Performance of Linear and Quadratic Interpolators for FFT Frequency Estimation of Single-Tone under Gaussian Multiplicative Noise

Zahraa C.Al-Rammahi [1], Zahir M. Hussain [2,3] , Salah K. Albermany [2]
[1] MSc Candidate; Faculty of Computer Science and Mathematics, University of Kufa
[2] Faculty of Computer Science and Mathematics, University of Kufa, Najaf, Iraq
[3] Professor (Adjunct), School of Engineering, Edith Cowan University, Australia
Emails: zahir.hussain@uokufa.edu.iq

***Abstract*— This work analyses the performance of linear and different quadratic interpolators (in terms of estimation error) for FFT frequency estimation of single tones under the effects of multiplicative noise. This method finds a quadratic fit in the neighborhood of the maximum of FFT with the three points, then apply different approximation methods: maximum of FFT, barycentric, and Quinn's Estimator. Numerical results showed that barycentric method is the best estimator under Gaussian multiplicative noise in terms of minimum mean squared estimation error, especially at high signal-to-noise ratios.**

Keywords—freqency estimation; DFT; multiplecative noise; maximam likelihood estimator ; quadratic interpolators

I. INTRODUCTION

In reality, in most application there is no noiseless signals. noise is the external energy that impair the signals Especially the multiplicative noise, so A major problem in signal analysis is noise. There is a wide range of applications involved frequency estimation e.g. Communications, Frequency of Doppler radar, Radio frequency identification and Resonance sensor systems. Multiplicative signal models are encountered in various applications, such as: time-selective fading in communication channels, where the amplitude variations are usually modeled as Rayleigh or Rician distributed [1].

There is a variety of approaches to the frequency and phase estimation problem, with differences in performance as regards frequency estimation accuracy and computational complexity [2].

The maximum likelihood estimator (MLE) to estimate the frequency of a sinusoid damaged by additive Gaussian noise was thoroughly studied by Rife and Boorstyn [3].

Yizheng Liao in [2] discussed five approximate maximum likelihood estimators and analyzed their performance in terms of the mean squared frequency and phase estimation errors as well as the computational complexity.

However, none of the literature compares the performance and accuracy of different quadratic interpolators for FFT estimation methods in present of multiplicative noise. Therefore, in our work presents a comparative study of the accuracy of these approach. we will test various methods using mean-squared error (MSE) under different signal to noise ratios (SNR). Best method is the one that gives less MSE.

Finally, we simulate the signal with different value of time vector and number of samples to show that best method is the one that achieve accuracy in less number of sample then it has less computation complexity

The paper is organized as follows: The signal and observation models as well as its distribution are defined in section II. In section III, frequency estimation based on Spectral Domain (Fourier Transform). In section IV, simulation result and performance comparison.

II. SIGNAL MODEL AND NOISY SIGNAL DISTRIBUTION

Let the signal to be noisy single-tone sinusoid as follows:

$$y(t) = n(t)\cos(\omega_o t + \emptyset o) + \epsilon(t) \quad (1)$$

where $n(t)$ is multiplicative white Gaussian noise with zero mean and variance σm^2. ω_o is the frequency of the signal, $\emptyset o$ is the initial phase and $\epsilon(t)$ is an additive white Gaussian noise with zero mean and variance σa^2. Where $n(t)$and $\epsilon(t)$ are independent process.

Consider $u(t) = n(t)e^{j\omega_o t}$ and $v(t) = n(t)e^{-j\omega_o t}$.

Then their autocorrelations are given by:

$$R_u = \mathcal{E}\{u(t)u^*(t+\tau)\} = \mathcal{E}\{n(t)n(t+\tau)e^{j\omega_o t}e^{-j\omega_o(t+\tau)}\}$$
$$= \mathcal{E}\{n(t)n(t+\tau)e^{-j\omega_o\tau}\}$$
$$= \mathcal{E}\{n(t)n(t+\tau)\}\cdot\mathcal{E}\{e^{-j\omega_o\tau}\} \quad \text{[independent processes]}$$
$$= R_n e^{-j\omega_o\tau}$$

where $\mathcal{E}$ is the statistical expectation functional. Similarly we have:

$R_v = R_n e^{+j\omega_o\tau}$.

Since $\cos(\omega_o t) = \frac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})$ [Euler], then

$R_y = \frac{1}{2}(R_u + R_v) = R_n\cos(\omega_o\tau)$.

Since $n(t)$ is wide-sense stationary (WSS) white noise, then $R_n(\tau) = \frac{\eta}{2}\delta(\tau)$; hence, $S_n(f) = \frac{\eta}{2}$, where $\frac{\eta}{2}$ is a constant that represents the double-sided power spectral density (PSD) of noise, $S_n(f)$. Note that according to Wiener-Kinchin Theorem (WKT) [4], $S_n(f) = \mathcal{F}_{\tau \to f}\{R_n(\tau)\}$. The psd of the signal y(t) is given by:

$$S_y(f) = \frac{1}{2}[S_n(f + f_o) + S_n(f - f_o)] = \frac{\eta}{2}$$

Hence, by WKT, $R_y(\tau) = \frac{\eta}{2}\delta(\tau)$, which means that y(t) is white noise with the same power σ^2 as n(t).

On the other hand, the autocorrelation of the signal y(t) can be obtained as:

$$R_y(\tau) = \frac{\eta}{2}\delta(\tau)\cos(\omega_o\tau)$$
$$= \frac{\eta}{2}\delta(\tau)\cos(0) = \frac{\eta}{2}\delta(\tau)$$

where the multiplication property of the delta function has been used [4,5]:
$x(t)\delta(t) = x(0)\delta(t)$ [on condition that x(t) is continuous at $t = 0$].

The mean of n(t) is zero, i.e., $\mathcal{E}\{n(t)\} = 0$.Hence,
$\mathcal{E}\{y(t)\} = \mathcal{E}\{n(t)\cos(\omega_o t)\} = \mathcal{E}\{n(t)\} \cdot \mathcal{E}\{\cos(\omega_o t)\} = 0$.

III. FREQUENCY ESTIMATION BASED ON SPECTRAL DOMAIN (FOURIER TRANSFORM)

A. *Approximate maximum likelihood estimator using Fast Fourier Transform (FFT) and no post-processing (FFT estimator)*

The maximum likelihood (ML) frequency estimator given the observation. [3]

$$w_{ML} = \arg(\max|X(w)|)$$

Using Fourier transform the frequency estimated by peak of the Fourier Spectrum X(f) of single tone sinusoidal signal, computed from the sampled signal X(K) by the DFT as $X(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x(n)e^{(-\frac{2\pi kn}{N})}$
However, the actual frequency may fall between DFT bins and the index of Fourier transform cannot be a non-integer number, hence we interpolate between points near the peak of the FFT by using several interpolation methods to improve the estimation accuracy.

B. *Approximate maximum likelihood estimator using Fast Fourier Transform (FFT) and post-processing (and Quadratic Interpolation)*

This method finds a quadratic fit y=a+bx+cx^2 in the neighborhood of the maximum max{X(f)} with the three points [2]:
$(K - 1, u_1 = |X_{K-1}|)$
$(K, u_2 = |X_K|)$
And $(K + 1, u_3 = |X_{K+1}|)$

Where $K = arg\,\{max_k[X(k)]\}$ is the index of absolute maximum magnitude of DFT, so actual frequency is $F = \frac{Kf_s}{N}$,f_s being the sampling frequency
Using this quadratic formula the maximum will be at the point $y = \frac{-b}{(2c)}$ as follow:
$d = (u_3 - u_1)/[2 * (2 * u_2 - u_1 - u_3)]$
$u = K + d$
Then estimated frequency is $F_O = \frac{uf_s}{N}$

1. *Barycentric method*

In this method, we use the same three point above as follow: [6]
$d = (u_3 - u_1)/(u_1 + u_2 + u_3)]$, $u = K + d$;

2. *Quinn's First Estimator [7]:*
   $(K - 1, u_1 = |X_{K-1}| = r_1 + im_1)$
   $(K, u_2 = |X_K| = r_2 + im_2)$
   And $(K + 1, u_3 = |X_{K+1}| = r_3 + im_3)$
   Then we perform following steps:
   $R = r_2^2 + im_2^2$;
   $S = (r_3 . r_2 + im_3 . im_2)/R$;
   $T = (-S) / ((1 - S))$;
   W=$(r_1 . r_2 + im_1 . im_2)/R$;
   $E = W/(1 - W)$;
   If $(S > 0)$ and $(W > 0)$ then, $d = S$
   Else, $d = W$;
   $u = K + d$;
3. *Quinn's second Estimator [8]:*
   For the same three points above we apply below steps:
   $d = \frac{S+w}{2} + H(S^2) + H(W^2)$
   Where $H(x) = \frac{\frac{1}{4}\ln(3x^2+6x+1) - \frac{\sqrt{6}}{24}\ln(x+1-\sqrt{\frac{2}{3}})}{x+1+\sqrt{\frac{2}{3}}}$
   Then
   $u = K + d$;

IV. SIMULATION RESULTS

We simulated the above algorithms with signal model with additive white Gaussian noise (AWGN) and multiplicative noise (MN) as per Equation (1) using MATLAB. The simulated signal has time lengths L=70s, sampling interval Ts=0.001s, and a number of samples $N = [L/Ts]$. The signal amplitude is $A = 1volt$,w_o is angler frequency $\omega_o = 2\pi f_o$, where $f_o = 23Hz$. We modeled MN as zero-mean Gaussian.

The signal-to-noise ratio (SNR) using the AWGN power and MN power is defined as follows $SNR = pxr/pn$, where $pxr = (px + pm)$,px is signal power, pm MN power, pn additive noise power. This is so because the multiplicative noise power is affecting the amplitude and phase of the signal.
Finally, we calculate the relative squared-error under each SNR and MN power as follows:

$$e = |\,((F_O - f_o)/f_o)|^2$$

and plot result for different value of time lengths L and number of samples N to study the effect of N on accuracy of frequency estimation.

also, we take different value of multiplicative noise MN in dB to show how FT estimation methods hold out against high MN.

Figure (1) shows the frequency estimated using four methods fe_1,fe_2,fe_3,fe_4refers to maximum of FFT, Quadratic, barycentric, and Quinn's Estimator respectively with multiplicative noise=-10 Db.

Figure (2) shows the frequency estimated using three methods vs mean square error(MSE). fe_1,fe_2,fe_3,fe_4refers to maximum of FFT, Quadratic, barycentric, and Quinn's Estimator respectively with multiplicative noise=-10dB.

Figure (3) shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn's Estimator for multiplicative noise power =-10 dB, N=101.it is clear that Quinn's is the worst in that small value of N.

Figure (4) shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn's Estimator for multiplicative noise power =-10 dB, N=1001.

Figure (5) shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn's Estimator for multiplicative noise power=-10 dB, N=10001.

Figure (6) shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn's Estimator for multiplicative noise power =-10 dB, N=70001. for this large value of N Quinn's is the same as maxFT.

It is clear that barycentric estimator is more accurate estimator under Gaussian multiplicative noise in terms of minimum mean squared estimation error, especially at high signal-to-noise ratios and it most accurate for all value of N so it has less computation complexity .

Figure (٧) and (8) shows the frequency estimation mean-squared error (MSE) versus SNR using barycentric estimator for multiplicative noise power=5,20 dB respectively, and different value of N. with MN power =20 dB barycentric estimator failed for small value of N.

It is clear that mean square error is inversely proportional with N .large value of N make barycentric Estimator hold out against high value of MN

V. CONCLUSIONS

From simulation results of implementing different quadratic interpolators: maximum of FFT, barycentric, and Quinn's Estimator using complex single-tone sinusoid affected by additive Gaussian and zero mean multiplicative Gaussian noise in MATLAB, we can compare between their performance in term of mean square error(MSE) and computation complexity as follow:

- Frequency estimated by the barycentric Estimator is nearest to actual frequency so it is more accurate than reset approach and has the least MSE error.
- Maximum FFT cannot achieve accurate frequency with small number of point then it has more computation complexity than other interpolations method.
- Barycentric is best for all different value of N. Quinn's is almost the same as maxFT for large N. for small N, Quinn is the worst. Hence, Barycentric has less computation complexity than other interpolations method.
- Barycentric estimator failed for MN power more than 20 dB but it can hold out if we increase N hence, this method be inefficient for large MN power because large value of N mean high computation complexity.

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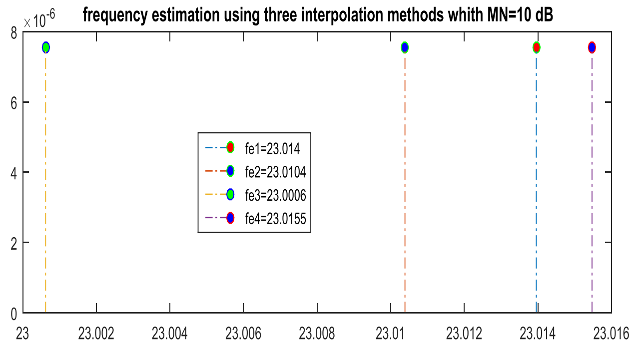

Fig. 1. interpolated FT peak method with three interpolation approaches , fe_1 using peak of FFT method, fe_2 using quadratic approach, fe_3 using barycentric approach, fe_4 using Quinn approach

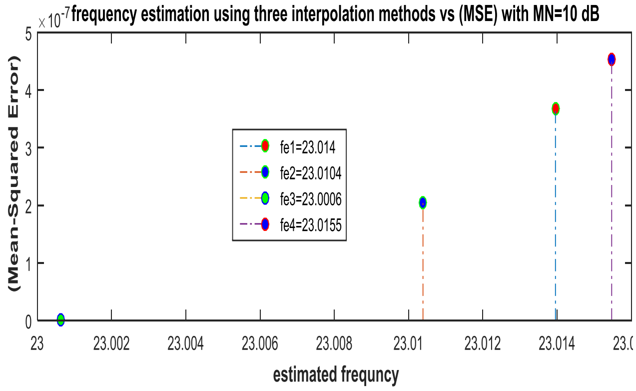

Fig. 2. interpolated FT peak method with three interpolation approaches , fe_1 using peak of FFT method, fe_2 using quadratic approach, fe_3 using barycentric approach, fe_4 using Quinn approach vs (MSE)

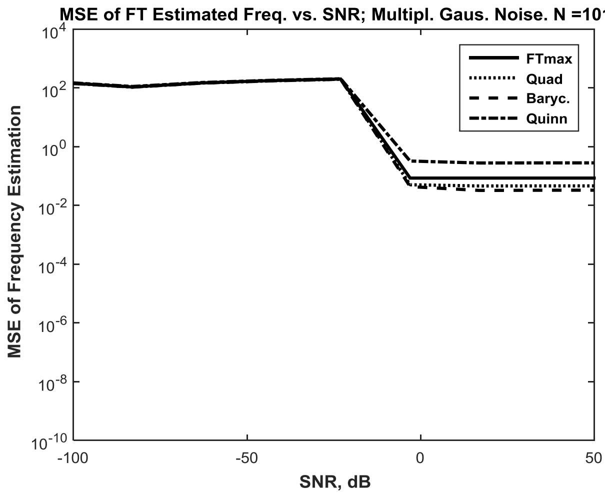

Fig. 3. shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn’s Estimator for multiplicative noise power=30 dB,N=101.

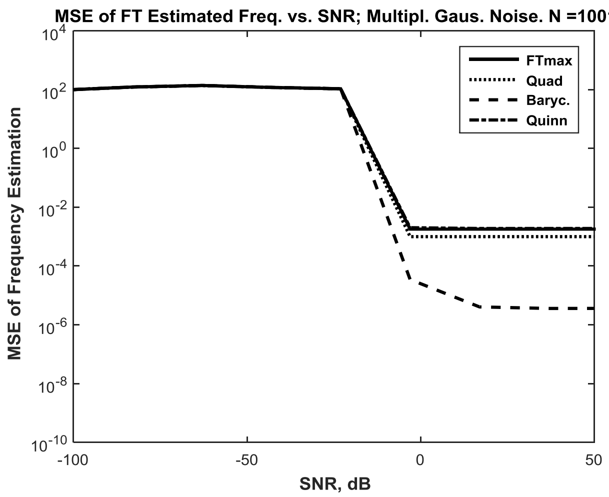

Fig. 4. shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn’s Estimator for multiplicative noise power=30 dB,N=1001.

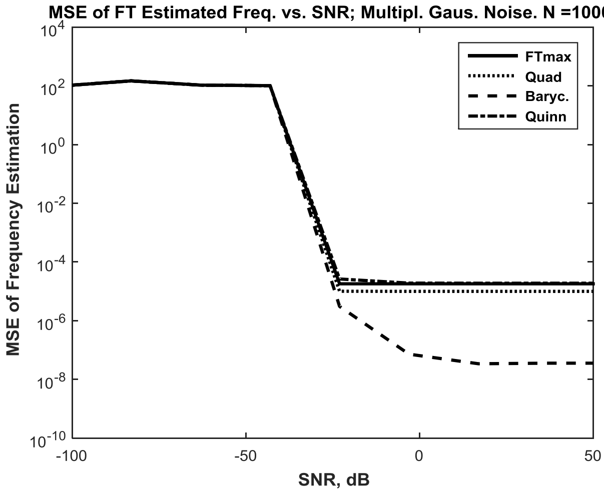

Fig. 5. shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn’s Estimator for multiplicative noise power=30 dB,N=10001.

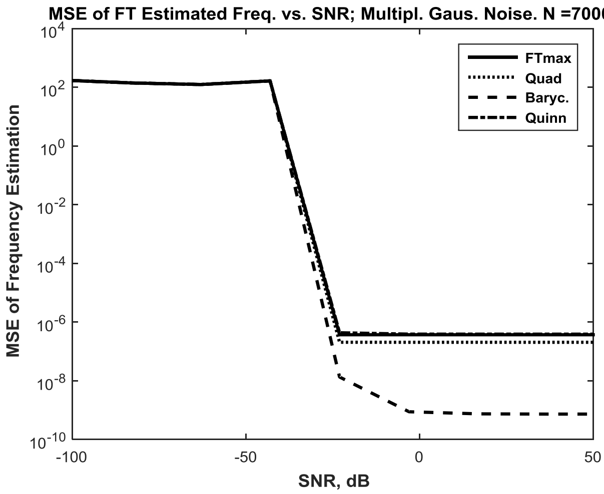

Fig. 6. shows the frequency estimation mean-squared error (MSE) versus SNR using FT peak and quadratic interpolators: maximum of FFT, barycentric, and Quinn’s Estimator for multiplicative noise power=30 dB,N=70001.

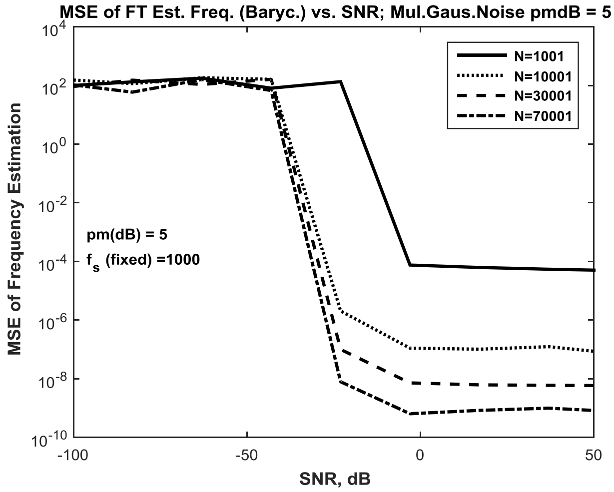

Fig. 7. shows the frequency estimation mean-squared error (MSE) versus SNR using using barycentric estimator for multiplicative noise power=5 dB respectively, and different value of N.

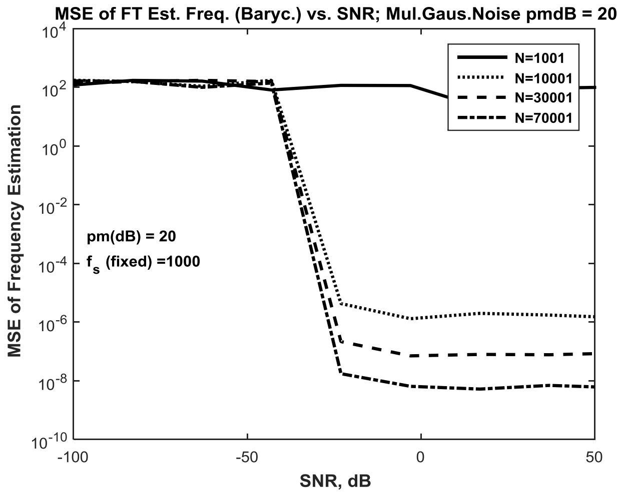

Fig. 8. shows the frequency estimation mean-squared error (MSE) versus SNR using using barycentric estimator for multiplicative noise power=20 dB respectively, and different value of N.