International Journal of Scientific & Engineering Research, Volume 7, Issue 4, April-2016
ISSN 2229-5518

Numerical solution of fractional integro-differential equations by least squares method and shifted Laguerre polynomials pseudo-spectral method

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Abstract— In this paper, we investigate the numerical solution of linear fractional integro-differential equations by least squares method with aid of shifted Laguerre polynomial. Some numerical examples are presented to illustrate the theoretical results.

Keywords — Linear fractional integro-differential equations; Caputo fractional derivatives; Pseudo-spectral method; Laguerre polynomials; least squares method.

1 INTRODUCTION

Fractional derivatives have recently played a significant role in many areas of sciences, engineering, fluid mechanics, biology, physics and economies Error! Reference source not found.. Many real-world physical systems display fractional order dynamics, that is their behavior is governed by fractional order differential equations. Consequently, considerable attention has been given to the solutions of fractional differential equations (FDEs) and integral equations of physical interest (Error! Reference source not found., Error! Reference source not found., Error! Reference source not found., Error! Reference source not found.). Most non-linear FDEs do not have exact analytic solutions, so approximate and numerical techniques (Error! Reference source not found., Error! Reference source not found.) must be used. Many mathematical problems in science and engineering are set in unbounded domains. There is a need to consider practical design and implementation issues in scientific computing for reliable and efficient solutions of these problems. Several numerical methods to solve the FDEs have been given such as variational iteration method Error! Reference source not found., homotopy perturbation method (Error! Reference source not found., Error! Reference source not found.), Adomian's decomposition method Error! Reference source not found., homotopy analysis method Error! Reference source not found. and collocation method (Error! Reference source not found., Error! Reference source not found., Error! Reference source not found.).

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of differential equations Error! Reference source not found.. In Error! Reference source not found., Khader introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials. In Error! Reference source not found. the generalized Laguerre polynomials were used to compute a spectral solution of a non-linear boundary value problems. The classical generalized Laguerre polynomials constitute a complete orthogonal sets of functions on the semi-infinite interval $[0,\infty)$ Error! Reference source not found.. Convolution structures of Laguerre polynomials were presented in Error! Reference source not found.. Also, other spectral methods based on other orthogonal polynomials are used to obtain spectral solutions on unbounded intervals Error! Reference source not found.. Spectral collocation methods are efficient and highly accurate techniques for numerical solution of non-linear differential equations. The basic idea of the spectral collocation method is to assume that the unknown solution $u(x)$ can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for a problem under consideration.

In this paper least squares method with aid of shifted Laguerre polynomial is applied to solving fractional Integro-differential equations. Least squares method has been studied in (Error! Reference source not found., Error! Reference source not found., Error! Reference source not found., Error! Reference source not found., Error! Reference source not found.).

In this paper, we are concerned with the numerical solution of the following linear fractional Integro-differential equation:

$$D^{\nu}\varphi(x) = f(x) + \int_0^1 K(x,t)\varphi(t)dt, \qquad 0 \le x,t \le 1, \qquad (1)$$

with the following supplementary conditions:

$$\varphi^{(i)}(0)=\delta_i, \qquad n-1<\nu\le n, \quad n\in N, \quad (2)$$

where $D^{\nu}\varphi(x)$ indicates the ν th Caputo fractional derivative of $\varphi(x); f(x); K(x,t)$ are given functions, x and t are real variables varying in the interval [0, 1], and $\varphi(x)$ is the unknown function to be determined.

The structure of this paper is arranged in the following way: In section 2, we introduce some basic definitions about Caputo fractional derivatives and properties of the classical generalized Laguerre polynomials. In section 3, we introduce the fundamental theorems for the fractional derivatives of the generalized Laguerre polynomials and its convergence analysis. In section 4, the procedure of solution of linear fractional integro-differential equation. In section 5, numerical example is given to solve the LFIDs and show the accuracy of the presented method. Finally, in section 6, the report ends with a brief conclusion and some remarks.

2 Preliminars and Notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1 The Caputo fractional derivative

Definition 1

The Caputo fractional derivative operator D^{ν} of order ν is defined in the following form:

$$D^{\nu}f(x)=\frac{1}{\Gamma(m-\nu)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}}dt, \quad \nu>0,$$

where $m-1<\nu\le m, m\in\mathbb{N}, x>0.$

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^{\nu}(\lambda f(x)+\mu g(x))=\lambda D^{\nu}f(x)+\mu D^{\nu}g(x),$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^{\nu}C=0, \qquad C \text{ is a constant}, \qquad (3)$$

$$D^{\nu}x^{n}=\begin{cases}0, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n<\lceil\nu\rceil;\\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\nu)}x^{n-\nu}, & \text{for} \quad n\in\mathbb{N}_0 \text{ and } n\ge\lceil\nu\rceil.\end{cases} \qquad (4)$$

We use the ceiling function $\lceil\nu\rceil$ to denote the smallest integer greater than or equal to ν, and $\mathbb{N}_0=\{0,1,2,...\}$. Recall that for $\nu\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see (Error! Reference source not found., Error! Reference source not found., Error! Reference source not found., Error! Reference source not found.).

2.2 The definition and properties of the classical Laguerre polynomials

The classical Laguerre polynomials $[L_i^{(\alpha)}(x)]_{i=0}^{\infty}$, $\alpha>-1$ are defined on the unbounded interval $(0,\infty)$ and can be determined with the aid of the following recurrence formula

$$(i+1)L_{i+1}^{(\alpha)}(x)+(x-2i-\alpha-1)L_i^{(\alpha)}(x)+(i+\alpha)L_{i-1}^{(\alpha)}(x)=0, \qquad i=1 \qquad (5)$$

where, $L_0^{(\alpha)}(x)=1$ and $L_1^{(\alpha)}(x)=\alpha+1-x$.

The analytic form of these polynomials of degree n is given by

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$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} x^k = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}, \tag{6}$$

$L_n^{(\alpha)}(0) = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$. These polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight function $w(x) = \frac{1}{\Gamma(1+\alpha)} x^{\alpha} e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} x^{\alpha} e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \begin{pmatrix} n+\alpha \\ n \end{pmatrix} \delta_{mn}. \tag{7}$$

Also, they satisfy the differentiation formula

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x), \quad k = 0,1,\ldots,n. \tag{8}$$

Any function $u(x)$ belongs to the space $L_w^2[0,\infty)$ of all square integrable functions on $[0,\infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x) = \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{9}$$

where the coefficients c_i are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)} \int_0^{\infty} x^{\alpha} e^{-x} L_i^{(\alpha)}(x) u(x) dx, \quad i = 0,1,\ldots. \tag{10}$$

Consider only the first $(m+1)$ terms classical Laguerre polynomials, so we can write

$$u_m(x) \cong \sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{11}$$

For more details on Laguerre polynomials, its definitions and properties see (Error! Reference source not found., Error! Reference source not found., Error! Reference source not found.).

3 THE APPROXIMATE FRACTIONAL DERIVATIVES OF $L_n^{(\alpha)}(x)$ AND ITS CONVERGENCE ANALYSIS

The main goal of this section is to introduce the following theorems to derive an approximate formula of the fractional derivatives of the classical Laguerre polynomials and study the truncating error and its convergence analysis.

Lemma 1.

Let $L_n^{(\alpha)}(x)$ be a generalized Laguerre polynomial then

$$D^{\nu} L_n^{(\alpha)}(x) = 0, \quad n = 0,1,\ldots,\lceil \nu \rceil - 1, \quad \nu > 0. \tag{12}$$

Proof. This lemma can be proved directly using a combination of Eqs.((3))-((4)).

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 1

Let $u(x)$ be approximated by the generalized Laguerre polynomials as ((11)) and also suppose $\nu > 0$ then, its Caputo fractional derivative can be written in the following form

$$D^{\nu}(u_m(x)) \cong \sum_{i=\lceil \nu \rceil}^{m} \sum_{k=\lceil \nu \rceil}^{i} c_i w_{i,k}^{(\nu)} x^{k-\nu}, \tag{13}$$

where $w_{i,k}^{(\nu)}$ is given by

$$w_{i,k}^{(\nu)} = \frac{(-1)^k}{\Gamma(k+1-\nu)} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix}. \tag{14}$$

Proof. Khader et. al. proved them in Error! Reference source not found..

For the Laguerre polynomials $L_n^{(\alpha)}(x)$, we have the following global uniform bounds estimates

$$|L_n^{(\alpha)}(x)| \le \begin{cases} \frac{(\alpha+1)_n}{n!} e^{x/2}, & for \quad \alpha \ge 0, x \ge 0, n = 0,1,2,...; \\ (2 - \frac{(\alpha+1)_n}{n!}) e^{x/2}, & for \quad -1 < \alpha \le 0, x \ge 0, n = 0,1,2,.... \end{cases} \tag{15}$$

Proof. These estimates were presented in Error! Reference source not found. and Error! Reference source not found., Szego proved them in Error! Reference source not found..

4. Procedure solution using Laguerre collocation method

In this section, the least squares method with aid of Laguerre collocation method is applied to study the numerical solution of the fractional Integro-differential ((1)).

The procedure of the implementation is given by the following steps:

1. Substitute by Eq.((11)) into Eq.((1)) we obtain Error! Reference source not found.:

$$D^{\nu}\left(\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x)\right) = f(x) + \int_0^1 K(x,t)\left(\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x)\right) dt. \tag{16}$$

2. Hence the residual equation is defined as

$$R(x, c_0, c_1, ..., c_n) = \sum_{i=0}^{m} c_i D^{\nu} L_i^{(\alpha)}(x) - f(x) - \int_0^1 K(x,t)\left(\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x)\right) dt. \tag{17}$$

3. Let

$$S(c_0, c_1, ..., c_n) = \int_0^1 \left(R(x, c_0, c_1, ..., c_n)\right)^2 \cdot \omega(x) dx. \tag{18}$$

where $\omega(x)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(x) = 1$ for simplicity.

4. Thus

$$S(c_0, c_1, ..., c_n) = \int_0^1 \left(\sum_{i=0}^{m} c_i D^{\nu} L_i^{(\alpha)}(x) - f(x) - \int_0^1 K(x,t)\left(\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x)\right) dt\right)^2 dx. \tag{19}$$

5. So, finding the values of $c_i, i = 0,1,...,n$, which minimize S is equivalent to finding the best approximation for the solution of the fractional Integro-differential equation ((1)).

6. The minimum value of S is obtained by setting

$$\frac{\partial S}{\partial c_i} = 0 \qquad i = 0,1,...,m. \tag{20}$$

7. Applying ((20)) to ((19)) we obtain

$$\int_0^1\left(\sum_{i=0}^{m}c_i D^{\nu}L_i^{(\alpha)}(x)-f(x)-\int_0^1 K(x,t)\left(\sum_{i=0}^{m}c_i L_i^{(\alpha)}(x)\right)dt\right)\times\left(D^{\nu}L_i^{(\alpha)}-\int_0^1 K(x,t)L_i^{(\alpha)dt}(x)\right)dx. \quad (21)$$

By evaluating the above equation for $i=0,1,\ldots,n$ we can obtain a system of $(m+1)$ linear equations with $(m+1)$ unknown coefficients c_i. This system can be formed by using matrices form as follows:

$$A=\begin{pmatrix}\int_0^1 R(x,c_0)h_0dx & \int_0^1 R(x,c_1)h_0dx & \ldots & \int_0^1 R(x,c_m)h_0dx\\ \int_0^1 R(x,c_0)h_1dx & \int_0^1 R(x,c_1)h_1dx & \ldots & \int_0^1 R(x,c_m)h_1dx\\ \ldots & \ldots & \ldots & \ldots\\ \int_0^1 R(x,c_0)h_mdx & \int_0^1 R(x,c_1)h_mdx & \ldots & \int_0^1 R(x,c_m)h_mdx\end{pmatrix},$$

$$B=\begin{pmatrix}\int_0^1 f(x)h_0dx\\ \int_0^1 f(x)h_1dx\\ \vdots\\ \int_0^1 f(x)h_mdx\end{pmatrix},$$

where

$$h_i=D^{\nu}L_i^{(\alpha)}(x)-\int_0^1 K(x,t)\sum_{i=0}^{m}c_i L_i^{(\alpha)}(x)dt,\ i=1,2,\ldots,m,$$

$$R(x,t)=\sum_{i=0}^{m}c_i D^{\nu}L_i^{(\alpha)}(x)-\int_0^1 K(x,t)\left(\sum_{i=0}^{m}c_i L_i^{(\alpha)}(x)\right)dt,\ i=0,1,\ldots,m$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solution of ((1)).

5. Applications and Numerical Results

In this section, we demonstrate the capability of the introduced approach. To achieve this aim, we solve two widely used examples from the literature. The introduced problems are stated in the traditional FLIDEs framework and then reformulated via our introduced methodology.

Example 1:

Consider the following fractional integrodifferential equation (Error! Reference source not found.)

$$D^{1/2}\varphi(x)=\frac{(8/3)x^{3/2}-2x^{1/2}}{\sqrt{\pi}}+\frac{x}{12}+\int_0^1 xt\varphi(t)dt,\qquad 0\le x,t\le 1, \quad (22)$$

subject to $\varphi(0)=0$ with the exact solution $\varphi(x)=x^2-x$.

Applying the least squares method with aid of Laguerre collocation method of $L_i^{(\alpha)}(x), i=0,1,\ldots,m$ at $m=7$, to the fractional integro-differential equation ((22)) we obtain a system of ((21)) linear equations with ((21)) unknown coefficients $c_i, i=0,1,\ldots,7$.

The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series ((11)). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results we can see that these solutions are in more accuracy of those obtained in (Error! Reference source not found..

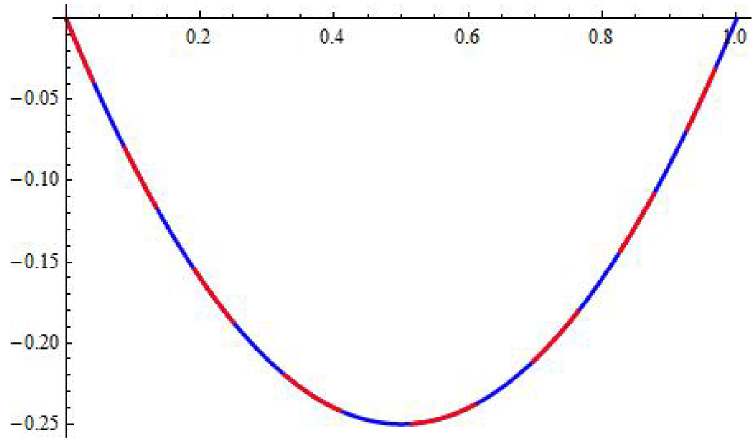

Figure 1 Comparison between the numerical solution and exact solution.

Example 2:

Consider the following fractional integrodifferential equation (Error! Reference source not found.)

$$D^{5/6}\varphi(x) = f(x) + \int_0^1 x e^t \varphi(t)dt, \qquad 0 \le x, t \le 1, \tag{23}$$

where $f(x) = \dfrac{-3x^{1/6}\Gamma(5/6)(-91+216x^2)}{91\pi} + (2-2e)x$,

subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x - x^3$.

Similarly as in Example 1 applying the least squares method with aid of Laguerre collocation method of $L_i^{(\alpha)}(x), i = 0,1,\ldots,m$ at $m = 7$, to the fractional integrodifferential equation ((23)) the numerical results are shown in Figures 5 and 6 and we obtain the approximate solution which is the same as the exact solution.

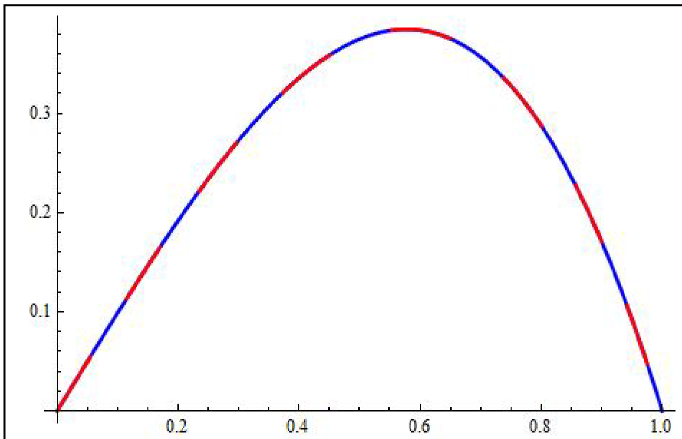

Figure 2 Comparison between the numerical solution and exact solution.

Example 3:

Consider the following fractional integrodifferential equation (Error! Reference source not found.)

$$D^{5/3}\varphi(x)=f(x)+\int_0^1(xt+x^2t^2)\varphi(t)dt, \qquad 0\le x,t\le 1, \tag{24}$$

where $f(x)=\dfrac{3\sqrt{3}\Gamma(2/3)x^{1/3}}{\pi}-x^2/5-x/4,$

subject to $\varphi(0)=0$ with the exact solution $\varphi(x)=x^2$.

Similarly as in Examples 1 and 2 applying the least squares method with aid of Laguerre collocation method of $L_i^{(\alpha)}(x), i=0,1,...,m$ at $m=7$, to the fractional integrodifferential equation ((24)) the numerical results are shown in Figures 7 and 8 and we obtain the approximate solution which is the same as the exact solution.

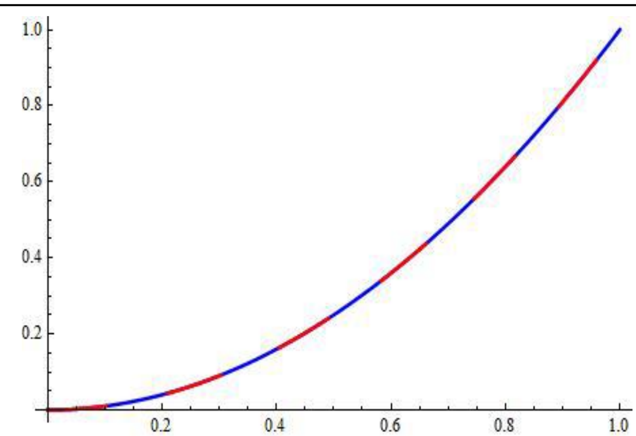

Figure 3 Comparison between the numerical solution and exact solution.

4 CONCLUSION

In this article, we introduced an accurate numerical technique for solving linear fractional integrodifferential equation. We have introduced an approximate formula for the Caputo fractional derivative of the generalized Laguerre polynomials in terms of classical Laguerre polynomials themselves. The fractional derivative is considered in the Caputo sense. The results show that the algorithm converges as the number of m terms is increased. The solution is expressed as a truncated Laguerre series and so it can be easily evaluated for arbitrary values of time using any computer program without any computational effort. From illustrative examples, we can conclude that this approach can obtain very accurate and satisfactory results. For all examples, the solution for the integer order case of the problem is also obtained for the purpose of comparison. In the end, from our numerical results using the proposed method, we can see that, the solutions are in excellent agreement with the exact solution and better than the numerical results obtained in Error! Reference source not found.. All computational calculations are made by Matlab program 12b and using Maple 16 programming.

ACKNOWLEDGMENT

Thank you for the referees their efforts. The authors would like to thank Prof.Dr. N. H. Sweilam, Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt, which provided support.

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