E_2 subject to the continuity of $f(tx)$ with respect to $t \in \mathbb{R}$ for each fixed $x \in E_1$. This Rassias' theorem implies Aoki's theorem as a special case. Let $f : E_1 \longrightarrow E_2$ be a function between Banach spaces. If f satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$, $0 \leq p \leq 1$ and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \longrightarrow E_2$ such that $\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$ for each $x \in E_1$. If in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear. In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [69] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [19] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 1990, Th.M. Rassias [56] observed that the proof of his stability theorem also holds true for $p < 0$. In 1991, Gajda [27] showed that the proof of Rassias' Theorem can be proved also for the case $p > 1$ by just replacing n by $-n$ in

$$g(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

In 2002, Czerwik [22] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

On the other hand, random theory is a setting in which uncertainty arising from problems in various fields of science, can be modeled. It is a practical tool for handling situations where classical theories fail to explain. In fact, there are many cases in which the norm of a vector is impossible to be determined exactly. In these cases the idea of random norm seems to be useful. Random theory has many application in several fields, for example, population dynamics, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, and so forth. The notion of random normed space goes back to Šherstnev in [67] and extended by Alsina, Schweizer and Sklar in [6].

In the sequel, several mathematicians have extensively studied stability theorems for several kinds of functional equations in various spaces. For example, in 2008, Baktash et al. [11] proved the stability theorem for this quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y),$$

in random normed spaces. In 2009, the general solution and the stability result for the following quadratic-quartic functional equation

$$f(2x+y)+f(2x-y)=4[f(x+y)+f(x-y)]+2[f(2x)-4f(x)]-6f(y),$$

was proved by M. Eshaghi Gordji, M. Bavand Savadkouhi and Choonkil Park [29]. In 2011, the stability problem for a cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x),$$

was proved by Saadati, Vaezpour and Park [62] in intuitionistic random normed spaces. In 2011, J. M. Rassias et al. [54] proved the stability for quartic functional equation

$$\begin{aligned}16f(x+4y)+f(4x-y)&=306[9f(x+\frac{y}{3})+f(x+2y)]\\&+136f(x-y)-1394f(x+y)+425f(y)-1530f(x),\end{aligned}$$

in non-Archimedean random normed spaces. In 2012, Afshin Erami et al. [25] proved the generalized Hyers-Ulam stability of the following cubic functional equation:

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y),$$

in random normed spaces via fixed point method. In 2014, J. Vahidi, S. J. Lee, F. Fallah, and R. Ahmadi [71] proved the stability of some functional equations in the random normed spaces under arbitrary t-norms. In 2016, Kim et al. [39] investigated stability of the general cubic functional equation

$$f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)=2k(k^2-1)f(y),$$

for fixed $k\in\mathbb{Z}^+$ with $k\geq 0$ via direct and fixed point methods in random normed spaces. In 2017, Yang-Hi Lee and Soon-Mo Jung [43] prove stability theorem for a class of functional equations including quadratic-additive functional equations. There are more examples which can be found in [1, 6, 11, 17, 25, 28, 29, 30, 39, 41, 43, 44, 45, 54, 58, 61, 63, 65, 66, 68, 73].

This thesis includes four chapters as follows. In Chpater 1, we will recall some introductory facts which are needed in the subsequent chapters. In Chapter 2, we prove Stability of a sextic functional eqution and an additive-quadratic funcional equation in random normed spaces via direct method under arbitrary t-norms and via fixed point method undrt min t-norm. In Chapter 3, we prove stability of the same sextic functional eqution and an additive-quadratic funcional equations in intuitionistic random normed spaces via direct and fixed point methods. In chapter 4, we prove stability of the same functional equtions in non-Archimedean random normed spaces via direct method. Finally, we would like to mention that the papers resulted from this thesis are:

1. Sh. Alshabbani, S.M. Vaezpour, R. Saadati, *Generalized Hyers–Ulam stability of sextic functional equation in random normed spaces*, J. Comput. Anal. Appl., **24** (2018), 370-381.

2. Sh. Alshabbani, S.M. Vaezpour, R. Saadati, *Generalized Hyers-Ulam stability of mixed type additive-quadratic functional equation in random normed spaces*, Journal of Mathematical Analysis, Accepted.

3. Sh. Alshabbani, S.M. Vaezpour, R. Saadati, *Generalized stability of an additive-quadratic functional equation in various random normed spaces*, submitted

4. Sh. Alshabbani, S.M. Vaezpour, R. Saadati, *Stability of the sextic functional equation in various spaces*, submitted

Chapter 1

Preliminaries

In this chapter, we recall definitions of t-norms, random normed spaces, intuitionistic random normed spaces, non-Archimedean random normed spaces. Also we will recall fixed point theorems in the last section. We shall adopt usual terminology, notation and conventions of the theory of random normed spaces, as in [6, 7].

1.1 t-norms

Definition 1.1.1 ([18, 32, 64])**.** A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions:

1. T is commutative and associative;
2. T is continuous;
3. $T(a,1) = a$ for all $a \in [0,1]$;
4. $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$.

Definition 1.1.2 ([32, 64])**.** If T is a t-norm, then its dual t-conorm $S : [0,1] \times [0,1] \longrightarrow [0,1]$ is given by

$$S(x,y) = 1 - T(1-x, 1-y).$$

It is obvious that a t-conorm is a commutative, associative, and monotone operation on [0,1] with unit element 0.

Example 1.1.1. ([32, 64])

The following are the four basic t-norms together with their dual t-conorms:

1. Minimum T_M and maximum S_M given by

$$T_M(x,y) = \min(x,y),$$

$$S_M(x,y) = \max(x,y).$$

2. Product T_P and probabilistic sum S_P given by

$$T_P(x,y) = x.y,$$

$$S_P(x,y) = x + y - x.y.$$

3. Lukasiewice t-norm T_L and Lukasiewicz t-conorm S_L given by

$$T_L(x,y) = \max(x+y-1,0),$$

$$S_L(x,y) = \min(x+y,1).$$

4. Weakest t-norm (drastic product) T_D and strongest t-conorm S_D given by

$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y)=1, \\ 0 & otherwise, \end{cases}$$

$$S_D(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y)=0, \\ 1 & otherwise. \end{cases}$$

If, for any two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, T_2 is stronger than T_1. Also we can prove the following ordering for four basic t-norm:

$$T_D < T_L < T_P < T_M.$$

Proposition 1.1.1. ([18])

1. The minimum t-norm T_M is the only t-norm satisfying $T(x,x) = x$ for all $x \in (0,1)$;
2. The weakest t-norm T_D is the only t-norm satisfying $T(x,x) = 0$ for all $x \in (0,1)$.

Proposition 1.1.2. *A t-norm T is continuous if and only if it is continuous in its first component, i.e., for all $y \in [0,1]$, if the one place function*

$$T(.,y) : [0,1] \longrightarrow [0,1],\ x \mapsto T(x,y),$$

is continuous. For example, the minimum T_M and Lukasiewicz t-norm T_L are continuous.

If T is a t-norm, then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \geq 1$. A t-norm T is said to be of Hadžić-type (denoted by $T \in \mathcal{H}$) if the family $\{x_T^{(n)}\}_{n\in N}$ is equicontinuous at $x = 1$, that is, for any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$x > 1 - \delta \Longrightarrow x_T^{(n)} > 1 - \varepsilon \quad \forall n \geq 1.$$

The t-norm T_M is a trivial example of Hadžić type but T_p is not of Hadžić type (see [18, 32]).

Other important triangular norms are (see [33]):

1. the Sugeno-Weber family $\{T_\lambda^{SW}\}_{\lambda\in[-1,\infty]}$, defined by $T_{-1}^{SW} = T_D$, $T_\infty^{SW} = T_P$ and
$$T_\lambda^{SW}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right), \quad \lambda \in (-1,\infty).$$
2. the Domby family $\{T_\lambda^{D}\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and
$$T_\lambda^{D}(x,y) = \frac{1}{1 + \left((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda\right)^{\frac{1}{\lambda}}}, \quad \lambda \in (0,\infty).$$
3. the Aczel-Alsina family $\{T_\lambda^{AA}\}_{\lambda\in[0,\infty]}$,defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and
$$T_\lambda^{AA}(x,y) = e^{-\left(|\log x|^\lambda + |\log y|^\lambda\right)^{\frac{1}{\lambda}}}, \quad \lambda \in (0,\infty).$$

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for $(x_1, x_2, ..., x_n) \in [0,1]^n$ the value $T(x_1, x_2, ..., x_n)$ defined by

$$T_{i=1}^0 x_i = 1, \quad T_{i=1}^n x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) = T(x_1, x_2, ..., x_n).$$

T can also be extended to a countable operation taking for any sequence $\{x_n\}_{n\in N}$ in $[0,1]$. Moreover,

$$T_{i=1}^\infty x_i = \lim_{n\longrightarrow\infty} T_{i=1}^n x_i. \tag{1.1.1}$$

The limit on the right-hand side of (1.1.1) exists since the sequence $\{T_{i=1}^{n} x_i\}_{n\in N}$ is nonincreasing and bounded from below.

Proposition 1.1.3. ([32, 33])

1. for $T \geq T_L$ the following implication hold:

$$\lim_{n\longrightarrow\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

2. If T is of Hadžić-type, then $\lim_{n\longrightarrow\infty} T_{i=1}^{\infty} x_{n+i} = 1$ for every sequence $\{x_n\}_{n\in N}$ in $[0,1]$ such that

$$\lim_{n\longrightarrow\infty} x_n = 1.$$

3. If $T \in \{T_\lambda^{AA}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then

$$\lim_{n\longrightarrow\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)^{\alpha} < \infty.$$

4. If $T \in \{T_\lambda^{SW}\}_{\lambda\in[-1,\infty)}$, then

$$\lim_{n\longrightarrow\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty.$$

1.2 Random normed spaces

Let $\triangle^+$ denote the space of all distribution functions, that is, the space of all mappings $f : \mathbb{R}\cup\{-\infty,+\infty\} \longrightarrow [0,1]$ such that f is monotone, nondecreasing, left continuous, $f(0) = 0$ and $f(+\infty) = 1$. D^+ is a subset of $\triangle^+$ consisting of all functions $f \in \triangle^+$ for which $\mathcal{L}^- f(+\infty) = 1$, where $\mathcal{L}^- f(x)$ denotes the left limit of the function f at the point x, that is, $\mathcal{L}^- f(x) = \lim_{t\longrightarrow x^-} f(t)$. The space $\triangle^+$ is partially ordered by the usual point wise ordering of functions , i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $\triangle^+$ in this order is the distribution function H_0

given by

$$H_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Example 1.2.1. *The function* $G(t)$ *defined by*

$$G(t) = \begin{cases} 0 & \textit{if } t \leq 0, \\ 1 - e^{-t} & \textit{if } t > 0, \end{cases}$$

is a distribution function. Since $\lim_{t \longrightarrow \infty} G(t) = 1, G \in D^+$.

Example 1.2.2. *The function* $F(t)$ *defined by*

$$F(t) = \begin{cases} 0 & \textit{if} \quad t \leq 0, \\ t & \textit{if} \quad 0 \leq t \leq 1, \\ 1 & \textit{if} \quad 1 \leq t. \end{cases}$$

is a distribution function. Since $\lim_{t \longrightarrow \infty} F(t) = 1, F \in D^+$. *See*[18, 32].

Definition 1.2.1 ([67])**.** A random normed space (briefly RN-space) is a triple (X, μ, T) where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

1. $\mu_x(t) = H_0(t)$ for all $t > 0$ iff $x = 0$;
2. $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;
3. $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Example 1.2.3. *Let* $(X, \|.\|)$ *be a linear normed space. Define a mapping*

$$\mu_x(t) = \begin{cases} 0 & \textit{if} \quad t \leq 0, \\ \frac{t}{t+\|x\|} & \textit{if} \quad t > 0. \end{cases}$$

Then (X, μ, T_p) *is a random normed space. Also* (X, μ, T_M) *is a random normed space.*

Example 1.2.4. *Let* $(X, \|.\|)$ *be a linear normed space. Define a mapping*

$$\mu_x(t) = \begin{cases} 0 & \textit{if} \quad t \leq 0, \\ e^{-(\frac{\|x\|}{t})} & \textit{if} \quad t > 0. \end{cases}$$

Then (X, μ, T_p) *is a random normed space.*

Definition 1.2.2. Let (X, μ, T) be an RN-space. We define the open ball $B_x(r, t)$ and the closed ball $B_x[r, t]$ with center $x \in X$ and radius $0 < r < 1$ for all $t > 0$ as follows:

$$B_x(r, t) = \{y \in X : \mu_{x-y}(t) > 1 - r\},$$

$$B_x[r, t] = \{y \in X : \mu_{x-y}(t) \geq 1 - r\},$$

respectively.

Theorem 1.2.1 ([18])**.** *Let (X, μ, T)be an RN-space. Every open ball $B_x(r, t)$ is open set.*

Different kinds of topologies can be introduced in a random normed space [64]. The (r, t)-topology is introduced by a family of neighborhoods

$$\{B_x(r, t)\}_{x \in X, t > 0, r \in (0,1)}.$$

In fact, every random norm μ on X generates a topology ($(r, t) - topology$) on X which has as a base the family of open sets of the form

$$\{B_x(r, t)\}_{x \in X, t > 0, r \in (0,1)}.$$

Theorem 1.2.2 ([18])**.** *Every RN-space (X, μ, T) is a Hausdorff space.*

Definition 1.2.3 ([18])**.** Let (X, μ, T) be an RN-space. A subset A of X is said to be R-bounded if there exist $t > 0$ and $r \in (0, 1)$ such that $\mu_{x-y}(t) > 1 - r$ for all $x, y \in A$.

Lemma 1.2.3. ([18]) If (X, μ, T) is an RN-space, then we have

1. The function $(x, y) \longrightarrow x + y$ is continuous;
2. The function $(\alpha, x) \longrightarrow \alpha x$ is continuous.

Theorem 1.2.4 ([18])**.** *Every compact subset A of an RN-space (X, μ, T) is R-bounded.*

Definition 1.2.4 ([45])**.** Let (X, μ, T) be an RN-space. Then

1. A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$, whenever $n \geq N$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$, whenever $n \geq m \geq N$.
3. An RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.2.5 ([64]). *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequnce such that $x_n \longrightarrow x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost every where.*

1.3 Intuitionistic random normed spaces

Definition 1.3.1 ([18, 54, 62]). A non-measure distribution function is a function $\nu : \mathbb{R} \longrightarrow [0, 1]$ which is non-increasing, right continuous, $inf_{x\in\mathbb{R}}\nu(x) = 1$ and $sup_{x\in\mathbb{R}}\nu(x) = 0$. We denote by B the collection of all non-measure distribution functions, and by G a special element of B defined by

$$G(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \longrightarrow B$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

Lemma 1.3.1. ([18]). Define the set L^* and the operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2, x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice ([59, 60]). We denote the units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

Definition 1.3.2 ([62]). A triangular norm (t-norm) on L^* is a mapping $\tau : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

1. $\forall x \in L^*$, $\tau(x, 1_{L^*}) = x$ (boundary condition);
2. $\forall (x, y) \in (L^*)^2$, $\tau(x, y) = \tau(y, x)$ (commutativity);
3. $\forall (x, y, z) \in (L^*)^3$, $\tau(x, \tau(y, z)) = \tau(\tau(y, x), z)$ (associativity);

4. $\forall (x, x', y, y') \in (L^*)^4, x \leq_{L^*} x', y \leq_{L^*} y' \Longrightarrow \tau(x, y) \leq_{L^*} \tau(x', y')$ (monotonicity).

Definition 1.3.3 ([62])**.** A continuous t-norm τ on L^* is said to be continuous t-representable if there exists a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\tau(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\}),$$

and

$$M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\}),$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ are continuous t-representable.

Definition 1.3.4 ([18])**.** Let μ and ν be measure and non-measure distribution functions from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$, where X is a real vector space. The triple $(X, \rho_{\mu,\nu}, \tau)$ is said to be an intuitionistic random normed spaces (briefly IRN-spaces) if X is a vector space, τ is a continuous t-representable, and $\rho_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \longrightarrow L^*$ satisfying the folowing conditions: for all $x, y \in X$ and $t, s > 0$,

1. $\rho_{\mu,\nu}(x, 0) = 0_{L^*}$;
2. $\rho_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;
3. $\rho_{\mu,\nu}(\alpha x, t) = \rho_{\mu,\nu}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
4. $\rho_{\mu,\nu}(x + y, t + s) \geq_{L^*} \tau(\rho_{\mu,\nu}(x, t), \rho_{\mu,\nu}(y, s))$.

In this case, $\rho_{\mu,\nu}$ is called an intuitionistic random norm. Here $\rho_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t))$.

Example 1.3.1 ([18])**.** *Let $(X, \|.\|)$ be a normed space. Let $\tau(a, b) = (a_1 b_1, min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be measure and non-measure distribution functions defined by*

$$\rho_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = (\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}),$$

$\forall t \in \mathbb{R}$. *Then $(X, \rho_{\mu,\nu}, \tau)$ is an IRN-space.*

Definition 1.3.5 ([54, 62]). A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \longrightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by $N_s(x) = 1 - x$, $\forall x \in [0, 1]$.

Definition 1.3.6 ([62]). Let $(X, \rho_{\mu,\nu}, \tau)$ be an IRN-space.

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ denoted by $(\{x_n\} \stackrel{\rho_{\mu,\nu}}{\longrightarrow} x)$ if, $\rho_{\mu,\nu}(x_n - x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for every $t > 0$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer $n_0 \in N$ such that $\rho_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\epsilon), \epsilon)$ $\forall n, m \geq n_0$ where N_s is a standard negator.
3. An IRN-space $(X, \rho_{\mu,\nu}, \tau)$ is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

1.4 Non-Archimeadean random normed spaces

By a non-Archimedean field we mean a field $\mathcal{K}$ equipped with a function (valuation) $|.|$ from $\mathcal{K}$ in to $[0, \infty)$ such that

1. $|r| = 0$ if and only if $r = 0$;
2. $|rs| = |r||s|$;
3. $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$.

clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. By the trivial valuation, we mean the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean nontrivial valuation $|.|$, that is, there exists $a_0 \in \mathcal{K}$ such that $|a_0|$ is not in $\{0, 1\}$.

The most important examples of non-Archimedean spaces are P-adic numbers. In

1897, Hensel [34] discovered the P-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p.

For any nonzero rational number x, there exsits a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}P^{n_x}$, where a and b are integers not divisible by P. Then $|x|_P := P^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_P$ is denoted by $\mathbb{Q}_p$, which is called the P-adic number field.

A function $\|.\| : X \to [0, \infty)$ is called a non-Archimedean if it satisfies the following conditions:

1. $\|x\| = 0$ if and only if $x = 0$;
2. for any $r \in \mathcal{K}$, $x \in X$, $\|rx\| = |r|\|x\|$;
3. the strong triangle inequality (ultrametric), namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

Then $(X, \|.\|)$ is called a non-Archimedeam normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\}$$

for all $n, m \geq 1$ with $n > m$, a sequence $\{x_n\}$ is a Cauchy sequence in X if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

Definition 1.4.1 ([18, 65])**.** A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a Linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

1. $\mu_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
2. $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;
3. $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$. It is easy to see that, if (3) holds, then so is
4. $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

Example 1.4.1. *As a classical example, if $(X, \|.\|)$ is a non-Archimedean normed linear space, then the triple (X, μ, T_M), where*

$$\mu_x(t) = \begin{cases} 0 & \text{if} \quad t \leq \|x\|; \\ 1 & \text{if} \quad t > \|x\|, \end{cases}$$

is a non-Archimedean RN-space.

Example 1.4.2. *Let $(X, \|.\|)$ be a non-Archimedean normed linear space. Define*

$$\mu_x(t) = \frac{t}{t + \|x\|},$$

for all $x \in X$ and $t > 0$. Then (X, μ, T_M) is a non-Archimedean RN-space.

Definition 1.4.2 ([18, 65])**.** Let (X, μ, T) be a non-Archimedean RN-space. Let $\{x_n\}$ be a sequence in X.

1. The sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that
$$\lim_{n \to \infty} \mu_{x_n - x}(t) = 1,$$
for $t > 0$. In this case, the point x is called the limit of the sequence $\{x_n\}$.
2. The sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$ and $p > 0$
$$\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon.$$
3. If each Cauchy sequence in X is convergent, then the random space is said to be complete and the non-Archimedean RN-space (X, μ, T) is called a non-Archimedean random Banach space.

Remark 1.4.1 ([18]). Let (X, μ, T_M) be a non-Archimedean RN-space. Then we have

$$\mu_{x_{n+P} - x_n}(t) \geq \min\{\mu_{x_{n+j+1} - x_{n+j}}(t) : j = 0, 1, 2, \cdots, P - 1\}.$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence in X if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\mu_{x_{n+1} - x_n}(t) > 1 - \varepsilon.$$

1.5 Fixed point theorems

The Banach fixed point theorem (also known as the Banach contraction principle) is an important tool in the theory of metric spaces because it guarantees the existence and uniqueness of fixed points of certain self mappings of metric spaces and provides a constructive method to find those fixed points. The theorem is named after Banach (1892-1945) and was first stated by him in 1922.

Theorem 1.5.1 (Banach [12])**.** *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a contraction, i.e., there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$. Then there exists a unique $a \in X$ such that $Ta = a$. Moreover, for all $x \in X$,

$$\lim_{n \longrightarrow \infty} T^n x = a$$

and, in fact, for all $x \in X$,

$$d(x, a) \leq \frac{1}{1 - \alpha} d(x, Tx).$$

Definition 1.5.1 ([64])**.** Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if it satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We now introduce one of the fundamental results of the fixed point theory.

Theorem 1.5.2. ([15, 23]) Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that

1. $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
3. y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
4. $d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

Chapter 2

Stability of certain functional equations in random normed spaces

In this chapter, we prove the stability of functional equations in random normed spaces under arbitrary t-norms via direct method and under min t-norm via fixed point method. It is necessary to mention that one of the results of this chapter has published in Ref. [2] and the other result in Ref. [3], has been sent for publication.

2.1 Introduction

A functional equation is called stable if any function satisfying the functional equation "approximately" is near to a true solution of the functional equation.
In the following we mention some examples of functional equations that Hyers-Ulam stability was investigated for them in several generalized spaces.
One of the most famous functional equation is the additive functional equation

$$f(x+y) = f(x) + f(y).$$

It was first solved by A.L. Cauchy in the class of continuous real-valued functions. The second famous functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{2.1.1}$$

is said to be quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (2.1.1). J.M. Rassias [53] introduced the following cubic functional equation

$$f(x+2y) + 3g(x) = 3g(x+y) + g(x-y) + 6g(y).$$

and investigated its Ulam stability problem. The quartic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4[f(x+y) + f(x-y) + 6f(y)].$$

was first introduced by J.M. Rassias [52], who solved its Ulam stability problem. The general solution of quintic functional equatin

$$f(x+3y)-5f(x+2y)+10f(x+y)-$$
$$10f(x)+5f(x-y)-f(x-2y)=120y,$$

and sextic functional equation

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)$$
$$-6f(x-2y)+f(x-3y)=720f(y),$$

was introduced and investigated the generalized Hyers-Ulam stability in quasi -β-normed spaces via fixed point method by Xu et al., [72].

Since the time the above stated results have been proved, several mathematicians have extensively studied stability theorems for several kinds of functional equations in random normed spaces. For example, Baktash et al., [11] proved the following stability theorem for quartic functional equation.

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

in random normed spaces.

Theorem 2.1.1. *Let X be a linear space, $(Z,\mu',\min)$ an RN-space, and $\varphi : X \times X \longrightarrow Z$ a function such that for some $0<\alpha<16$,*

$$\mu'_{\varphi(2x,0)}(t) \geq \mu'_{\alpha\varphi(x,0)}(t), \quad \forall x \in X, t>0,$$

$f(0)=0$ and $\lim_{n\longrightarrow\infty}\mu'_{\varphi(2^n x,2^n y)}(16^n t)=1$ for all $x,y \in X$ and all $t>0$. Let $(Y,\mu,\min)$ be a complete RN-space. If $f : X \longrightarrow Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) \geqslant \mu'_{\varphi(x,y)}(t), \quad \forall x,y \in X, t>0,$$

then there exists a unique quartic mapping $Q : X \longrightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\varphi(x,0)}(2(16-\alpha)t).$$

In 2016, Kim et al. [39] investigated stability of the general cubic functional equation

$$f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)=2k(k^2-1)f(y) \tag{2.1.2}$$

for fixed $k\in\mathbb{Z}^+$ with $k\geq 0$ via direct method in random normed spaces as follows:

Theorem 2.1.2. *Let X be a real linear space, (X,μ',T_M) be an RN-space and (Y,μ,T_M) be a complete RN-space and let $\varphi: X^2\longrightarrow\mathbb{Z}$ be an even function such that, for some $0<\alpha<k^3$*

$$\mu'_{\varphi(kx,ky)}(t)\geq\mu'_{\alpha\varphi(x,0)}(t),\quad \forall x\in X, t>0,$$

$\lim_{n\longrightarrow\infty}\mu'_{\varphi(k^nx,k^ny)}(k^{3n}t)=1$ *for all $x,y\in X$ and all $t>0$. If $f: X\longrightarrow Y$ is a mapping with $f(0)=0$ such that*

$$\mu_{Df(x,y)}(t)\geq\mu'_{\varphi(x,y)}(t),$$

for all $x,y\in X$ and $t>0$, where

$$Df(x,y)=f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-2k(k^2-1)f(y)$$

for all $x,y\in X$ and $k\in\mathbb{Z}^+$ with $k\geq 2$, then there exists a unique cubic mapping $C: X\longrightarrow Y$ such that

$$\mu_{f(y)-C(x)}(t)\geq\mu'_{\varphi(0,y)}\Big(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}\Big),\quad \forall x\in X, t>0.$$

See also, Cho et al.,[20], Kenary et al., [38], Mohamadi et al., [47],...] .

2.2 Stability of sextic functional equation via direct method

In this section, using the direct method, we prove the generalized stability of the sextic functional equation (2.2.1) and the additive-quadratic functional equation (2.3.1) in complete RN-spaces. The functional equation

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] \\ +2(n^6-n^4-n^2+1)[f(x)+f(y)], \tag{2.2.1}$$

is called the sextic functional equation since the function $f(x) = cx^6$ is a solution for this equation, where c is a constant. The following theorem states a stability result for the sextic functional equation (2.2.1) in complete RN-spaces.

Theorem 2.2.1. *Let X be a real linear space, (Y, μ, T) a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{2.2.2}$$

where

$$\begin{aligned} D_s f(x, y) := {} & f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) \\ & - (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)] \end{aligned}$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{m\to\infty} T_{i=1}^{\infty}(\phi_{n^{i+m-1}x,0}(n^{6m+5i}t)) = 1, \tag{2.2.3}$$

and

$$\lim_{m\to\infty} \phi_{n^m x, n^m y}(n^{6m}t) = 1 \tag{2.2.4}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and the inequality*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{n^{i-1}x,0}(n^{5i}t) \tag{2.2.5}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (4.2.1), we get

$$\mu_{f(nx)-n^6 f(x)}(t) \geq \phi_{x,0}(2t) \geq \phi_{x,0}(t) \tag{2.2.6}$$

for all $x \in X$. Then we get

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(n^6 t), \tag{2.2.7}$$

therefore,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}}-\frac{f(n^k x)}{n^{6k}}}(t) \geq \phi_{n^k x,0}(n^{6k+6}t), \tag{2.2.8}$$

that is,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}}-\frac{f(n^k x)}{n^6 k}}(\frac{t}{n^{k+1}}) \geq \phi_{n^k x,0}(n^{5(k+1)}t) \tag{2.2.9}$$

for every $k \in N$, $t > 0$, n positive integer, $n > 1$. As

$$1 > \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ... + \frac{1}{n^k},$$

by the triangle inequality it follows:

$$\begin{aligned}
\mu_{\frac{f(n^m x)}{n^{6m}} - f(x)}(t) &\geq \mu_{\frac{f(n^m x)}{n^{6m}} - f(x)}(\sum_{k=0}^{m-1} \frac{1}{n^{k+1}} t) \\
&\geq T_{k=0}^{m-1} \left(\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}} (\frac{1}{n^{k+1}} t) \right) \\
&\geqslant T_{k=0}^{m-1} (\phi_{n^k x, 0}(n^{5k+5} t) \\
&= T_{i=1}^{m} \left(\phi_{n^{i-1}x,0}(n^{5i} t) \right),
\end{aligned} \tag{2.2.10}$$

$x \in X$, $t > 0$, and $n > 1$. In order to prove the convergence of the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$, we replace x by $n^j x$, and multiplying the left-hand side of (3.3.9) by $\frac{n^{6j}}{n^{6j}}$, we get

$$\mu_{\frac{f(n^{m+j}x)}{n^{6m+6j}} - \frac{f(n^j x)}{n^{6j}}}(t) \geq T_{i=1}^{m} \left(\phi_{n^{j+i-1}x,0}(n^{6j+5i} t) \right). \tag{2.2.11}$$

Since the right-hand side of the inequality (3.3.10) tends to 1 as m and j tend to infinity, the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(n^j x)}{n^{6j}}$$

for all $x \in X$.

Replacing x, y by $n^m x$ and $n^m y$, respectively, in (4.2.1), then multiplying the right hand-side by $\frac{n^{6m}}{n^{6m}}$, it follows that

$$\mu_{\frac{1}{n^{6m}} D_s f(n^m x, n^m y)}(t) \geq \phi_{n^m x, n^m y}(n^{6m} t)$$

for all $x, y \in X$, and positive integer n, $n > 1$. Taking the limit as $m \to \infty$ we find that S satisfies (2.2.1), that is, S is a sextic map. To prove (4.2.4) take the limit as $m \to \infty$ in (3.3.9).

Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a sextic function r which satisfies (4.2.4) and equation (2.2.1). Therefore

$$\begin{aligned}
\mu_{r(x)-s(x)}(t) &= \mu_{r(x) - \frac{f(n^j x)}{n^{6j}} + \frac{f(n^j)}{n^{6j}} - s(x)}(t) \\
&\geq T(\mu_{r(x) - \frac{f(n^j x)}{n^{6j}}}(\frac{t}{2}), \mu_{\frac{f(n^j x)}{n^{6j}} - s(x)}(\frac{t}{2})).
\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\mu_{r(x)-s(x)}(t) = 1$. Therefore $r = s$. □

Corollary 2.2.2. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{2.2.12}$$

for all $x \in X$, $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(1 - \frac{\|x\|}{n^{4i+1}t + \|x\|})$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.2.1. □

Corollary 2.2.3. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{n^{5i}t}{n^{5i}t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.2.1. □

Corollary 2.2.4. *Let X be a real linear space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and let $L \geq 0$ and p be a real number with $0 < p < 5$ and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\frac{t}{t + Ln^{i(p-5)-p} \| x \|^p})$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.2.1. □

Example 2.2.1. *Let* $(X, \|.\|)$ *be a Banach algebra and*

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

for all $x, y \in X$ *and* $t > 0$. *Let*

$$\varphi_{x,y}(t) = \begin{cases} \max\{1 - \frac{(8n^6)(\|x\|+\|y\|)}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We note that $\varphi_{x,y}(t)$ *is a distribution function and* $\lim_{j\to\infty} \varphi_{n^j x, n^j y}(n^{6j}t) = 1$ *for all* $x, y \in X$ *and* $t > 0$.

It is easy to show that (X, μ, T_L) *is an RN-space (this was essentially proved by Mushtari in* ([48]), see also ([57])). Indeed, $\mu_x(t) = 1$, $\forall t > 0$ implies $\frac{\|x\|}{t} = 0$ and hence $x = 0$ for all $x \in X$ and $t > 0$. Obviously, $\mu_{\lambda x}(t) = \mu_x(\frac{t}{\lambda})$ for all $x \in X$ and $t > 0$. Next, for all $x, y \in X$ and $t, s > 0$, we have

$$\begin{aligned} \mu_{x+y}(t+s) &= \max\{1 - \frac{\|x+y\|}{t+s}, 0\} \\ &= \max\{1 - \|\frac{x+y}{t+s}\|, 0\} \\ &= \max\{1 - \|\frac{x}{t+s} + \frac{y}{t+s}\|, 0\} \\ &\geq \max\{1 - \|\frac{x}{t}\| - \|\frac{y}{s}\|, 0\} \\ &= T_L(\mu_x(t), \mu_y(s)). \end{aligned}$$

It is easy to see that (X, μ, T_L) is complete, for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t}, \quad \forall x, y \in X,$$

and $t > 0$ and $(X, \|.\|)$ is complete. Define a mapping $f : X \longrightarrow X$ by $f(x) = x^6 + \|x\| x_0$ for all $x \in X$, where x_0 is a unit vector in X. A simple computation shows that

$$\begin{aligned}
&\|f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)\\
&\quad-(n^4+n^2)[f(x+y)+f(x-y)]-2(n^6-n^4-n^2+1)[f(x)+f(y)]\|\\
&=|\;\|\; nx+y\;\|+\|\; nx-y\;\|+\|\; x+ny\;\|+\|\; x-ny\;\|\\
&\quad-(n^2+n^4)[\|\; x+y\;\|+\|\; x-y\;\|]\\
&\quad-2(n^6-n^4-n^2+1)[\|\; x\;\|+\|\; y\;\|]|\\
&\leq 2(n^6+n+2)(\|\; x\;\|+\|\; y\;\|)\leq 8n^6(\|\; x\;\|+\|\; y\;\|)
\end{aligned}$$

for all $x, y \in X$. Hence $\mu_{D_s f(x,y)}(t) \geq \phi_{x,y}(t)$ for all $x, y \in X$ and $t > 0$. Fix $x \in X$ and $t > 0$. Then it follows that,

$$\begin{aligned}
(T_L)_{i=1}^{\infty}\left(\phi_{n^{i+j-1}x,0}(n^{6j+5i)}t)\right) &= \max\left\{\sum_{i=1}^{\infty}\left(\phi_{n^{i+j-1}x,0}(n^{6j+5i)}t)-1\right)+1,0\right\}\\
&= \max\left\{1-\frac{8n^5\|x\|}{n^{5j}(n^4-1)t},0\right\}
\end{aligned}$$

for all $x \in X$, $n \in \mathbb{N}$ and $t > 0$. Hence

$$\lim_{j\to\infty}(T_L)_{i=1}^{\infty}\left(\varphi_{n^{i+j-1}x,0}(n^{6j+5i)}t)\right)=1$$

for all $x \in X$ and $t > 0$. Thus, all the conditions of Theorem 3.2.1 hold. Since

$$(T_L)_{i=1}^{\infty}\left(\phi_{n^{i-1}x,0}(n^{5i}t)\right)=\max\{1-\frac{8n^5\|x\|}{(n^4-1)t},0\}$$

for all $x \in X$ and $t > 0$, we can deduce that $S(x) = x^6$ is the unique sextic mapping $S : X \longrightarrow X$ such that

$$\mu_{f(x)-s(x)}(t)\geq \max\{1-\frac{8n^5\|x\|}{(n^4-1)t},0\}$$

for all $x \in X$ and $t > 0$.

2.3 Stability of additive-quadratic functioal equation via direct method

The generalized stability of different mixed type functional equations in random normed spaces and generalized spaces has been studied by many authors. For example, Madjid Eshaghi Gordji and Meysam Bavand Savadkouhi [31] in 2011, proved the stability of the additive, quadratic and cubic functional equation

$$f(x+3y)+f(x-3y)=9(f(x+y)-f(x-y))-16f(x),$$

in random normed space under arbitrary t-norms. The general solution and the Hyers-Ulam stability of the following quartic-additive functional equation

$$f(x+2y)-4f(x+y)-4f(x-y)+f(x-2y)=\frac{12}{7}(f(2y)-2f(y))-6f(x),$$

in random normed space was proved by Abasalt Bodaghia [13] in 2014. (See also, e.g., [9, 36, 47, 49, 58, 65]).

Now, the functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)]-[f(y)+f(-y)], \quad (2.3.1)$$

is called the additive-quadratic functional equation since the function $f(x)=ax^2+bx$ is a solution for this equation where a and b are constants. One can easily show that an even mapping $f: X \longrightarrow Y$ satisfies equation (2.3.1) if and only if the even mapping $f: X \longrightarrow Y$ is a quadratic mapping, that is,

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+4f(x)-2f(y).$$

Also, one can easily show that an odd mapping $f: X \longrightarrow Y$ satisfies equation (2.3.1) if and only if the odd mapping $f: X \longrightarrow Y$ is an additive mapping, that is,

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)].$$

For a given mapping $f: X \longrightarrow Y$, we define

$$\begin{aligned} D_s f(x,y) := {} & f(2x+y)+f(2x-y)-2[f(x+y)+f(x-y)] \\ & -2[f(x)+f(-x)]+[f(y)+f(-y)], \end{aligned}$$

for all $x, y \in X$ and $t > 0$.

The following theorem states the generalized stability of the additive-quadratic functional equation (2.3.1) in complete RN-spaces. Also, we present an illustrative example under the min t-norm.

Theorem 2.3.1. *Let X be a real linear space and (Y, μ, T) be a complete RN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{2.3.2}$$

for all $x, y \in X$ and $t > 0$, if

$$\lim_{j\to\infty} T_{i=1}^{\infty}(\phi_{2^{i+j-1}x,0}(2^{i+2j+1}t)) = 1, \tag{2.3.3}$$

and

$$\lim_{m\to\infty} \phi_{2^m x,2^m y}(2^{2m}t) = 1, \tag{2.3.4}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfies equation (2.3.1) *and the inequality*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{2^{i-1}x,0}(2^{i+1}t), \tag{2.3.5}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (4.2.1) we get

$$\mu_{2f(2x)-8f(x)}(t) \geq \phi_{x,0}(t), \tag{2.3.6}$$

for all $x \in X$. Then we get

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \phi_{x,0}(8t), \tag{2.3.7}$$

therefore,

$$\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}}-\frac{f(2^k x)}{2^{2k}}}(t) \geq \phi_{2^k x,0}(2^{2k+3}t), \tag{2.3.8}$$

that is

$$\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}}-\frac{f(2^k x)}{2^2 k}}(\frac{t}{2^{k+1}}) \geq \phi_{2^k x,0}(2^{k+2}t), \tag{2.3.9}$$

for every $k \in N$, $t > 0$. As

$$1 > \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k},$$

by the triangle inequality it follows:

$$\begin{aligned}
\mu_{\frac{f(2^n x)}{2^{2n}} - f(x)}(t) &\geq \mu_{\frac{f(2^n x)}{2^{2n}} - f(x)}(\sum_{k=0}^{n-1} \frac{1}{2^{k+1}} t) \\
&\geq T_{k=0}^{n-1} \left(\mu_{\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}}(\frac{1}{2^{k+1}} t) \right) \\
&\geqslant T_{k=0}^{n-1} (\phi_{2^k x, 0}(2^{k+2} t) \\
&= T_{i=1}^{n} \left(\phi_{2^{i-1}x,0}(2^{i+1} t) \right). \qquad (2.3.10)
\end{aligned}$$

$x \in X$, $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$, we replace x with $2^j x$ and multiplying the left hand of (3.3.9) by $\frac{2^{2j}}{2^{2j}}$,

$$\mu_{\frac{f(2^{n+j}x)}{2^{2(n+j)}} - \frac{f(2^j x)}{2^{2j}}}(t) \geq T_{i=1}^{n} \left(\phi_{2^{j+i-1}x,0}(2^{i+2j+1} t) \right). \qquad (2.3.11)$$

Since the right hand side of the inequality (3.3.10) tends to 1 as i and j tend to infinity, the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(2^j x)}{2^{2j}},$$

for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $S : X \longrightarrow Y$ is an even mapping. Replacing x, y with $2^m x$ and $2^m y$, respectiveiy, in (4.2.1) then multiplying the right hand side by $\frac{2^{2m}}{2^{2m}}$, it follows that:

$$\mu_{\frac{1}{2^{2m}} D_s f(2^m x, 2^m y)}(t) \geq \phi_{2^m x, 2^m y}(2^{2m} t).$$

for all $x, y \in X$. Taking the limit as $m \to \infty$ we find that S satisfies (2.2.1), that is, S is a quadratic map. To prove (4.2.4) take the limit as $n \to \infty$ in (3.3.9).
Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a quadratic function r which satisfies (4.2.4) and equation (2.2.1). Therefore

$$\begin{aligned}
\mu_{r(x)-s(x)}(t) =& \mu_{r(x) - \frac{f(2^j x)}{2^{2j}} + \frac{f(2^j x)}{2^{2j}} - s(x)}(t) \\
&\geq T(\mu_{r(x) - \frac{f(2^j x)}{2^{2j}}}(\frac{t}{2}), \mu_{\frac{f(2^j x)}{2^{2j}} - s(x)}(\frac{t}{2})).
\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\mu_{r(x)-s(x)}(t) = 1$. Therefore $r = s$. □

In Theorem (4.2.1), if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 2.3.2. *Let X be a real linear space and (Y, μ, T) be a complete RN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{2.3.12}$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{j\to\infty} T_{i=1}^{\infty}(\phi_{2^{i+j-1}x,0}(2^{j+1}t)) = 1, \tag{2.3.13}$$

and

$$\lim_{m\to\infty} \phi_{2^m x,2^m y}(2^m t) = 1, \tag{2.3.14}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfies equation (2.2.1) *and the inequality*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{2^{i-1}x,0}(2t), \tag{2.3.15}$$

for all $x \in X$ and $t > 0$.

Corollary 2.3.3. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{2.3.16}$$

for all $x \in X$, $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(1 - \frac{\|x\|}{4t + \|x\|}),$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem (4.2.1). □

Corollary 2.3.4. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, and $t > 0$ and $\varepsilon > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{2^{i+1}t}{2^{i+1}t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem (4.2.1). □

Corollary 2.3.5. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and let $L \geq 0$ and p be a real number with $p < 1$ and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\frac{2^{i+1}t}{2^{i+1}t + L2^{(i-1)p}\|x\|^p}),$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$, in Theorem (4.2.1). □

In corollary (2.3.5) if

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)L},$$

then the result is similar.

Example 2.3.1. *Let* $(X, \|.\|)$ *be a Banach algebra and*

$$\mu_x(t) = \begin{cases} 1 - \frac{\|x\|}{t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases},$$

for all $x, y \in X$ *and* $t > 0$. *Let*

$$\varphi_{x,y}(t) = \begin{cases} 1 - \frac{12(\|x\|+\|y\|)}{t} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases}.$$

We note that $\varphi_{x,y}(t)$ *is a distribution function and* $\lim_{j\to\infty} \varphi_{2^j x, 2^j y}(2^{2j}t) = 1$ *for all* $x, y \in X$ *and* $t > 0$.

It is easy to show that (X, μ, T_M) *is a RN-space. Indeed,* $\mu_x(t) = 1 \ \forall t > 0 \Longrightarrow \frac{\|x\|}{t} = 0$ *and hence* $x = 0$ *for all* $x \in X$ *and* $t > 0$. *Obviously,* $\mu_{\lambda x}(t) = \mu_x(\frac{t}{\lambda})$ *for all* $x \in X$ *and* $t > 0$. *Now let*

$$1 - \frac{\|x\|}{t} \leq 1 - \frac{\|y\|}{s},$$

for all $x, y \in X$.

if $x = y$, *we have* $s \geq t$. *Thus, otherwise, we have*

$$\frac{\|x+y\|}{t+s} \leq \frac{\|x\|}{t+s} + \frac{\|y\|}{t+s} \leq 2\frac{\|x\|}{t+s} \leq \frac{\|x\|}{t}.$$

Then

$$1 - \frac{\|x+y\|}{t+s} \geq 1 - \frac{\|x\|}{t}$$

and so

$$\mu_{x+y}(t+s) \geq T_M(1 - \frac{\|x\|}{t}, 1 - \frac{\|y\|}{s}) = T_M(\mu_x(t), \mu_y(s)).$$

It is easy to see that (X, μ, T_M) *is complete, for*

$$\mu_{x-y}(t) = 1 - \frac{\|x-y\|}{t} \quad \forall x, y \in X$$

and $t > 0$ *and* $(X, \|.\|)$ *is complete. Define a mapping* $f : X \longrightarrow X$ *by* $f(x) = x^2 + \|x\|x_0$ *for all* $x \in X$, *where* x_0 *is a unite vector in X. A simple computation shows that*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 4f(x) + 2f(y)\| = \\ &\quad |\|2x+y\| + \|2x-y\| - 2\|x+y\| - 2\|x-y\| - 4\|x\| + 2\|y\|| \\ &\quad \leq 12(\|x\| + \|y\|), \end{aligned}$$

for all $x, y \in X$. Hence $\mu_{D_s f(x,y)}(t) \geq \phi_{x,y}(t)$ for all $x, y \in X$ and $t > 0$. Fix $x \in X$ and $t > 0$, then it follows that,

$$(T_M)_{i=1}^{\infty} \left(\phi_{2^{i+j-1}x,0}(2^{2j+i+1)}t)\right) = 1 - \frac{12\|x\|}{2^{j+2}t},$$

for all $x \in X$, $n \in \mathbb{N}$ and $t > 0$. Hence

$$\lim_{j\to\infty} (T_M)_{i=1}^{\infty} \left(\varphi_{2^{i+j-1}x,0}(2^{1+2j+i)}t)\right) = 1,$$

for all $x \in X$ and $t > 0$. Thus, all the conditions of theorem (4.2.1) hold. Since

$$(T_M)_{i=1}^{\infty} \left(\phi_{2^{i-1}x,0}(2^{1+i}t)\right) = 1 - \frac{12.2^{i-1}\|x\|}{2^{i+1}t} = 1 - \frac{3\|x\|}{t},$$

for all $x \in X$ and $t > 0$. We can deduce that $S(x) = x^2$ is the unique quadratic mapping $S : X \longrightarrow X$ such that

$$\mu_{f(x)-s(x)}(t) \geq 1 - \frac{3\|x\|}{t},$$

for all $x \in X$ and $t > 0$.

Similar to what we had for an even mapping, the following corollaries can be proved.

Corollary 2.3.6. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, and $t > 0$ and $\varepsilon > 0$. Then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfying (2.2.1) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{2t}{2t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x, y \in X$ and $t > 0$, in Theorem (4.2.2). □

Corollary 2.3.7. *Let X be a real linear space and (Y, μ, T) be a complete RN-space such that $T = T_M$, or T_p and let $L \geq 0$ and p be a real number with $p \leq 0$ and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $S : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}\left(\frac{2t}{2t + L2^{(i-1)p} \parallel x \parallel^p}\right),$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + L(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$, in Theorem (4.2.2). □

In corollary (2.4.3) if

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)L},$$

then the result is similar.

Stability of certain functional equations via fixed point method

Fixed point theorems play important roles in proving our main theorems. All stability results for functional equations were proved by applying direct method. The direct method sometimes does not work. In consequence, the fixed point method for studying the stability of functional equations was used for the first time by Baker in 1991 [10]. Next, in 2003, V. Radu [51] gave a lecture at seminar on fixed point theory Cluj-Napoca and proved the Hyers-Ulam-Rassias stability of functional equation by fixed method. Then, in 2003, Cadariu and Radu [15, 16] considered Jensen functional equation and proved a stability result via fixed point method. Jung and Chang [37] proved the stability of a cubic type functional equation with the fixed point alternative. Since then, some authors (see e.g., [14, 23, 24, 32, 33, 42, 46]) considered some important functional equations and proved the stability results via fixed point method in several spaces.

Jin and Lee [36], Ebadian et al [24], [18, chapter 5] investigated the stability in the setting of random normed spaces by fixed point method. In 2012, Afshin, Erami et al. [25] proved the generalized Hyers-Ulam stability of the following cubic functional equation:

$$3f(x+3y) + f(3x-y) = 15f(x+y) + 15f(x-y) + 80f(y),$$

in random normed spaces via fixed point method as follows:

Theorem 2.3.8. *Let X be a real linear space, (Z, μ', min) be an RN-space and $\varphi : X^2 \longrightarrow Z$ be a function such that there exists $0 < \alpha < \frac{1}{27}$ such that*

$$\mu'_{\varphi(\frac{x}{3},\frac{y}{3})}(t) \geq \mu'_{\alpha\varphi(x,y)}(t)$$

for all$x, y \in X$ and $t > 0$ and $\lim_{n\longrightarrow\infty} \mu'_{27^n\alpha(\frac{x}{3^n},\frac{y}{3^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. Let (Y, μ, min) be a complete RN-space. If $f : X \longrightarrow Y$ is a mapping with $f(0) = 0$ and such that

$$\mu_{3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y)}(t) \geq \mu'_{\varphi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, then the limit $C(x) = \lim_{n\longrightarrow\infty} 27^n f(\frac{x}{3^n})$ exist for all $x \in X$and defines a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\frac{\alpha\varphi(x,0)}{1-27\alpha}}(t).$$

for all $x \in X$ and $t > 0$.

The following theorem was proved by Kim [39] in random normed spaces by fixed point method.

Theorem 2.3.9. *Let X be a real linear space, (X, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-space and let $\varphi : X^2 \longrightarrow D^+$, ($\varphi(x, y)$ is denoted by $\varphi_{(x,y)}$) be an even function such that, for some $0 < \alpha < k^3$*

$$\varphi_{(x,y)}(t) \leq \varphi_{(kx,ky)}(\alpha t) \quad \forall x \in X, t > 0.$$

If $f : X \longrightarrow Y$ is a mapping with $f(0) = 0$ which satisfies

$$\mu_{Df(x,y)}(t) \geq \varphi_{(x,y)}(t),$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\mu_{f(y)-C(x)}(t) \geq \varphi_{(0,y)}(\frac{2k(k^2-1)(k^3-\alpha)t}{k^3+\alpha}), \quad \forall x \in X, t > 0,$$

for all $x, y \in X$ and $t > 0$.

2.4 Fixed point method and sextic functional equation

In this section, using the fixed point method, we prove the generalized stability of the sextic functional equation (2.2.1) in complete RN-spaces.

Theorem 2.4.1. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t) \tag{2.4.1}$$

for all $x, y \in X$, and $t > 0$, where

$$\begin{aligned} D_s f(x,y) := {}& f(nx+y) + f(nx-y) + f(x+ny) + f(x-ny) \\ & - (n^4+n^2)[f(x+y) + f(x-y)] - 2(n^6-n^4-n^2+1)[f(x)+f(y)] \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(n^6-\alpha)t) \tag{2.4.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. Let $y = 0$ in (3.5.1); we get

$$\mu_{2f(nx)-2n^6 f(x)}(t) \geq \phi_{x,0}(t) \tag{2.4.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\mu_{\frac{f(nx)}{n^6} - f(x)}(t) \geq \phi_{x,0}(2n^6 t). \tag{2.4.4}$$

Consider the set

$$E := \{g : X \to Y : \ g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \texttt{inf}\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [44, Lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(nx)}{n^6}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{n^6}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g, h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(\frac{\epsilon \alpha t}{n^6}) &= \mu_{\frac{g(nx)}{n^6} - \frac{h(nx)}{n^6}}(\frac{\epsilon \alpha t}{n^6}) \\
&= \mu_{g(nx)-h(nx)}(\alpha \varepsilon t) \\
&\geq \phi_{nx,0}(2\alpha n^6 t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

we have

$$\mu_{Jg(x)-Jh(x)}(\frac{\epsilon \alpha t}{n^6}) \geq \phi_{x,0}(2n^6 t),$$

that is,

$$d_G(g,h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{n^6}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{n^6} d_G(g,h),$$

for all $g, h \in E$. Next, from

$$\mu_{\frac{f(nx)}{n^6} - f(x)}(t) \geq \phi_{x,0}(2n^6 t),$$

it follows that $d_G(f, Jf) \leq 1$. Using Theorem (1.5.2), we show the existence of a fixed point of J, that is, the existence of a mapping $g : X \longrightarrow Y$ such that $g(nx) = n^6 g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u,v) < \epsilon \Longrightarrow \mu_{u(x)-v(x)}(t) \geq \phi_{x,0}(\frac{2n^6 t}{\epsilon}),$$

it follows from $d_G(J^n f, g) \longrightarrow 0$ that $\lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}} = g(x)$ for all $x \in X$. Also from

$$d_G(f,g) \leq \frac{1}{1-L} d(f, Jf)$$

for all $g, h \in E$, we have $d_G(f,g) \leq \frac{1}{1-\frac{\alpha}{n^6}}$, and it immediately follows that

$$\mu_{g(x)-f(x)}(\frac{n^6}{n^6-\alpha} t) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This means that

$$\mu_{g(x)-f(x)}(t) \geqslant \phi_{x,0}(2(n^6-\alpha)t)$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ satisfying

$$\mu_{g(x)-f(x)}(Ct) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This completes the proof. □

Corollary 2.4.2. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{2.4.5}$$

for all $x \in X$, $t > 0$. *Then there exists a unique sextic mapping* $s : X \longrightarrow Y$ *satisfying* (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(n^6 - \alpha)t + \|x\|}$$

for every $x \in X$, $t > 0$, *and* n *positive integer. Moreover, we have*

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x \in X$ and $t > 0$ in Theorem 3.5.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 2.4.3. *Let* X *be a real linear space,* (Y, μ, T_M) *a complete RN-space and* $f : X \longrightarrow Y$ *a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, *and* $\varepsilon > 0$. *Then there exists a unique sextic mapping* $s : X \longrightarrow Y$ *satisfying* (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(n^6 - \alpha)t}{2(n^6 - \alpha)t + \varepsilon\|x_0\|}$$

for every $x \in X$, $t > 0$, *and* n *positive integer. Moreover, we have*

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x \in X$, and $t > 0$ in Theorem 3.5.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 2.4.4. *Let* X *be a real linear space,* (Y, μ, T_M) *a complete RN-space and* $f : X \longrightarrow Y$ *a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $0 < p < 6$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(n^6-\alpha)t}{2(n^6-\alpha)t+\theta \parallel x \parallel^p}$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t+\theta(\parallel x \parallel^p + \parallel y \parallel^p)}$$

for all $x, y \in X$ and $t > 0$ in Theorem 3.5.1. Then we can choose $n^p < \alpha < n^6$ and so we get the desired result. □

2.5 Fixed point method and additive-quadratic functional equation

In this section, using the fixed point method, we prove the generalized stability of the additive-quadratic functional equation (2.3.1) in complete RN-spaces.

Theorem 2.5.1. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 4,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{2.5.1}$$

for all $x, y \in X$, and $t > 0$. Then there exists a unique quadratic mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(4-\alpha)t), \tag{2.5.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. Let $y = 0$ in (4.3.1); we get

$$\mu_{2f(2x)-8f(x)}(t) \geq \phi_{x,0}(t), \tag{2.5.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \phi_{x,0}(8t). \tag{2.5.4}$$

Consider the set

$$E := \{g : X \to Y : \; g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g,h) = \texttt{inf}\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(8t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [44, Lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(2x)}{4}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{4}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g,h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(8t)$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{4}) &= \mu_{\frac{g(2x)}{4}-\frac{h(2x)}{4}}(\frac{\epsilon\alpha t}{4}) \\
&= \mu_{g(2x)-h(2x)}(\alpha\varepsilon t) \\
&\geq \phi_{2x,0}(\alpha 8t),
\end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 4,$$

we have

$$\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{4}) \geq \phi_{x,0}(8t),$$

that is,

$$d_G(g,h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{4}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{4} d_G(g,h),$$

for all $g, h \in E$. Next, from

$$\mu_{\frac{f(2x)}{4}-f(x)}(t) \geq \phi_{x,0}(8t),$$

it follows that $d_G(f, Jf) \leq 1$. Using Theorem 1.5.2, we show the existence of a fixed point of J, that is, the existence of a mapping $g : X \longrightarrow Y$ such that $g(2x) = 4g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u, v) < \epsilon \Longrightarrow \mu_{u(x)-v(x)}(t) \geq \phi_{x,0}(\frac{8t}{\epsilon}),$$

it follows from $d_G(J^n f, g) \longrightarrow 0$ that $\lim_{m\longrightarrow\infty} \frac{f(2^n x)}{4^n} = g(x)$ for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $g : X \longrightarrow Y$ is an even mapping.

Also from

$$d_G(f, g) \leq \frac{1}{1-L} d(f, Jf),$$

for all $g, h \in E$, we have $d_G(f, g) \leq \frac{1}{1-\frac{\alpha}{4}}$, and it immediately follows that

$$\mu_{g(x)-f(x)}(\frac{4}{4-\alpha}t) \geqslant \phi_{x,0}(8t),$$

for all $x \in X$ and $t > 0$. This means that

$$\mu_{g(x)-f(x)}(t) \geqslant \phi_{x,0}(2(4-\alpha)t),$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ satisfying

$$\mu_{g(x)-f(x)}(Ct) \geqslant \phi_{x,0}(8t),$$

for all $x \in X$ and $t > 0$. This completes the proof. □

In Theorem (4.3.1), if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 2.5.2. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{2x,2y}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < 2,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{2.5.5}$$

for all $x, y \in X$, *and* $t > 0$. *Then there exists a unique an additive mapping* $g : X \longrightarrow Y$ *such that*

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(2-\alpha)t), \tag{2.5.6}$$

for all $x \in X$ *and* $t > 0$. *Moreover, we have*

$$g(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Corollary 2.5.3. *Let* X *be a real linear space,* (Y, μ, T_M) *a complete RN-space, and* $f : X \longrightarrow Y$ *an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{2.5.7}$$

for all $x \in X$, $t > 0$. *Then there exists a unique quadratic mapping* $s : X \longrightarrow Y$ *satisfying* (2.2.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(4-\alpha)t + \|x\|},$$

for every $x \in X$, $t > 0$, *and* n *positive integer. Moreover, we have*

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x \in X$ and $t > 0$ in Theorem 2.5.1. Then we can choose $2 \leq \alpha < 4$ and so we get the desired result. □

Corollary 2.5.4. *Let* X *be a real linear space,* (Y, μ, T_M) *a complete RN-space and* $f : X \longrightarrow Y$ *an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, *and* $\varepsilon > 0$. *Then there exists a unique quadratic mapping* $s : X \longrightarrow Y$ *satisfying* (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(4-\alpha)t}{2(4-\alpha)t + \varepsilon\|x_0\|},$$

for every $x \in X$, $t > 0$, *and* n *positive integer. Moreover, we have*

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|},$$

for all $x \in X$, and $t > 0$ in Theorem 2.5.1. Then we can choose $1 \leq \alpha < 4$ and so we get the desired result. □

Corollary 2.5.5. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $p \leq 1$. Then there exists a unique quadrtic mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(4-\alpha)t}{2(4-\alpha)t + \theta\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$ in Theorem 2.5.1. Then we can choose $2^p \leq \alpha < 4$ and so we get the desired result. □

Corollary 2.5.6. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and let $z_0 \geq 0$ and p be a real number with $p < 1$ and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)z_0},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(4-\alpha)t}{2(4-\alpha)t + z_0\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)z_0},$$

for all $x, y \in X$ and $t > 0$, in Theorem 2.5.1. Then we can choose $2^{2p} \leq \alpha < 4$ and so we get the desired result. □

Similar to what we had for an even mapping, the following corollaries can be proved.

Corollary 2.5.7. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|}, \tag{2.5.8}$$

for all $x \in X$, $t > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(2-\alpha)t + \|x\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|},$$

for all $x \in X$ and $t > 0$ in Theorem 3.5.2. Then we can choose $\alpha = 2$ and so we get the desired result. □

Corollary 2.5.8. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(2-\alpha)t}{2(2-\alpha)t + \varepsilon\|x_0\|},$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon \|x_0\|},$$

for all $x \in X$, and $t > 0$ in Theorem 3.5.2. Then we can choose $1 \leq \alpha < 2$ and so we get the desired result. □

Corollary 2.5.9. *Let X be a real linear space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $p < 1$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(2-\alpha)t}{2(2-\alpha)t + \theta\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)},$$

for all $x, y \in X$ and $t > 0$ in Theorem 3.5.2. Then we can choose $2^p \leq \alpha < 2$ and so we get the desired result. □

Corollary 2.5.10. *Let X be a real linear space and (Y, μ, T_M) be a complete RN-space and let $z_0 \geq 0$ and p be a real number with $p \leq 0$ and $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p\|y\|^p)z_0},$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $s : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(2-\alpha)t}{2(2-\alpha)t + z_0\|x\|^p},$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + (\|x\|^p + \|y\|^p + \|x\|^p \|y\|^p) z_0},$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.5.2. Then we can choose $2^{2p} \leq \alpha < 2$ and so we get the desired result. □

Chapter 3

Stability of certain functioal equations in intuitionistic random normed spaces

In this chapter, we prove the stability of certain functional equations in intuitionistic random normed spaces under arbitrary t-norms via direct method and under min t-norm via fixed point method. It is necessary to mention the results of this chapter, in Ref. [4] and Ref. [5], has been sent for publication.

3.1 Introduction

There are many interesting results concerning intuitionistic random normed spaces. For example, in 2011, the stability problem for a cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{3.1.1}$$

was proved by Saadati, Vaezpour and Park [62] in intuitionistic random normed spaces as follows:

Theorem 3.1.1. *Let X be a real linear space and $(Y,\rho_{\mu,\nu},\tau)$ be a complete IRN-space and $f: X \longrightarrow Y$ be a mapping with $f(0)=0$ for which there are maps $\xi,\zeta : X^2 \longrightarrow D^+$. $\xi(x,y)$ is denoted by $\xi_{x,y}$, $\zeta(x,y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t),\zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x,y,t)$ with the property*

$$\rho_{\mu,\nu}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t) \geqslant_{L^*} Q_{\xi,\zeta}(x,y,t).$$

If

$$\tau_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{n+i-1}x,0,2^{3n+2i+1}t)=1_{L^*},$$

and

$$\lim_{n\to\infty} Q_{\xi,\zeta}(2^n x,2^n y,2^{3n}t)=1_{L^*},$$

for all $x,y\in X$ and $t>0$, then there exists a unique cubic mapping $C: X\longrightarrow Y$ satisfying equation (3.1.1) *and the inequality*

$$\rho_{\mu,\nu}(f(x)-C(x),t) \geq_{L^*} \tau_{i=1}^{\infty}(2^{i-1}x,0,2^{2i+1}t),$$

for all $x\in X$ and $t>0$.

In 2012, Choonkil Park, Madjid Eshaghi Gordji, at el., [50] investigated the Hyers-Ulam stability of the additive-quadratic functional equation

$$\sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf(\frac{1}{n}\sum_{i=1}^{n} x_i) \quad (n \geq 2)$$

in intuitionistic random normed spaces (see also,[54, 73]).

3.2 Stability of sextic functional equation via direct method

In this section, using the direct method, we prove the generalized stability of the sextic functional equation (2.2.1) in complete IRN-spaces. Also, we present corollary and illustrative example under the t-representable norm M related to our results .

Theorem 3.2.1. *Let X be a real liner space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t), \tag{3.2.1}$$

where

$$(D_s f(x, y), t) := (f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) \\ - (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)], t),$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{j \longrightarrow \infty} \tau_{i=1}^{\infty}(Q_{\xi,\zeta}(n^{i+j-1}x, 0, 2n^{6j+5i}t) = 1_{L^*}, \tag{3.2.2}$$

and

$$\lim_{m \to \infty} Q_{\xi,\zeta}(n^m x, n^m y, n^{6m}t) = 1_{L^*}, \tag{3.2.3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying equation (2.2.1) and the inequality

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \tau_{i=1}^{\infty}(n^{i-1}x, 0, 2n^{5i}t), \tag{3.2.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (4.2.1) we get

$$\rho_{\mu,\nu}(f(nx) - n^6 f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2t), \tag{3.2.5}$$

for all $x \in X$. Then we get

$$\rho_{\mu,\nu}(\frac{f(nx)}{n^6} - f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t), \tag{3.2.6}$$

therefore,

$$\rho_{\mu,\nu}(\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}, t) \geq_{L^*} Q_{\xi,\zeta}(n^k x, 0, 2n^{6k+6} t), \tag{3.2.7}$$

that is

$$\rho_{\mu,\nu}(\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^6 k}, \frac{t}{n^{k+1}}) \geq_{L^*} Q_{\xi,\zeta}(n^k x, 0, 2n^{5(k+1)} t), \tag{3.2.8}$$

for every $k \in N$, $t > 0$, n positive integer, $n > 1$. As

$$1 > \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ... + \frac{1}{n^k},$$

by the triangle inequality for $x \in X$, $t > 0$, $n > 1$ it follows:

$$\begin{aligned}
\rho_{\mu,\nu}(\frac{f(n^m x)}{n^{6m}} - f(x), t) &\geq_{L^*} \rho_{\mu,\nu}(\frac{f(n^m x)}{n^{6m}} - f(x), \sum_{k=0}^{m-1} \frac{1}{n^{k+1}} t) \\
&\geq_{L^*} \tau_{k=0}^{m-1} \left(\rho_{\mu,\nu}(\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}, \frac{1}{n^{k+1}} t) \right) \\
&\geq_{L^*} \tau_{k=0}^{m-1} (Q_{\xi,\zeta}(n^k x, 0, 2n^{5k+5} t)) \\
&= \tau_{i=1}^{m} \left(Q_{\xi,\zeta}(n^{i-1} x, 0, 2n^{5i} t) \right).
\end{aligned} \tag{3.2.9}$$

In order to prove the convergence of the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$, we replace x with $n^j x$ and multiply the left hand of (3.3.9) by $\frac{n^{6j}}{n^{6j}}$,

$$\rho_{\mu,\nu}\left(\frac{f(n^{m+j}x)}{n^{6m+6j}} - \frac{f(n^j x)}{n^{6j}}, t \right) \geq_{L^*} \tau_{i=1}^{m} \left(Q_{\xi,\zeta}(n^{j+i-1} x, 0, 2n^{6j+5i} t) \right). \tag{3.2.10}$$

Since the right hand side of the inequality (3.3.10) tends to 1 as m and j tend to infinity, the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(n^j x)}{n^{6j}},$$

for all $x \in X$.

Replacing x, y with $n^m x$ and $n^m y$, respectively, in (4.2.1) then multiplying the right hand side by $\frac{n^{6m}}{n^{6m}}$, it follows that:

$$\rho_{\mu,\nu}(\frac{1}{n^{6m}} D_s f(n^m x, n^m y), t) \geq_{L^*} Q_{\xi,\zeta}(n^m x, n^m y, n^{6m} t),$$

for all $x, y \in X$, and positive integer n, $n > 1$. Taking the limit as $m \to \infty$ we find that S satisfies (2.2.1), that is, S is a sextic mapping. To prove (4.2.4) take the limit as $m \to \infty$ in (3.3.9).

Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a sextic function r which satisfies (4.2.4) and equation (2.2.1). Therefore

$$\begin{aligned}\rho_{\mu,\nu}(r(x) - S(x), t) &= \rho_{\mu,\nu}(r(x) - \frac{f(n^j x)}{n^{6j}} + \frac{f(n^j x)}{n^{6j}} - S(x), t) \\ &\geq_{L^*} \tau(\rho_{\mu,\nu}(r(x) - \frac{f(n^j x)}{n^{6j}}, \frac{t}{2}), \rho_{\mu,\nu}(\frac{f(n^j x)}{n^{6j}} - S(x), \frac{t}{2})).\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\rho_{\mu,\nu}(r(x) - S(x), t) = 1_{L^*}$. Therefore $r = S$. □

Corollary 3.2.2. *Let $(X, \rho'_{\mu',\nu'}, \tau)$ be an IRN- space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space. If $f : X \longrightarrow Y$ be a mapping satisfying*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} \rho'_{\mu',\nu'}(x + y, t), \tag{3.2.11}$$

for all $x, y \in X$, $t > 0$ in which

$$\lim_{j \longrightarrow \infty} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}, (x, 0, 2n^{4i+5j+1} t)) = 1_{L^*}, \tag{3.2.12}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geqslant_{L^*} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}(x, 0, 2n^{4i+1} t)),$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \rho'_{\mu',\nu'}(x + y, t),$$

for all $x, y \in X$ and $t > 0$, the corollary immediate from Theorem 3.2.1. □

Example 3.2.1. *Let $(X, \|.\|)$ be a Banach algebra space and $(X, \rho'_{\mu',\nu'}, M)$ be an IRN-space in which*

$$\rho'_{\mu',\nu'}(x, t) = (\frac{t}{t + |2(n^2 + n^4 - 2n^6)|(\|x\| + 1)}, \frac{|2(n^2 + n^4 - 2n^6)|(\|x\| + 1)}{t + |2(n^2 + n^4 - 2n^6)|(\|x\| + 1)}),$$

for all $x, y \in X$ and $t > 0$ and let $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space in which

$$\rho_{\mu,\nu}(x,t) = (\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}),$$

for all $x, y \in X$ and $t > 0$. Define the mapping $f : X \longrightarrow Y$ by $f(x) = x^6 + x_0$ for all $x \in X$ where x_0 is a unit vector in X. A straightforward computation shows that

$$\rho_{\mu,\nu}(D_s f(x,y), t) \geq_{L^*} \rho'_{\mu',\nu'}(x+y, t),$$

for all $x, y \in X$ and $t > 0$. Also we have

$$\begin{aligned}\lim_{j\longrightarrow\infty} M_{i=1}^{\infty}(\rho'_{\mu',\nu'}(x, 0, 2n^{4i+5j+1}t)) &= \lim_{j\longrightarrow\infty}\lim_{m\longrightarrow\infty} M_{i=1}^{m}(\rho'_{\mu',\nu'}(x, 0, 2n^{4i+5j+1}t)) \\ &= \lim_{j\longrightarrow\infty}\lim_{m\longrightarrow\infty} \rho'_{\mu',\nu'}(x, 0, 2n^{5+5j}t)) \\ &= \lim_{j\longrightarrow\infty} \rho'_{\mu',\nu'}(x, 0, 2n^{5+5j}t)) = 1_{L^*},\end{aligned}$$

for all $x \in X$ and $t > 0$. Therefore, there exists a unique sextic mapping $S : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \rho'_{\mu',\nu'}(x, 0, 2n^5 t)$$

for all $x \in X$ and $t > 0$.

3.3 Stability of mixed type functioal equation via direct method

In this section, using the direct method, we prove the generalized stability of the additive-quadratic functional equation (2.3.1) in complete IRN-spaces. Also, we present an illustrative example.

Theorem 3.3.1. *Let X be a real linear space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$\rho_{\mu,\nu}(D_s f(x,y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t), \qquad (3.3.1)$$

if

$$\lim_{j\longrightarrow\infty} \tau_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{i+j-1}x, 0, 2^{2j+i+1}t) = 1_{L^*}, \qquad (3.3.2)$$

and

$$\lim_{m\to\infty} Q_{\xi,\zeta}(2^m x, 2^m y, 2^{2m} t) = 1_{L^*}, \tag{3.3.3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quadratic mapping $S : X \longrightarrow Y$

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \tau_{i=1}^{\infty}(2^{i-1}x, 0, 2^{i+1}t), \tag{3.3.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (4.2.1) we get

$$\rho_{\mu,\nu}(2f(2x) - 8f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, t), \tag{3.3.5}$$

for all $x \in X$. Then we get

$$\rho_{\mu,\nu}(\frac{f(2x)}{4} - f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t), \tag{3.3.6}$$

therefore,

$$\rho_{\mu,\nu}(\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}, t) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, 2^{2k+3}t), \tag{3.3.7}$$

that is

$$\rho_{\mu,\nu}(\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^2 k}, \frac{t}{2^{k+1}}) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, 2^{k+2}t), \tag{3.3.8}$$

for every $k \in N$, $t > 0$. As

$$1 > \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k},$$

by the triangle inequality for $x \in X$, $t > 0$, it follows:

$$\begin{aligned}
\rho_{\mu,\nu}(\frac{f(2^n x)}{2^{2n}} - f(x), t) &\geq_{L^*} \rho_{\mu,\nu}(\frac{f(2^n x)}{2^{2n}} - f(x), \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} t) \\
&\geq_{L^*} \tau_{k=0}^{n-1}\left(\rho_{\mu,\nu}(\frac{f(2^{k+1}x)}{2^{2k+2}} - \frac{f(2^k x)}{2^{2k}}, \frac{1}{2^{k+1}} t)\right) \\
&\geq_{L^*} \tau_{k=0}^{n-1}(Q_{\xi,\zeta}(2^k x, 0, 2^{k+2}t)) \\
&= \tau_{i=1}^{n}\left(Q_{\xi,\zeta}(2^{i-1}x, 0, 2^{i+1}t)\right).
\end{aligned} \tag{3.3.9}$$

$x \in X$, $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$, we replace x with $2^j x$ and multiplying the left hand of (3.3.9) by $\frac{2^{2j}}{2^{2j}}$,

$$\rho_{\mu,\nu}\left(\frac{f(2^{n+j}x)}{2^{2n+2j}} - \frac{f(2^j x)}{2^{2j}}, t\right) \geq_{L^*} \tau_{i=1}^{n}\left(Q_{\xi,\zeta}(2^{j+i-1}x, 0, 2^{2j+i+1}t)\right). \tag{3.3.10}$$

Since the right hand side of the inequality (3.3.10) tends to 1 as i and j tend to infinity, the sequence $\{\frac{f(2^j x)}{2^{2j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(2^j x)}{2^{2j}},$$

for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $S : X \longrightarrow Y$ is an even mapping.

Replacing x, y with $2^m x$ and $2^m y$, respectiveiy, in (4.2.1) then multiplying the right hand side by $\frac{2^{2m}}{2^{2m}}$, it follows that:it follows that:

$$\rho_{\mu,\nu}(\frac{1}{2^{2m}} D_s f(2^m x, 2^m y), t) \geq_{L^*} Q_{\xi,\zeta}(2^m x, 2^m y, 2^{2m} t),$$

for all $x, y \in X$. Taking the limit as $m \to \infty$ we find that S satisfies (2.3.1), that is, S is a quadratic map. To prove (4.2.4) take the limit as $n \to \infty$ in (3.3.9).

Finally, to prove the uniqueness of the quadratic function S, let us assume that there exists a quadratic function r which satisfies (4.2.4) and equation (2.3.1). Therefore

$$\begin{aligned}\rho_{\mu,\nu}(r(x) - S(x), t) &= \rho_{\mu,\nu}(r(x) - \frac{f(2^j x)}{2^{2j}} + \frac{f(2^j x)}{2^{2j}} - S(x), t) \\ &\geq_{L^*} \tau(\rho_{\mu,\nu}(r(x) - \frac{f(2^j x)}{2^{2j}}, \frac{t}{2}), \rho_{\mu,\nu}(\frac{f(2^j x)}{2^{2j}} - S(x), \frac{t}{2})).\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\rho_{\mu,\nu}(r(x) - s(x), t) = 1$. Therefore $r = s$. □

In Theorem (4.2.1), if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 3.3.2. *Let X be a real linear space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t), \qquad (3.3.11)$$

if

$$\lim_{j \longrightarrow \infty} \tau_{i=1}^{\infty}(Q_{\xi,\zeta}(2^{i+j-1} x, 0, 2^{i+1} t) = 1_{L^*}, \qquad (3.3.12)$$

and

$$\lim_{m \to \infty} Q_{\xi,\zeta}(2^m x, 2^m y, 2^m t) = 1_{L^*}, \qquad (3.3.13)$$

for all $x, y \in X$ and $t > 0$, then there exists a unique additive mapping $S : X \longrightarrow Y$

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \tau_{i=1}^{\infty}(2^{i-1}x, 0, 2t), \tag{3.3.14}$$

for all $x \in X$ and $t > 0$.

Corollary 3.3.3. *Let $(X, \rho'_{\mu',\nu'}, \tau)$ be an IRN- space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space. If $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} \rho'_{\mu',\nu'}(x + y, t), \tag{3.3.15}$$

for all $x, y \in X$, $t > 0$ in which

$$\lim_{j \longrightarrow \infty} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}, (x, 0, 2^{j+2}t)) = 1_{L^*}, \tag{3.3.16}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geqslant_{L^*} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}(x, 0, 4t)),$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \rho'_{\mu',\nu'}(x + y, t),$$

for all $x, y \in X$ and $t > 0$, the corollary immediate from Theorem (4.2.1). □

Corollary 3.3.4. *Let $(X, \rho'_{\mu',\nu'}, \tau)$ be an IRN- space and $(Y, \rho_{\mu,\nu}, \tau)$ be a complete IRN-space. If $f : X \longrightarrow Y$ be an odd mapping satisfying*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} \rho'_{\mu',\nu'}(x + y, t), \tag{3.3.17}$$

for all $x, y \in X$, $t > 0$ in which

$$\lim_{j \longrightarrow \infty} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}, (x, 0, 2^{2-j}t)) = 1_{L^*}, \tag{3.3.18}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique additive mapping $S : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geqslant_{L^*} \tau_{i=1}^{\infty}(\rho'_{\mu',\nu'}(x, 0, 2^{2-i}t)),$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \rho'_{\mu',\nu'}(x + y, t),$$

for all $x, y \in X$ and $t > 0$, the corollary immediate from Theorem (3.3.2). □

Example 3.3.1. *Let* $(X, \|.\|)$ *be a Banach algebra space and* $(X, \rho'_{\mu',\nu'}, M)$ *be an IRN-space in which*

$$\rho'_{\mu',\nu'}(x,t) = \left(\frac{t}{t+4(\|x\|+1)}, \frac{4(\|x\|+1)}{t+4(\|x\|+1)}\right),$$

for all $x, y \in X$ *and* $t > 0$ *and let* $(Y, \rho_{\mu,\nu}, M)$ *be a complete IRN-space in which*

$$\rho_{\mu,\nu}(x,t) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right),$$

for all $x, y \in X$ *and* $t > 0$. *Define the mapping* $f : X \longrightarrow Y$ *by* $f(x) = x^2 + x_0$ *for all* $x \in X$ *where* x_0 *is a unit vector in* X. *A straightforward computation shows that*

$$\rho_{\mu,\nu}(D_s f(x,y), t) \geq_{L^*} \rho'_{\mu',\nu'}(x+y, t),$$

for all $x, y \in X$ *and* $t > 0$. *Also we have*

$$\begin{aligned}
\lim_{j\longrightarrow\infty} M_{i=1}^{\infty}(\rho'_{\mu',\nu'}(x,0,2^{j+2}t)) &= \lim_{j\longrightarrow\infty}\lim_{m\longrightarrow\infty} M_{i=1}^{m}(\rho'_{\mu',\nu'}(x,0,2^{j+2}t)) \\
&= \lim_{j\longrightarrow\infty}\lim_{m\longrightarrow\infty} \rho'_{\mu',\nu'}(x,0,2^{j+2}t)) \\
&= \lim_{j\longrightarrow\infty} \rho'_{\mu',\nu'}(x,0,2^{j+2}t)) = 1_{L^*},
\end{aligned}$$

for all $x \in X$ *and* $t > 0$. *Therefore, there exists a unique quadratic mapping* $S : X \longrightarrow Y$ *such that*

$$\rho_{\mu,\nu}(f(x) - S(x), t) \geq_{L^*} \rho'_{\mu',\nu'}(x, 0, 4t)$$

for all $x \in X$ *and* $t > 0$.

Stability of certain functioal equations via fixed point method

In this section, using the fixed point method, we prove the generalized stability of the sextic functional equation (2.2.1) and the additive-quadratic functional equation (2.3.1) in IRN-spaces.

3.4 Stability of sextic functioal equation via fixed point method

In this section, using the fixed point method, we prove the generalized stability of the sextic functional equation (2.2.1) in complete IRN-spaces. Also, we present some corollaries related to our result.

Theorem 3.4.1. *Let X be a real linear space and $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$Q_{\xi,\zeta}(nx, ny, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), \quad 0 < \alpha < n^6$$

and

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t) \tag{3.4.1}$$

for all $x, y \in X$, and $t > 0$. Where

$$\begin{aligned}(D_s f(x, y), t) &:= (f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny)\\ &- (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)], t)\end{aligned}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $g : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - g(x), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2(n^6 - \alpha)t) \tag{3.4.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. Let $y = 0$ in (3.5.1) we get

$$\rho_{\mu,\nu}(2f(nx) - 2n^6 f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, t), \tag{3.4.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\rho_{\mu,\nu}(\frac{f(nx)}{n^6} - f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t). \tag{3.4.4}$$

Consider the set

$$E := \{g : X \to Y : \; g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g,h) = \inf \left\{ \epsilon > 0 : \rho_{\mu,\nu}(g(x) - h(x), \epsilon t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t) \right\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [44, lemma 2.1]) . Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(nx)}{n^6}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz costant $k = \frac{\alpha}{n^6}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g,h) < \epsilon$. Then we have

$$\rho_{\mu,\nu}(g(x) - h(x), \epsilon t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t),$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned} \rho_{\mu,\nu}(Jg(x) - Jh(x), \frac{\epsilon\alpha t}{n^6}) &= \rho_{\mu,\nu}(\frac{g(nx)}{n^6} - \frac{h(nx)}{n^6}, \frac{\epsilon\alpha t}{n^6}) \\ &= \rho_{\mu,\nu}(g(nx) - h(nx), \alpha\varepsilon t) \\ &\geq_{L^*} Q_{\xi,\zeta}(nx, 0, 2\alpha n^6 t), \end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$Q_{\xi,\zeta}(nx, ny, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), \quad 0 < \alpha < n^6,$$

we have

$$\rho_{\mu,\nu}(Jg(x) - Jh(x), \frac{\epsilon\alpha t}{n^6}) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t),$$

that is,

$$d_G(g,h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{n^6}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{n^6} d_G(g,h),$$

for all $g, h \in E$. Next, from

$$\rho_{\mu,\nu}\left(\frac{f(nx)}{n^6} - f(x), t\right) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t),$$

follows that $d_G(f, Jf) \leq 1$. Using the Theoreme (3.5.1), there exists a fixed point of J, that is, there is a mapping $g : X \longrightarrow Y$such that $g(nx) = n^6 g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u, v) < \epsilon \Longrightarrow \rho_{\mu,\nu}(u(x) - v(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, \frac{2n^6 t}{\epsilon}).$$

It follows from $d_G(J^n f, g) \longrightarrow 0$ that $\lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}} = g(x)$ for all $x \in X$. Also from

$$d_G(f, g) \leq \frac{1}{1 - L} d(f, Jf),$$

for all $g, h \in E$. Then $d_G(f, g) \leq \frac{1}{1-\frac{\alpha}{n^6}}$. It immediately follows that

$$\rho_{\mu,\nu}\left(g(x) - f(x), \frac{n^6}{n^6 - \alpha} t\right) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t),$$

for all $x \in X$ and $t > 0$. This means that

$$\rho_{\mu,\nu}(g(x) - f(x), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2(n^6 - \alpha)t),$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ such that

$$\rho_{\mu,\nu}(g(x) - f(x), Ct) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2n^6 t),$$

for all $x \in X$ and $t > 0$. This completes the proof. □

Corollary 3.4.2. *Let $(X, \rho'_{\mu',\nu'}, M)$ be an IRN-space and $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} \left(\frac{t}{t + \|x + y\|}, \frac{\|x + y\|}{t + \|x + y\|}\right), \tag{3.4.5}$$

for all $x, y \in X$, $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$satisfying (2.3.1) *and*

$$\rho_{\mu,\nu}(f(x) - s(x), t) \geqslant_{L^*} \left(\frac{2(n^6 - \alpha)t}{2(n^6 - \alpha)t + \|x\|}, \frac{\|x\|}{2(n^6 - \alpha)t + \|x\|}\right),$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$S(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \left(\frac{t}{t + \|x + y\|}, \frac{\|x + y\|}{t + \|x + y\|}\right),$$

for all $x \in X$, and $t > 0$ in Theorem (3.5.1). Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

3.5 Stability of mixed type functional equation via fixed point method

In this section, using the fixed point method, we prove the generalized stability of the mixed type functional equation (2.3.1) in complete IRN-spaces. We recall a fundamental result in fixed point theory.

Theorem 3.5.1. *Let X be a real linear space and $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space and $f : X \longrightarrow Y$ be an even mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$Q_{\xi,\zeta}(2x, 2y, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), \quad 0 < \alpha < 4$$

and

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t) \tag{3.5.1}$$

for all $x, y \in X$, and $t > 0$. Then there exists a unique quadratic mapping $g : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - g(x), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2(4 - \alpha)t) \tag{3.5.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. Let $y = 0$ in (3.5.1); we get

$$\rho_{\mu,\nu}(2f(2x) - 8f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, t), \tag{3.5.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\rho_{\mu,\nu}(\frac{f(2x)}{4} - f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t). \tag{3.5.4}$$

Consider the set

$$E := \{g : X \to Y : \ g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \inf \{\epsilon > 0 : \rho_{\mu,\nu}(g(x) - h(x), \epsilon t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [44, lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(2x)}{4}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz costant $k = \frac{\alpha}{4}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g, h) < \epsilon$. Then we have

$$\rho_{\mu,\nu}(g(x) - h(x), \epsilon t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t),$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned}\rho_{\mu,\nu}(Jg(x) - Jh(x), \frac{\epsilon\alpha t}{4}) &= \rho_{\mu,\nu}(\frac{g(2x)}{4} - \frac{h(2x)}{4}, \frac{\epsilon\alpha t}{4}) \\ &= \rho_{\mu,\nu}(g(2x) - h(2x), \alpha\varepsilon t) \\ &\geq_{L^*} Q_{\xi,\zeta}(2x, 0, \alpha 8t),\end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$Q_{\xi,\zeta}(2x, 2y, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), \quad 0 < \alpha < 4,$$

we have

$$\rho_{\mu,\nu}(Jg(x) - Jh(x), \frac{\epsilon\alpha t}{4}) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t),$$

that is,

$$d_G(g, h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{4}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{4} d_G(g, h),$$

for all $g, h \in E$. Next, from

$$\rho_{\mu,\nu}\left(\frac{f(2x)}{4} - f(x), t\right) \geq_{L^*} Q_{\xi,\zeta}(x, 0, 8t),$$

follows that $d_G(f, Jf) \leq 1$. Using the Theorem (1.5.2), there exists a fixed point of J, that is, there is a mapping $g : X \longrightarrow Y$such that $g(2x) = 4g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u, v) < \epsilon \Longrightarrow \rho_{\mu,\nu}(u(x) - v(x), t) \geq_{L^*} Q_{\xi,\zeta}(x, 0, \frac{8t}{\epsilon}).$$

It follows from $d_G(J^n f, g) \longrightarrow 0$ that $\lim_{m\longrightarrow\infty} \frac{f(2^n x)}{4^n} = g(x)$ for all $x \in X$. Since $f : X \longrightarrow Y$ is even, $g : X \longrightarrow Y$ is an even mapping. Also from

$$d_G(f, g) \leq \frac{1}{1 - L} d(f, Jf),$$

for all $g, h \in E$. Then $d_G(f, g) \leq \frac{1}{1-\frac{\alpha}{4}}$. It immediately follows that

$$\rho_{\mu,\nu}\left(g(x) - f(x), \frac{4}{4-\alpha}t\right) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 8t),$$

for all $x \in X$ and $t > 0$. This means that

$$\rho_{\mu,\nu}(g(x) - f(x), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2(4-\alpha)t),$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ such that

$$\rho_{\mu,\nu}(g(x) - f(x), Ct) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 8t),$$

for all $x \in X$ and $t > 0$. This completes the proof. □

In Theorem (3.5.1), if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 3.5.2. *Let X be a real linear space and $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space and $f : X \longrightarrow Y$ be an odd mapping with $f(0) = 0$ for which there is a map $\xi : X^2 \longrightarrow D^+$ and a map ζ from X^2 to the space of non-measure distribution functions. $\xi(x, y)$ is denoted by $\xi_{x,y}$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ is denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property*

$$Q_{\xi,\zeta}(2x, 2y, \alpha t) \geq_{L^*} Q_{\xi,\zeta}(x, y, t), \quad 0 < \alpha < 2$$

and

$$\rho_{\mu,\nu}(D_s f(x, y), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, y, t) \tag{3.5.5}$$

for all $x, y \in X$, and $t > 0$. Then there exists a unique additive mapping $g : X \longrightarrow Y$ such that

$$\rho_{\mu,\nu}(f(x) - g(x), t) \geqslant_{L^*} Q_{\xi,\zeta}(x, 0, 2(2-\alpha)t) \tag{3.5.6}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = \lim_{n \longrightarrow \infty} \frac{f(2^n x)}{2^n}.$$

Corollary 3.5.3. *Let $(X, \rho'_{\mu',\nu'}, M)$ be an IRN-space and $(Y, \rho_{\mu,\nu}, M)$ be a complete IRN-space and $f : X \longrightarrow Y$ be an even mapping satisfying*

$$\rho_{\mu,\nu}(D_s f(x,y), t) \geqslant_{L^*} \left(\frac{t}{t + \|x+y\|}, \frac{\|x+y\|}{t + \|x+y\|} \right), \tag{3.5.7}$$

for all $x, y \in X$, $t > 0$. Then there exists a unique quadratic mapping $S : X \longrightarrow Y$ satisfying (2.3.1) *and*

$$\rho_{\mu,\nu}(f(x) - s(x), t) \geqslant_{L^*} \left(\frac{2(4-\alpha)t}{2(4-\alpha)t + \|x\|}, \frac{\|x\|}{2(4-\alpha)t + \|x\|} \right),$$

for every $x \in X$, $t > 0$. Moreover, we have

$$S(x) = \lim_{m \longrightarrow \infty} \frac{f(2^n x)}{4^n}.$$

Proof. It is enough to put,

$$Q_{\xi,\zeta}(x, y, t) = \left(\frac{t}{t + \|x+y\|}, \frac{\|x+y\|}{t + \|x+y\|} \right),$$

for all $x \in X$, and $t > 0$ in Theorem (3.5.1). Then we can choose $2 \leq \alpha < 4$ and so we get the desired result. □

Chapter 4

Stability of certain functional equations in non-Archimedean random normed spaces.

In this chapter, we prove the stability of the sextic functional equation 2.2.1 and the additive-quadratic functional equation 2.3.1 in non-Archimedean random normed spaces via direct method under arbitrary t-norms. It is necessary to mention that the results of this chapter in Ref. [4] and Ref. [5] has been sent for publication.

4.1 Introduction

Hyers-Ulam stability has been proved for several functional equations in non-Archimedean random normed spaces. See for example [18, chapter 6] and ([65, 62]). In 2011, J. M. Rassias et al. [54] proved the following theorem for quartic functional equation

$$\begin{aligned} 16f(x+4y)+f(4x-y) &= 306[9f(x+\frac{y}{3})+f(x+2y)] \\ &+136f(x-y)-1394f(x+y)+425f(y)-1530f(x), \end{aligned} \tag{4.1.1}$$

in non-Archimeadean randon normed spaces as follows:

Theorem 4.1.1. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and (Y,μ,T) be a non-Archimedean random Banach space over $\mathcal{K}$. Let $f: X \to Y$ be a ψ-approximately quartic function. If for some $\alpha \in \mathbb{R}$ with $\alpha > 0$ and for some positive integer k, $k > 3$ with $|4^k| < \alpha$.*

$$\psi(4^{-k}x, 4^{-k}y, t) \geq \psi(x, y, \alpha t), \quad x \in X, t > 0,$$

and

$$\lim_{n\to\infty} T_{j=n}^{\infty} M(x, \frac{\alpha^j t}{|4|^{kj}}) = 1,$$

for all $x \in X$ and $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that:

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M(x, \frac{\alpha^{i+1} t}{|4|^{ki}}), \tag{4.1.2}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T(\psi(x,0,t), \psi(4x,0,t), \cdots, \psi(4^{k-1}x,0,t)),$$

for all $x \in X$ and $t > 0$.

Yeol Je Cho and Reza Saadati [21] in 2011, proved the generalized Hyers-Ulam stability of the following additive-cubic-quartic functional equation

$$\begin{aligned}11f(x+2y)+11f(x-2y) &= 44f(x+y)+44f(x-y)+12f(3y)\\ &\quad -48f(2y)+60f(y)-66f(x)\end{aligned}$$

in various complete lattictic random normed spaces as followes:

Theorem 4.1.2. *Let K be a non-Archimedean field, X be a vector space over K and (Y,μ,T)be a non-Archimedean complete LRN-space over KLet $f : X \longrightarrow Y$ be an odd and ψ-approximately mixed ACQ mapping. If, for some $\alpha \in \mathbb{R}, \alpha > 0$, and some integer $k, k > 3$ with $|2^k| < \alpha$*

$$\psi_{(2^{-k}x,2^{-k})}(t) \geq \psi_{(x,y)}(\alpha t), \quad x \in X, t > 0,$$

and

$$\lim_{n\to\infty} T_{j=n}^{\infty} M(x, \frac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}},$$

then there exists a unique cubic mapping $C : X \longrightarrow Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq_{\mathcal{L}} T_{i=1}^{\infty} M(x, \frac{\alpha^{i+1} t}{|2|^{ki}}),$$

where

$$M(x,t) := T(\psi_{x,0}(t), \psi_{2x,0}(t), \cdots, \psi_{2^{k-1}x,0}(t)),$$

for all $x \in X$ and $t > 0$.

4.2 Sextic functional equation in non-Archimedean random normed spaces.

Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and (Y,μ,T) be a non-Archimedean random Banach space over $\mathcal{K}$. We investigate the stability of (2.2.1), where f is a mapping from X to Y and $f(0) = 0$. It is well known that a function f satisfies the functional equation (2.2.1) if and only if it is sextic. Next we define a random approximately sextic mapping. Let ψ be a distribution function on $X \times X \times [0, \infty)$ such that $\psi(x, y, .)$ is nondecreasing and

$$\psi(cx, cy, t) \geq \psi(x, x, \frac{t}{|c|}) \quad \forall x \in X, c \neq 0$$

.

Definition 4.2.1. A mapping $f : X \to Y$ is said to be ψ-approximately sextic if

$$\mu_{D_s f(x,y)}(t) \geq \psi(x, y, t), \quad \forall x, y \in X, t > 0, \tag{4.2.1}$$

where

$$\begin{aligned} D_s f(x, y) := & f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) \\ & - (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)], \end{aligned}$$

for all $x, y \in X$ and $t > 0$. In this section, we assume that $n \neq 0$ (i.e. the characteristic of $\mathcal{K}$ is not 0).

Theorem 4.2.1. *Let $f : X \to Y$ be a ψ-approximately sextic function. If, for some $\alpha \in \mathbb{R}$ with $\alpha > 0$ and for some positive integer k with $|n^k| < \alpha$, $n \geq 2$, $n \in N$.*

$$\psi(n^{-k}x, n^{-k}y, t) \geq \psi(x, x, \alpha t), \tag{4.2.2}$$

and

$$\lim_{m\to\infty} T_{j=m}^{\infty} M\left(x, \frac{\alpha^j t}{|n|^{kj}}\right) = 1, \tag{4.2.3}$$

for all $x \in X$ and $t > 0$, then there exists a unique sextic mapping $Q : X \to Y$ such that:

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|n|^{ki}}\right), \tag{4.2.4}$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T\left(\psi(x, 0, t), \psi(nx, 0, t), \cdots, \psi(n^{k-1}x, 0, t)\right),$$

for all $x \in X$ and $t > 0$.

Proof. First we show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$

$$\mu_{f(n^j x)-n^{6j}f(x)}(t) \geq M_j(x,t) = T\left(\psi(x,0,t), \psi(2x,0,t), \cdots, \psi(n^{j-1}x,0,t)\right), \quad (4.2.5)$$

putting $y = 0$ in (4.2.1) we have

$$\mu_{2f(nx)-2n^6 f(x)}(t) \geq \psi(x,0,t),$$

then

$$\mu_{f(nx)-n^6 f(x)}(t) \geq \psi(x,0,2t) \geq \psi(x,0,t) \quad \forall x \in X, t > 0.$$

This prove (4.2.5) for $j = 1$. Assume that (4.2.5) hold for some $j > 1$. Replacing y by 0 and x by $n^j x$ in (4.2.1) we get

$$\mu_{f(n^{j+1}x)-n^6 f(n^j x)}(t) \geq \psi(n^j x,0,t), \quad \forall x \in X, t > 0. \quad (4.2.6)$$

Since $|n^6| \leq 1$ for $n \geq 2$, it follows that

$$\begin{aligned}
\mu_{f(n^{j+1}x)-n^{6(j+1)}f(x)}(t) &\geq T\left(\mu_{f(n^{j+1}x)-n^6 f(n^j x)}(t), \mu_{n^6 f(n^j x)-n^{6(j+1)}f(x)}(t)\right) \\
&= T\left(\mu_{f(n^{j+1}x)-n^6 f(n^j x)}(t), \mu_{f(n^j x)-n^{6j}f(x)}(\frac{t}{|n^6|})\right) \\
&\geq T\left(\mu_{f(n^{j+1}x)-n^6 f(n^j x)}(t), \mu_{f(n^j x)-n^{6j}f(x)}(t)\right) \\
&\geq T\left(\psi(n^j x,0,t), M_j(x,t)\right) \\
&= M_{j+1}(x,t), \quad \forall x \in X, t > 0.
\end{aligned}$$

So

$$\mu_{f(n^j x)-n^{6j}f(x)}(t) \geq M(x,t),$$

holds for all $j \geq 1$, in particular, we have

$$\mu_{f(n^k x)-n^{6k}f(x)}(t) \geq M(x,t), \quad \forall x \in X, t > 0. \quad (4.2.7)$$

Replacing x by $n^{-(km+k)}x$ in (4.2.7) and using the inequality (4.2.2), we have

$$\mu_{f\left(\frac{x}{n^{km}}\right)-n^{6k}f\left(\frac{x}{n^{k+km}}\right)}(t) \geq M\left(\frac{x}{n^{k+km}}, t\right) \geq M\left(x, \alpha^{m+1}t\right),$$

for all $x \in X$, $t > 0$ and $m \geq 0$. Then we have

$$\mu_{(n^{6k})^m f\left(\frac{x}{n^{km}}\right)-n^{6k}\left(n^{6k}\right)^m f\left(\frac{x}{n^{k(m+1)}}\right)}(t) \geq M\left(x, \frac{\alpha^{m+1}t}{|n^{6k}|^m}\right) \geq M\left(x, \frac{\alpha^{m+1}t}{|n^k|^m}\right),$$

for all $x \in X$, $t > 0$ and $m \geq 0$. So

$$\mu_{(n^{6k})^m f\left(\frac{x}{n^{km}}\right) - n^{6k(m+1)} f\left(\frac{x}{n^{k(m+1)}}\right)}(t) \geq M\left(x, \frac{\alpha^{m+1} t}{|n^k|^m}\right),$$

for all $x \in X$ and $t > 0$.

$$\begin{aligned}\mu_{(n^{6k})^m f\left(\frac{x}{n^{km}}\right) - n^{6k(m+p)} f\left(\frac{x}{n^{k(m+p)}}\right)}(t) &\geq T_{j=m}^{m+(p-1)}\left(\mu_{(n^{6k})^j f\left(\frac{x}{n^{kj}}\right) - n^{6k(j+1)} f\left(\frac{x}{n^{k(j+1)}}\right)}(t)\right) \\ &\geq T_{j=m}^{m+(p-1)} M\left(x, \frac{\alpha^{j+1}}{|n^k|^j} t\right),\end{aligned}$$

for all $x \in X$, $t > 0$. Since $\lim_{m\to\infty} T_{j=m}^{\infty} M\left(x, \frac{\alpha^{j+1}}{|n^k|^j} t\right) = 1$, for all $x \in X$, $t > 0$, it follows that $\{(n^{6k})^m f\left(\frac{x}{(n^k)^m}\right)\}$ is a Cauchy sequence in the non-Archimedean random Banach space (Y, μ, T). Hence, we can define a mapping $Q : X \to Y$ such that

$$\lim_{m\to\infty} \mu_{(n^{6k})^m f\left(\frac{x}{(n^k)^m}\right) - Q(x)}(t) = 1,$$

for all $x \in X$, $t > 0$. It follows that for all $m \geq 1$, $x \in X$ and $t > 0$.

$$\begin{aligned}\mu_{f(x) - (n^{6k})^m f\left(\frac{x}{(n^k)^m}\right)}(t) &= \mu_{\sum_{i=0}^{m-1} (n^{6k})^i f\left(\frac{x}{(n^k)^i}\right) - (n^{6k})^{i+1} f\left(\frac{x}{(n^k)^{i+1}}\right)}(t) \\ &\geq T_{i=0}^{m-1}(\mu_{(n^{6k})^i f\left(\frac{x}{(n^k)^i}\right) - (n^{6k})^{i+1} f\left(\frac{x}{(n^k)^{i+1}}\right)}(t)) \\ &\geq T_{i=0}^{m-1}\left(M\left(x, \frac{\alpha^{i+1}}{|n^k|^i} t\right)\right),\end{aligned}$$

and so

$$\begin{aligned}\mu_{f(x)-Q(x)}(t) &\geq T\left(\mu_{f(x) - (n^{6k})^m f\left(\frac{x}{(n^k)^m}\right)}(t), \mu_{(n^{6k})^m f(\frac{x}{(n^k)^m}) - Q(x)}(t)\right) \\ &\geq T\left(T_{i=0}^{m-1} M\left(x, \frac{\alpha^{i+1}}{|n^k|^i} t\right), \mu_{(n^{6k})^m f\left(\frac{x}{(n^k)^m}\right) - Q(x)}(t)\right),\end{aligned}$$

taking $m \to \infty$ we have

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1}}{|n^k|^i} t\right),$$

which prove (4.2.4). Since T is continuous, from a well-known result in probabilistic metric space (see e.g., [64, Chapter 12]) it follows that
$\lim_{m\to\infty} \mu_{D_Q f(x,y)}(t) =$

$$\mu_{Q(nx+y)+Q(nx-y)+Q(x+ny)+Q(x-ny)-(n^4+n^2)[Q(x+y)+Q(x-y)]-2(n^6-n^4-n^2+1)[Q(x)+Q(y)]}(t),$$

for all $x, y \in X$, $t > 0$, where

$$\begin{aligned} D_Q f(x,y) =& (n^{6k})^m f(\frac{nx+y}{n^{km}}) + (n^{6k})^m f(\frac{nx-y}{n^{km}}) + \\ & (n^{6k})^m f(\frac{x+ny}{n^{km}}) + (n^{6k})^m f(\frac{x-ny}{n^{km}}) \\ & + (n^4+n^2)(n^{6k})^m [f(\frac{x+y}{n^{km}}) + (f(\frac{x-y}{n^{km}})] \\ & - 2(n^6-n^4-n^2+1)(n^{6k})^m [f(\frac{x}{n^{km}}) + f(\frac{y}{n^{km}})]. \end{aligned}$$

On the other hand, replacing x, y by $n^{-km}x$, $n^{-km}y$ in (4.2.1) and using (4.2.2) we get

$$\begin{aligned} \mu_{D_Q f(x,y)}(t) &\geq \psi(n^{-km}x, n^{-km}y, \frac{t}{|n^{6k}|^m}) \\ &\geq \psi(n^{-km}x, n^{-km}y, \frac{t}{|n^k|^m}) \\ &\geq \psi(x, y, \frac{\alpha^m t}{|n^k|^m}), \end{aligned}$$

for all $x, y \in X$, $t > 0$. Since $\lim_{m\to\infty} \psi(x, y, \frac{\alpha^m t}{|n^k|^m}) = 1$, we show that Q is a sextic mapping. Finally if $Q' : X \to Y$ is a nother sextic mapping such that

$$\mu_{Q'(x)-f(x)}(t) \geq M(x,t), \quad \forall x \in X, t > 0,$$

then, for all $m \in N$, $x \in X$ and $t > 0$,

$$\mu_{Q(x)-Q'(x)}(t) \geq T(\mu_{Q(x)-(n^{6k})^m f(\frac{x}{|n^k|^m})}, \mu_{(n^{6k})^m f(\frac{x}{|n^k|^m})-Q'(x)}(t)),$$

Therefor, we conclude that $Q = Q'$ this completes the proof. □

Corollary 4.2.2. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and (Y, μ, T) be non-Archimedean random Banach space over $\mathcal{K}$ under the t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be a ψ-approximately sextic mapping. If, for some $\alpha \in \mathbb{R}$ with $\alpha > 0$, and some positive integer k with $|n^k| < \alpha$, $n \geq 2$.*

$$\psi(n^{-k}x, n^{-k}y, t) \geq \psi(x, y, \alpha t),$$

for all $x \in X$ and $t > 0$. Then there exists a unique sextic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M(x, \frac{\alpha^{i+1}}{|n^k|^i}),$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T(\psi(x, 0, t), \psi(nx, 0, t), \cdots, \psi(n^{k-1}x, 0, t)),$$

for all $x \in X$ and $t > 0$.

Proof. Since

$$\lim_{j\to\infty} M(x, \frac{\alpha^j t}{|n^k|^j}) = 1,$$

for all $x \in X$, $t > 0$ and T is of Hadžić type, it follows that

$$\lim_{m\to\infty} T_{j=m}^{\infty} M(x, \frac{\alpha^j t}{|n^k|^j}) = 1,$$

for all $x \in X$ and $t > 0$. Now, if we can apply Theorem 4.2.1, then we can get the conclusion. □

Example 4.2.1. *Let (X, μ, T_M) be a non-Archimedean random normed space in which*

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$ and (Y, μ, T_M) be a complete non-Archimedean random normed space. Define

$$\psi(x, y, t) = \frac{t}{1 + t}.$$

It is easy to see that (4.2.2) holds for $\alpha = 1$. Also, since $M(x, t) = \frac{t}{1+t}$, we have

$$\lim_{m\to\infty} T_{M,j=m}^{\infty}(x, \frac{\alpha^j t}{|n|^{kj}}) = \lim_{m\to\infty}(\lim_{i\to\infty} T_{M,j=m}^{i} M(x, \frac{t}{|n|^{kj}}))$$
$$= \lim_{m\to\infty}\lim_{i\to\infty}(\frac{t}{t + |n|^{km}}) = 1,$$

for all $x \in X$ and $t > 0$.

Let $f : X \to Y$ be a ψ-approximately sextic mapping. Thus all the conditions of Theorem (4.2.1) hold and so there exists a unique sextic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{t}{t + |n^k|}.$$

4.3 Mixed type functional equation in non-Archimedean random normed spaces.

In this section we investigate the stability of the additive-quadratic functional equation (2.3.1), where f is a mapping from X to Y and $f(0) = 0$. since f is a sum of an even function and an odd function, therefore f satisfies the functional equation (2.3.1) if and only if it is additive-quadratic. Next we define a random approximately additive-quadratic mapping. Let ψ be a distribution function on $X \times X \times [0, \infty)$ such that $\psi(x, y, .)$ is nondecreasing and

$$\psi(cx, cy, t) \geq \psi(x, x, \frac{t}{|c|}) \quad \forall x \in X, c \neq 0$$

.

Definition 4.3.1. A mapping $f : X \to Y$ is said to be ψ-approximately additive-quadratic if

$$\mu_{D_s f(x,y)}(t) \geq \psi(x, y, t), \quad \forall x, y \in X, t > 0, \tag{4.3.1}$$

where

$$\begin{aligned} D_s f(x, y) := {} & f(2x + y) + f(2x - y) - 2[f(x + y) + f(x - y)] \\ & - 2[f(x) + f(-x)] + [f(y) + f(-y)], \end{aligned}$$

for all $x, y \in X$ and $t > 0$.

In this section, we assume that $2 \neq 0$ (i.e. the characteristic of $\mathcal{K}$ is not 2).

Theorem 4.3.1. *Let $f : X \to Y$ be an even and ψ-approximately additive-quadratic function. If, for some $\alpha \in \mathbb{R}$ with $\alpha > 0$ and for some positive integer k with $|2^k| < \alpha$.*

$$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x, x, \alpha t), \tag{4.3.2}$$

and

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1, \tag{4.3.3}$$

for all $x \in X$ and $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that:

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right), \tag{4.3.4}$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T\left(\psi(x,0,t), \psi(2x,0,t), \cdots, \psi(2^{k-1}x,0,t)\right),$$

for all $x \in X$ and $t > 0$.

Proof. The proof is similar to prove of Theorem (4.2.1) □

In Theorem (4.3.1), if f is an odd mapping, then the following theorem can be proved similarly.

Theorem 4.3.2. *Let $f : X \to Y$ be an odd and ψ-approximately additive-quadratic function. If, for some $\alpha \in \mathbb{R}$ with $\alpha > 0$ and for some positive integer k with $|2^k| < \alpha$.*

$$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x, x, \alpha t),$$

and

$$\lim_{n\to\infty} T_{j=n}^{\infty} M\left(x, \frac{\alpha^j t}{|2|^{kj}}\right) = 1,$$

for all $x \in X$ and $t > 0$, then there exists a unique additive mapping $Q : X \to Y$ such that:

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M\left(x, \frac{\alpha^{i+1} t}{|2|^{ki}}\right),$$

for all $x \in X$ and $t > 0$, where

$$M(x,t) := T\left(\psi(x,0,t), \psi(2x,0,t), \cdots, \psi(2^{k-1}x,0,t)\right),$$

for all $x \in X$ and $t > 0$.

Corollary 4.3.3. *Let $\mathcal{K}$ be a non-Archimedean field, X be a vector space over $\mathcal{K}$ and (Y, μ, T) be non-Archimedean random Banach space over $\mathcal{K}$ under the t-norm $T \in \mathcal{H}$. Let $f : X \to Y$ be an even and ψ-approximately additive-quadratic mapping. If, for some $\alpha \in \mathbb{R}$ with $\alpha > 0$, and some positive integer k with $|2^k| < \alpha$.*

$$\psi(2^{-k}x, 2^{-k}y, t) \geq \psi(x, y, \alpha t),$$

for all $x \in X$ and $t > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty} M(x, \frac{\alpha^{i+1}}{|2^k|^i}),$$

for all $x \in X$ and $t > 0$, where

$$M(x, t) := T(\psi(x, 0, t), \psi(2x, 0, t), \cdots, \psi(2^{k-1}x, 0, t)),$$

for all $x \in X$ and $t > 0$.

Proof. Since

$$\lim_{j\to\infty} M(x, \frac{\alpha^j t}{|2^k|^j}) = 1,$$

for all $x \in X$, $t > 0$ and T is of Hadžić type, it follows that

$$\lim_{n\to\infty} T_{j=n}^{\infty} M(x, \frac{\alpha^j t}{|2^k|^j}) = 1,$$

for all $x \in X$ and $t > 0$. Now, if we can apply Theorem (4.2.1), then we can get the conclusion. □

Example 4.3.1. *Let (X, μ, T_M) be a non-Archimedean random normed space in which*

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$ and (Y, μ, T_M) be a complete non-Archimedean random normed space. Define

$$\psi(x, y, t) = \frac{t}{1 + t}.$$

It is easy to see that 4.3.2 holds for $\alpha = 1$. Also, since $M(x, t) = \dfrac{t}{1 + t}$, we have

$$\lim_{n\to\infty} T_{M,j=n}^{\infty}(x, \frac{\alpha^j t}{|2|^{kj}}) = \lim_{n\to\infty} (\lim_{i\to\infty} T_{M,j=n}^{i} M(x, \frac{t}{|2|^{kj}}))$$
$$= \lim_{n\to\infty} \lim_{i\to\infty} (\frac{t}{t + |2|^{kn}}) = 1,$$

for all $x \in X$ and $t > 0$.

Let $f : X \to Y$ be an even and ψ-approximately additive-quadratic mapping. Thus all the conditions of Theorem (4.3.1) hold and so there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \frac{t}{t + |2^k|}.$$

Conclusion

In this thesis we conclude that it is possible to prove stability of this sextic function

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] + 2(n^6-n^4-n^2+1)[f(x)+f(y)],$$

and this additive-quadratic functional equation

$$f(2x+y)+f(2x-y)=2[f(x+y)+f(x-y)]+2[f(x)+f(-x)]-[f(y)+f(-y)],$$

in random normed spaces and various random normed spaces by direct method and fixed point method.

In Chapter 2, we prove stability of a sextic functional eqution and an additive-quadratic funcional equation above in random normed spaces via direct method under arbitrary t-norms and via fixed point method undrt min t-norm. In Chapter 3, we prove stability of the same sextic functional eqution and an additive-quadratic funcional equations in intuitionistic random normed spaces via direct and fixed point methods. In chapter 4, we prove stability of the same functional equtions in non-Archimedean random normed spaces via direct method.

Acronyms

AQCQ .... additive-quadratic-cubic-quartic

AQ .... additive-quadratic

$\mathcal{H}$ .... Hadžić-type

IRN-space .... intuitionistic random normed space

RN-space .... random normed space

$\triangle^+$ .... space of all distribution functions

$\mathcal{N}$ .... involutive negation

τ .... binary operation on $\triangle^+$

t-conorm .... triangular conorm

t-norm .... triangular norm

T_L .... Lukasiewicz t-norm

T_M .... minimum t-norm

(X, μ, T) .... random normed space

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Symbols

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bibliography

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