Some properties of a class of Meromorphic Univalent Functions

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Abstract: The in this paper, we to study the class of meromorphich univalent functions. We obtain some geometric properties, like, coefficient boundes, convex linear combinations, distortion bounds, extreme points, closure, Convexity Theorem, and neighborhood property .

Keywords: univalent function, coefficient bounds, convex linear combinations, distortion bounds, extreme points, closure, Convexity Theorem and neighborhood property.

1.Introduction:

Let S denote the class of functions analytic and meromorphic in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U/\{0\}$ and let $\sum$ denote the subclass of S consisting of functions of the form :-

$$f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n, (a_n \geq 0 \,; n \in \mathbb{N} = \{1,2, \dots\}) \qquad (1.1)$$

which are analytic and meromorphic univalent in the punctured unit disk U^*.

A function $f \in \sum$ is said to be meromorphically starlike of order β if

$$Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \beta, \qquad (z \in U = U^* \cup \{0\}, 0 \leq \beta < 1), \qquad (1.2)$$

and a function $f \in \sum$ is said to be meromorphically convex of order β if

$$Re\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \beta, (z \in U = U^* \cup \{0\}, 0 \leq \beta < 1). \quad (1.3)$$

We denote by $S^*(\beta), S(\beta)$, respectively, the classes of univalent meromorphic starlike functions of order β and univalent meromorphically convex functions of order β . Similar classes have been extensively studied by Clunie[3], Miller[6] and Atshan[1],[2].

Definition(1): A function $f \in \sum$ is said to be in the class $\sum(\alpha, t)$ if the following condition is satisfied:-

$$\left|\frac{\frac{tz^2 f''(z)}{zf'(z)}+t}{(1+2t)-\frac{2tz^2 f''(z)}{zf'(z)}}\right| < \alpha\,,\ where\ 0 < t < 1, 0 < \alpha < 1\,. \quad (1.4)$$

2. Coefficient estimates.

The following theorem gives a necessary and sufficient condition for a function f to be in the class $\sum(\alpha, t)$.

Theorem(1): Let $f \in \sum$. Then $f \in \sum(\alpha, t)$ if and only if

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_n \leq \alpha + 6\text{t}\alpha - \text{t}\,,\ where\ 0 < t < 1, 0 < \alpha < 1 \quad (2.1)$$

The result is sharp for the function f given by

$$f(z) = z^{-1} - \frac{\alpha+6\text{t}\alpha-\text{t}}{n[n-\alpha+2t\alpha n]}\ z^n, n \in \mathbb{N} \quad (2.2)$$

Proof: Suppose that the inequality (2.1) holds true and $|z| = 1$, then we have

$$|tz^2 f''(z) + tzf'(z)| - \alpha|(1+2t)zf'(z) - 2tz^2 f''(z)| =$$

$$\left|tz^{-1} + \sum_{n=1}^{\infty} n^2 a_n z^n\right| - \alpha\left|(1+6t)z^{-1} + \sum_{n=1}^{\infty} n[1-2tn]a_n z^n\right| \leq$$

$$t + \sum_{n=1}^{\infty} n^2 a_n - \alpha\,(1+6t) - \alpha\sum_{n=1}^{\infty} n[1-2tn]a_n \leq$$

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_n - [\alpha+6\text{t}\alpha-\text{t}] \leq 0\,,$$

by hypothesis. Thus by maximum modulus principle, $f \in \sum(\alpha, t)$.

Conversely, assume that

$$\left|\frac{\frac{tz^2 f''(z)}{zf'(z)} + t}{(1+2t) - \frac{2tz^2 f''(z)}{zf'(z)}}\right| = \left|\frac{tz^2 f''(z) + tzf'(z)}{(1+2t) - 2tz^2 f''(z)}\right| = \left|\frac{tz^{-1} + \sum_{n=1}^{\infty} n^2 a_n z^n}{-(1+6t)z^{-1} + \sum_{n=1}^{\infty} n[1-2tn] a_n z^n}\right| \leq \alpha.$$

Since $Re(z) \leq |z|$ for all z, we have

$$Re\left\{\frac{tz^{-1} + \sum_{n=1}^{\infty} n^2 a_n z^n}{-(1+6t)z^{-1} + \sum_{n=1}^{\infty} n[1-2tn] a_n z^n}\right\} < \alpha. \qquad (2.3)$$

We choose the value of z on the real axis and $z \to 1^{-}$. Through real values, we obtain inequality (2.3).

Corollary(1): Let $f \in \sum(\alpha, t)$. Then

$$a_n \leq \frac{\alpha + 6\mathrm{t}\alpha - \mathrm{t}}{n[n - \alpha + 2t\alpha n]}, \quad (n \geq 1). \qquad (2.4)$$

3.Convex linear Combination.

In the next theorem, we show that the class $\sum(\alpha, t)$ is closed under convex linear combination.

Theorem(2): The class $\sum(\alpha, t)$ is closed under convex linear combination.

Proof: Let $f_1(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,1} z^n$ and $f_2(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,2} z^n$ belong

to the class $\sum(\alpha, t)$, for $0 \leq \Upsilon \leq 1$.

We must show that the function h defined by $h(z) = \Upsilon f_1(z) + (1-\Upsilon) f_2(z) \in \sum(\alpha, t)$.

Since f_1 and $f_2 \in \sum(\propto, t)$, then by Theorem(1), we have $\sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n] a_{n,1} \leq \alpha + 6\mathrm{t}\alpha - \mathrm{t}$, $\sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n] a_{n,2} \leq \alpha + 6\mathrm{t}\alpha - \mathrm{t}$.

Now, $h(z) = \Upsilon f_1(z) + (1-\Upsilon) f_2(z) = z^{-1} + \sum_{n=1}^{\infty} \left[\Upsilon a_{n,1} + (1-\Upsilon) a_{n,2}\right] z^n$.

Then

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n][\Upsilon a_{n,1}+(1-\Upsilon)a_{n,2}] = \Upsilon\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,1}+(1-\Upsilon)\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,2}$$

$$\leq \Upsilon(\alpha+6t\alpha-t)+(1-\Upsilon)(\alpha+6t\alpha-t)=\alpha+6t\alpha-t.$$

Then by Theorem(1), we have $h(z) \in \sum(\alpha, t)$ and the proof is complete.

4. Closure Theorem.

Theorem(3): Let the function f_k defined by

$f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,k}\, z^n, \left(a_{n,k} \geq 0, n \in \mathbb{N}, k = 1,2, \dots, \ell\right)$

be in The class $\sum(\alpha, t)$ for every $k = 1,2, \dots, \ell$. Then the function h defined by

$h(z) = z^{-1} + \sum_{n=1}^{\infty} e_n\, z^n, (e_n \geq 0, n \in \mathbb{N}, n \geq 1)$, also belongs to the class $\sum(\alpha, t)$, where $e_n = \frac{1}{\ell}\sum_{k=1}^{\ell} a_{n,k}$.

Proof: Since $f_k(z) \in \sum(\alpha, t)$, we have

$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_k \leq \alpha + 6\text{t}\alpha - \text{t}$, for every $i = 1,2, \dots, \ell$.

Hence $\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]e_n = \sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n](\frac{1}{\ell}\sum_{k=1}^{\ell} a_{n,k})$

$$= \frac{1}{\ell}\sum_{k=1}^{\ell}(\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,k})$$

$$= \frac{1}{\ell}\sum_{k=1}^{\infty}(\alpha+6t\alpha-t) = \alpha+6t-t.$$

Therefore, by Theorem(1.1), we have $h \in \sum(\alpha, t)$.

5. Distortion Theorem.

In the following theorem, we prove distortion bounds associated with the class introduced in (1.4).

Theorem(4): If $f \in \sum(\alpha, t)$, then

$$\frac{1}{r}-\frac{\alpha+6t\alpha-t}{[1-\alpha+2t\alpha]}\leq|f(z)|\leq\frac{1}{r}+\frac{\alpha+6t\alpha-t}{[1-\alpha+2t\alpha]},(0<|z|=r<1). \tag{5.1}$$

The result is sharp for the function f given by (2.2).

Proof: Let $f(z)\in\sum(\alpha,t)$. Then by Theorem(1), we get

$$\sum_{n=1}^{\infty}a_n\leq\frac{\alpha+6\text{t}\alpha-\text{t}}{[1-\alpha+2t\alpha]}, \tag{5.2}$$

since $f(z)=z^{-1}-\sum_{n=1}^{\infty}a_n z^n$,

then $|f(z)|\leq\frac{1}{|z|}+\sum_{n=1}^{\infty}a_n|z|^n\leq\frac{1}{|z|}+|z|\sum_{n=1}^{\infty}a_n\leq\frac{1}{r}+\frac{\alpha+6\text{t}\alpha-\text{t}}{[1-\alpha+2t\alpha]}r.$ (5.3)

Similarly $|f(z)|\geq\frac{1}{r}-\frac{\alpha+6\text{t}\alpha-\text{t}}{[1-\alpha+2t\alpha]}r$ (5.4)

From (5.3) and (5.4), we get (5.1) and the proof is complete.

6.Extreme points.

In the next theorem, we obtain extreme points for the class $\sum(\alpha,t)$.

Theorem(5): Let $f_0(z)=z^{-1}$ and $f_n(z)=z^{-1}+\frac{\alpha+6\text{t}\alpha-\text{t}}{n[n-\alpha+2t\alpha n]}z^n$. Then $f(z)$ in the class $\sum(\alpha,t)$ if and only if it can be expressed in the form $f(z)=\sum_{n=0}^{\infty}M_n f_n(z)$, where $M_n\geq 0,(n\geq 0)$ and $\sum_{n=0}^{\infty}M_n=1$.

Proof: Assume that $f(z)=\sum_{n=0}^{\infty}M_n f_n(z)=M_0 z^{-1}+\sum_{n=1}^{\infty}\frac{(\alpha+6\text{t}\alpha-\text{t})M_n}{n[n-\alpha+2t\alpha n]}z^n$.

Then it follows that $\sum_{n=1}^{\infty}\frac{n[n-\alpha+2t\alpha n]}{(\alpha+6\text{t}\alpha-\text{t})}M_n\frac{(\alpha+6\text{t}\alpha-\text{t})}{n[n-\alpha+2t\alpha n]}=\sum_{n=1}^{\infty}M_n\leq 1.$

Therefore $f\in\sum(\alpha,t)$.

Conversely, assume that $f\in\sum(\alpha,t)$, then by(2.1), we have

$$a_n\leq\frac{\alpha+6\text{t}\alpha-\text{t}}{n[n-\alpha+2t\alpha n]},\qquad(n\geq 1).$$

Setting

$M_n=\frac{n[n-\alpha+2t\alpha n]}{(\alpha+6\text{t}\alpha-\text{t})}a_n$ and $M_{0=1-}\sum_{n=1}^{\infty}M_n$.

Hence, $f(z) = \sum_{n=0}^{\infty} M_n f_n(z) = M_0 f(z) + \sum_{n=1}^{\infty} M_n f_n(z)$.

This completes the proof .

7. Convexity .

In the following theorem, we obtain the radius of convexity for the functions class $\sum(\alpha, t)$.

Theorem(6): Let $f \in \sum(\alpha, t)$. Then f is univalent meromorphic convex of order α $(0 < \alpha < 1)$ in the disk $|z| < R$, where

$$R = \inf_{n\geq 1} \left\{ \frac{(1-\alpha)[n - \alpha + 2t\alpha n]}{(n - \alpha + 2)[\propto + 6t\alpha - t]} \right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function f given by (2.2).

Proof: It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \alpha \quad \text{for} |z| < R. \tag{7.1}$$

But

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \frac{zf''(z) + 2f'(z)}{f'(z)} \right| \leq \frac{\sum_{n=1}^{\infty} n(n+1) a_k |z|^{n-1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n-1}}.$$

Thus, (7.1) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} n(n+1) a_k |z|^{n-1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n-1}} \leq 1 - \alpha,$$

or if

$$\sum_{n=1}^{\infty} \frac{n(n - \alpha + 2)}{1 - \alpha} a_n |z|^{n-1} \leq 1. \tag{7.2}$$

Since $f \in \sum(\alpha, t)$, we have

$$\sum_{n=1}^{\infty} \frac{n[n - \alpha + 2t\alpha n]}{\alpha + 6\mathrm{t}\alpha - \mathrm{t}} a_n \leq 1.$$

Hence, (7.1) will be true if

$$\frac{n(n-\alpha+2)}{1-\alpha}|z|^{n-1} \leq \frac{n[n-\alpha+2t\alpha n]}{\alpha+6\mathrm{t}\alpha-\mathrm{t}} ,$$

or equivalently

$$|z| \leq \left\{\frac{(1-\alpha)[n-\alpha+2t\alpha n]}{(n-\alpha+2)[\alpha+6t\alpha-t]}\right\}^{\frac{1}{n-1}} (n \geq 1),$$

which follows the result.

8. Neighborhood property.

Next, we determine the inclusion relation involving (n,δ) –neighborhoods. Following the earlier works on neighborhoods of analytic functions by Goodman[4], Ruscheweyh[8] and Raina and Srivastava[7]but for meromorphic function studied by Liu and Srivastava[5] and Atshan[1],

We define the (n,δ) –neighborhoods of a function $f(z) \in \sum$ by

$$N_{n,\delta}(f) = \{g \in \sum : g(z) = z^{-1} - \sum_{n=1}^{\infty} b_n z^n \ and \ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \delta > 0\}. \qquad (8.1)$$

Definition(8.1): A function $g(z) \in \sum$ is said to be in the class $\sum^{\sigma}(\alpha,t)$ if there exists a function $f(z) \in \sum(\alpha,t)$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| \leq 1-\sigma, (z \in U, 0 \leq \sigma < 1). \qquad (8.2)$$

Theorem(7): Let $f(z) \in \sum(\alpha,t)$ and

$$\sigma = 1 - \frac{\delta(1-\alpha+2t\alpha)}{1-2\alpha-4t\alpha+t}. \qquad (8.3)$$

Then $N_{n,\delta}(f) \subset \sum^{\sigma}(\alpha,t)$.

Proof: Let $g(z) \in N_{n,\delta}(f)$. Then, we have from (2.44) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}).$$

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}).$$

Also, since $f(z) \in \sum(\alpha, t)$, we have from Theorem(1)

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha+6t\alpha-t}{1-\alpha+2t\alpha},$$

so that

$$\left|\frac{g(z)}{f(z)}-1\right| = \left|\frac{\sum_{n=1}^{\infty}(a_n-b_n)z^n}{z^{-1}-\sum_{n=1}^{\infty} a_n z^n}\right| \leq \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty}|a_n|} \leq \frac{\delta(1-\alpha+2t\alpha)}{1-2\alpha-4t\alpha+t} = 1-\sigma.$$

Thus, by Definition(8.1), $g(z) \in \sum^{\sigma}(\alpha, t)$ for σ given by(8.3).

This completes the proof.

References:

[1] W. G. Atshan , Subclass of meromorphic functions with positive coefficient defined by Ruscheweyh derivative II , J. Surveys in Math. With its Applications, 3(1)(2008),67-77.

[2] W. G. Atshan and S. R. Kulkarni, Subclass of meromorphic functions withpositive coefficients defined by Ruscheweyh derivative I, J. Rajasthan Acad. Phys. Sci., 6(2)(2007),129-140.

[3] J. G. Clunie, On meromorphic schicht functions, J. London Math. Soc. ,34(1959), 215-216.

[4] A. W. Goodman, Univalent functions and non-analytic curves, Proc. Amer. Math. Soc. , 8(1975), 598-601.

[5] J. L. Liu and H. M. Srivastava, A linear operator and associated families ofmeromorphically multivalent functions, J. Math. Anal. Appl. 259(2001),566-581.

[6] J. E. Miller, Convex meromorphic mapping and retated function, proc. Amer. Math. Soc., 25(1970), 220-228.

[7] R. K. Raina and H. M. Srivastava, Inclusion and neighborhoods properties of some analytic and multivalent functions, J. Inequal. Pure Appl. Math.,7(1)(2006), Art. 5, 1-6.

[8] S. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc. ,

81(1981), 521-527.