UNIVERSITY OF AL-QADISIYAH

COLLEGE OF COMPUTER

SCIENCE AND MATHEMATICS

DEPARTMENT OF MATHEMATICS

SOME RESULTS OF SPECTRAL THEORY IN FUZZY NORMED SPACES

A THESIS

SUBMITTED TO THE COUNCIL OF THE COLLEGE OF COMPUTER SCIENCE AND MATHEMATICS

UNIVERSITY OF AL-QADISIYAH

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF SCIENCE

IN

MATHEMATICS

BY

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2016 A.D **1437 A.H**

بسم الله الرحمن الرحيم

وَأَن لَّيْسَ لِلْإِنسَانِ إِلَّا مَا سَعَى ﴿39﴾ وَأَنَّ سَعْيَهُ

سَوْفَ يُرَى ﴿40﴾ ثُمَّ يُجْزَاهُ الْجَزَاءَ

الْأَوْفَى ﴿41﴾

صدق الله العلي العظيم

سورة النجم

الآية ﴿39-41﴾

Acknowledgments

Thanks to Allah for helping and giving me the power to complete my thesis.

I am indebted a great deal to my devoted supervisor ***Asst. Prof. Dr. Noori Farhan Al-Mayahi*** for his efforts, invaluable instruction, guiding the research track and support during the writing of this thesis. I wish him a good health and success.

I would like to thank all my instructors in the Msc.Course . I wish to express my sincere thanks to the dearest individuals in my life: my father, my mother, my sisters and my brothers .

Finally,I would like to thank all my friends for their help, encouragement and love.

ABBAS

DEDICATION

To

Ali Ibn Abi Talib

my imam, the crown of my head and the light of my eyes …

I certify that this thesis, entitled **Some Results of Spectral Theory in Fuzzy Normed Spaces**, by **Abbas Mohammed Abbas**, has been prepared under my supervision at the University of Al-Qadisiyah, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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We certify that we have read this thesis " **Some Results of Spectral Theory in Fuzzy Normed Spaces** " and as an examining committee, we examined the study in its content, and that in our opinion it meets the standards of a thesis for the Degree of Master of Science in Mathematics.

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LIST OF PUBLICATIONS

1- Some Properties of Spectral Theory in Fuzzy Hilbert Spaces , Journal of AL-Qadisiyah for computer science and mathematics, Vol.(8), No.(2), (2016) (Accepted for publication).

2- Spectral Properties Of Fuzzy Compact Linear Operator On Fuzzy Normed Spaces, International Journal of Science and Research (IJSR), Volume 6 ,Issue 4 ,(2017) (Accepted for publication).

ABSTRACT

This thesis study deals with addressing some properties of spectral theory of linear operator defined on a fuzzy normed spaces which is considered as an expansion for the spectral theory of linear operator defined on normed spaces.

This also introduces some definitions of eigenvalue and eigenvectors regular values ,resolvent set,spectrum,point spectrum,continuous spectrum and residual spectrum of linear operator defined on fuzzy normed spaces and some of their properties ,also we will introduce definition fuzzy compact operator on fuzzy normed space which is considered as an expansion for the compact operators on normed space and its relationship spectral theory in fuzzy normed space (Spectral properties of fuzzy compact linear operators on fuzzy normed spaces).

TABLE OF SYMBOLS

Symbols	Meaning
$\boldsymbol{I}$	The closed interval [0,1]
$\boldsymbol{I}^{\boldsymbol{X}}$	The set of all functions defined from X in to I
$*, \circ$	Binary operation from $I \times I$ to I
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^{+}$	The set of all positive real numbers
$\boldsymbol{F}$	Field (real or complex numbers)
$\mathbb{Z}$	The set of all integer numbers
$\mathbb{Z}^{+}$	The set of all positive integer numbers
$\forall$	For all
$\ni$	Such that
$\mathbb{C}$	The set of all complex numbers
$\overline{\boldsymbol{A}}$	The closure of a fuzzy set A
Sup	Supermum (least upper bound)
Inf	Infemum (great lower bound)
$\|.\|$	Normed function from X to $\mathbb{R}$
$\boldsymbol{N}$	Normed function from $X \times (0, \infty)$ to [0,1]
$\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{t})$	Open ball in fuzzy normed spaces
$\boldsymbol{\rho}(\boldsymbol{T})$	The set of all regular values λ of T

$\boldsymbol{\sigma}(\boldsymbol{T})$	The spectrum of T
$\boldsymbol{\rho}_{\mathbf{p}}(\boldsymbol{T})$	The point spectrum of T
$\boldsymbol{\rho}_{\mathbf{c}}(\boldsymbol{T})$	The continuous spectrum of T
$\boldsymbol{\sigma}_{\mathbf{r}}(\boldsymbol{T})$	The residual spectrum of T
$\mathbf{R}_{\boldsymbol{\lambda}}(\boldsymbol{T})$	The resolvent operator of T
$\boldsymbol{L}(\boldsymbol{X})$	The space of all linear operator from $\boldsymbol{X}$ in to $\boldsymbol{X}$
$\boldsymbol{FB}(\boldsymbol{X}, \boldsymbol{Y})$	The space of all fuzzy bounded linear operator from $\boldsymbol{X}$ in to $\boldsymbol{Y}$

TABLE OF CONTENTS

INTRODUCTION

The theory of fuzzy sets was introduced by L. A. Zadeh [24] in 1965.Aftar the pioneer work of Zadeh ,many researchers have extended this concept in various branches ,many other mathematicians have studied fuzzy normed space from several points of view [23],[18], [14]. Fuzzy Hilbert spaces is an extension to the Hilbert space. The definition of a fuzzy Hilbert space has been introduced by M. Goudarzi and S. M. Vaezpour [9] in 2009 . T. Bag and Samanta [4] in 2003 have definition compact set in fuzzy normed space. In 2005, T. Bag and Samanta [5] introduced the concept of continuity and boundedness of linear operator with respect to their fuzzy norm.

The present thesis consists of three chapters.

Chapter one ,deals with the concept of fuzzy sets and concept of binary operations t-norm and t-conorm and fuzzy normed space and some their properties .The concept of fuzzy Hilbert spaces and some their properties .

Chapter two, deals with the concept of eigenvalue and eigenvectors and regular value ,resolvent, spectrum, point spectrum, continuous spectrum and residual spectrum of linear operator defined on fuzzy normed and some their properties and we give an interdiction to spectral theory of linear operator on fuzzy normed space and fuzzy Hilbert space.

Chapter three, deals with the concept of fuzzy compact linear operator on fuzzy normed space and some of their properties . Also ,we consider spectral properties of fuzzy compact linear operator T on a fuzzy normed space.

CHAPTER ONE
Basic Concepts

This chapter deals with fuzzy normed space and fuzzy pre-Hilbert space. It consists of four sections. Section one deals with the concepts of fuzzy sets and some of their properties. Section two discusses the concepts of t-norm and t-conorm and some of their basic properties and the relationship between them, in addition to some examples. The concepts of the fuzzy normed space is dealt with in section three. Section four deals with the concepts of fuzzy pre-Hilbert space .

1.1 Fuzzy Sets

This section deals with the basic concepts of fuzzy sets and some of their properties.

Let X be a non-empty set, and I denote for the closed interval $[0,1]$ of real numbers, i.e., $I = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and I^X denotes all functions from X in to I, i.e., $I^X = \{ u : u$ is a function from X to $I\}$.

Definition (1.1.1) : [24] A fuzzy set u in X (or a fuzzy subset from X) is a function from X in to I, i.e., $u \in I^X$.

If u is a fuzzy set in X then u is a described as characteristic function which is connect every $x \in X$ by real number $u(x)$ in the interval I. $u(x)$ is the grade of membership function to x in u. u can be described completely as:

$$u = \{(x, u(x)) : x \in X, 0 \leq u(x) \leq 1\},$$

where $u(x)$ is called the membership function for the fuzzy set u. Also, the fuzzy set u may be termed as:

$$u = \left\{\frac{u(x)}{x} : x \in X\right\}$$

Example (1.1.2) : Let $X = \{a, b, c\}$, and let the function $u : X \to I$ which is define as: $u(a) = \frac{1}{5}$, $u(b) = \frac{2}{7}$, $u(c) = \frac{2}{3}$ represent a fuzzy set in X.

While the function $v : X \to \mathbb{R}$ which is define as: $v(a) = \frac{1}{8}$, $v(b) = \frac{5}{2}$, $v(c) = 5$ not represent a fuzzy set in X because $v(b), v(c) > 1$.

Example (1.1.3) : [12] Let $\mathbb{R}$ be the set of real numbers, and let u be fuzzy set in $\mathbb{R}$. Then we can define grade of membership function mathematical as:

$$u(x) = \begin{cases} \frac{x-1}{x} & ,x > 1 \\ 0 & ,x \leq 1 \end{cases}$$

Remark (1.1.4) : [12] If we want to know difference between fuzzy sets and regular sets, we note if u is a regular set then define of grade of membership take only two valuables 0,1. i.e.

$$u(x) = \begin{cases} 1 & ,x \in u \\ 0 & ,x \notin u \end{cases}$$

Therefore $u(x) \in \{0,1\}$. While if u is fuzzy set in X, then $0 \leq u(x) \leq 1$ for all $x \in X$, and thus the regular set become special case for the fuzzy sets.

Definition (1.1.5) : [6] Let u be a fuzzy set in X :

(1) The support of the fuzzy set u is denoted by u^*or $Supp(u)$ and is defined as: $u^*= Supp\ (u) = \{x \in X{:}\, u(x) > 0\}$.

(2) The point $x \in X$ is the crossover point for the fuzzy set u if $u(x) = \frac{1}{2}$.

(3)A fuzzy set u is called normal if there exists $x_0 \in X$such that $u(x_0) = 1$, i.e. $\{x \in X : u(x) = 1\} \neq \emptyset$.

(4) The height of the fuzzy set u is denoted by $ht(u)$ and defined as: $ht(u) = \sup\ \{u(x){:}\, x \in X\}$.

And special case if u is normal, then $ht(u) = 1$, and so u is said to be finite if u^* is finite set, inverse that u is said to be infinite.

Example (1.1.6) : Let $X = \{a, b, c\}$ and let u be fuzzy set in X defined as:

$$u\ (a) = \frac{1}{2}, u(b) = \frac{1}{3}, u(c) = 0,$$

$$u^* = \{\ x \in X : u(x) > 0\} = \{a, b\}.$$

The point a is a crossover point for the fuzzy set u because $u\ (a) = \frac{1}{2}$ and u is not normal because there is no element whose image equal to 1 and

$$ht\ (u) = \sup\ \{u(x): x \in\ X\} = \sup\{\frac{1}{2}, \frac{1}{3}, 0\} = \frac{1}{2}.$$

Definition (1.1.7) : [6] Let X be a non-empty set. A fuzzy set $u : X \to I$ is defined by $u(x)\ = 0$, for all $x \in X$ is called an empty fuzzy set and is denoted by $\emptyset$ or 0. The fuzzy set $u : X \to I$ is called a non-empty if there exists at least $x \in X$ such that $u(x) \neq 0$.

Definition (1.1.8) : [6] A fuzzy set $u : X \to I$ which is defined as: $u(x) = 1$, for all $x \in X$ is called a universal fuzzy set and denoted by X or 1.

Definition (1.1.9) : [24] Let u, v be a two fuzzy sets in X. Denoted to union of two sets u, v by $u \cup v$ and defined as:

$$(u \cup v)(x) = \max\ \{u(x), v(x)\}$$

for all $x \in X$, and so denoted to intersection of two sets u, v by $u \cap v$ and defined as:

$$(u \cap v)(x) = \min\ \{u(x), v(x)\}.$$

Remark (1.1.10) : [24] If u, v are two fuzzy sets in X, then $u \cup v$, $u \cap v$ are two fuzzy sets in X.

Example (1.1.11) : Let $X = \{a, b, c\}$ and u, v be two fuzzy sets in X such that $u(a) = \frac{1}{6}$, $u(b) = \frac{1}{2}$, $u(c) = \frac{1}{10}$, $v(a) = \frac{1}{5}$, $v(b) = \frac{1}{7}$, $v(c) = \frac{1}{3}$. Then:

$$(u \cup v)(a) = \max \{u(a), v(a)\} = \max\{\frac{1}{6}, \frac{1}{5}\} = \frac{1}{5}$$

$$(u \cup v)(b) = \max \{u(b), v(b)\} = \max\{\frac{1}{2}, \frac{1}{7}\} = \frac{1}{2}$$

$$(u \cup v)(c) = \max \{u(c), v(c)\} = \max\{\frac{1}{10}, \frac{1}{3}\} = \frac{1}{3}$$

$$(u \cap v)(a) = \min \{u(a), v(a)\} = \min\{\frac{1}{6}, \frac{1}{4}\} = \frac{1}{6}$$

$$(u \cap v)(b) = \min \{u(b), v(b)\} = \min\{\frac{1}{2}, \frac{1}{7}\} = \frac{1}{7}$$

$$(u \cap v)(c) = \min \{u(c), v(c)\} = \min\{\frac{1}{10}, \frac{1}{3}\} = \frac{1}{10}.$$

1.2 Norms and Their Complements of the Type t

This section discusses the concepts of t-norm and t-conorm and some of their basic properties and the relationship between them. The section also includes some examples.

Definition (1.2.1) : [12] Let $*$ be a binary operation on the set I, i.e., $* : I \times I \to I$ is a function. Then $*$ is said to be t-norm (triangular-norm) on the set I if the following axioms are satisfied :

(1) $a * 1 = a$, for all $a \in I$.

(2) $*$ is commutative (i.e. $a * b = b * a$, for all $a, b \in I$).

(3) $*$ is monotone (i.e. if $b, c \in I$ such that $b \leq c$, then $a * b \leq a * c$, for all $a \in I$).

(4) $*$ is associative (i.e. $a * (b * c) = (a * b) * c$, for all $a, b, c \in I$).

If, in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

The following theorem introduces the characteristics of the t-norm :

Theorem (1.2.2) : [12] Let $*$ be a t-norm on the set I. Then

(1) $1 * 1 = 1$.

(2) $0 * 1 = 0$.

(3) $1 * 0 = 0$.

(4) $0 * 0 = 0$.

(5) $a * a \leq a$, for all $a \in I$.

(6) If $a \leq c,\ b \leq d$, then $a * b \leq c * d$ for all $a, b, c, d \in I$.

Example (1.2.3) : [12] The basic t-norms are :

(i) A binary operation $*_m$ on I, which is defined by $a\ *_m b = \min\{a, b\}$ for all $a, b \in I$ is a t-norm, and called the standard intersection.

(ii) A binary operation $*_p$ on I, which is defined by $a\ *_p b = a.b$ for all $a, b \in I$ is a t-norm, and called the algebraic product.

(iii) A binary operation $*_b$ on I, which is defined by $a\ *_b b = \max\{0, a + b - 1\}$ for all $a, b \in I$ is a t-norm, and called the bounded sum or bounded difference.

(iv) A binary operation $*_d$ on I, which is defined by

$$a *_d b = \begin{cases} a & , b = 1 \\ b & , a = 1 \\ 0 & , \ o.w. \end{cases}$$

for all $a, b \in I$ is a t-norm, and called the drastic intersection.

Theorem (1.2.4) : [12] $a\ *_d b \leq\ a\ *_b b\ \leq\ a\ *_p b\ \leq\ a\ *_m\ b$ for all $a, b \in I$.

Theorem (1.2.5) : [12] Let $*$ be a t-norm on a set I. Then $*_d \leq * \leq *_m$.

Definition (1.2.6) : [12] Let $\circ$ be a binary operation on the set I i.e. , $\circ$: $I \times I \to I$ is a function). $\circ$ is said to be (t-conorm) on the set I if the following axioms are satisfied :

(1) $a \circ 0 = a$, for all $a \in I$.

(2) $\circ$ is commutative (i.e. $a \circ b = b \circ a$, for all $a, b \in I$).

(3) $\circ$ is monotone (i.e. if $b, c \in I$ such that $b \leq c$, then $a \circ b \leq a \circ c$ for all $a \in I$).

(4) $\circ$ is associative (i.e.$a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in I$).

If, in addition, $\circ$ is continuous then $\circ$ is called a continuous t-conorm.

The following theorem introduces the characteristics of t-conorm:

Theorem (1.2.7) : [12] Let $\circ$ be t-conorm on the set I. Then

(1) $0 \circ 0 = 0$.

(2) $1 \circ 0 = 1$.

(3) $0 \circ 1 = 1$.

(4) $1 \circ 1 = 1$.

(5) $a \circ a \geq a$ for all $a \in I$.

(6) If $a \leq b,\ c \leq d\ then\ a \circ c \leq b \circ d$.

Example (1.2.8) : [12] The basic t-conorm is :

(i) A binary operation $\circ_m$ on I, which is defined b $a \circ_m b = \max\ \{a, b\}$ for all $a, b \in I$ conorm, and called the standard union.

(ii) A binary operation $\circ_p$ on I, which is defined $a \circ_p b = a + b - ab$ for all $a, b \in I$ is t-conorm, and called the algebraic product.

(iii) A binary operation $\circ_b$ on I, which is defined $a \circ_b b = \min\{1, a + b\}$ for all $a, b \in I$ is t-conorm, and called bounded sum.

(iv) A binary operation $\circ_d$ on I, which is defined by

$$a \circ_d b = \begin{cases} a & , b = 0 \\ b & , a = 0 \\ 1 & , \ o.w. \end{cases}$$

for all $a, b \in I$ is t-conorm, and called drastic union.

Theorem (1.2.9) : [12] $a \circ_m b \leq a \circ_p b \leq a \circ_b b \leq a \circ_d b$ for all $a, b \in I$.

Theorem (1.2.10) : [12] Let $\circ$ be a t-conorm on a set I then $\circ_m \leq \circ \leq \circ_d$ for all $a, b \in I$.

Definition (1.2.11) : [12] Let $*$ be t-norm, and $\circ$ be t-conorm. Then $*$ and $\circ$ are said to be dual if they satisfies the following axioms :

(a) $a * b = 1 - \big((1-a) \circ (1-b)\big)$ for all $a, b \in I$.

(b) $a \circ b = 1 - \big((1-a) * (1-b)\big)$ for all $a, b \in I$.

Theorem (1.2.12) : [12] Let $*$ be t-norm, and $\circ$ be t-conorm .Then

(1) $*_m, \circ_m$ are dual.

(2) $*_b, \circ_b$ are dual.

(3) $*_p, \circ_p$ are dual.

(4) $*_d, \circ_d$ are dual.

1.3 Fuzzy Normed Spaces

This section deals with the concept of fuzzy normed space and some of its properties.

Definition (1.3.1) : [14] let X be a vector space over F, where F is either the field of real numbers or the field of complex numbers.
A norm on X is a function $\|.\|: X \to \mathbb{R}$ having the following properties:

(1) $\|x\| \geq 0$, for all $x \in X$.

(2) $\|x\| = 0$ if and only if $x = 0$.

(3) $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in X$ and $\lambda \in F$.

(4) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

The vector X over F together with $\|.\|$ is called a normed space and is denoted by $(X, \|.\|)$ or simply X.

Definition (1.3.2) : [23] Let X be a vector space over F, $*$ be a continuous t-norm on I, a function $N: X \times (0, \infty) \to [0,1]$ is called fuzzy norm if it satisfies the following conditions : for all $x, y \in X$ and $t, s > 0$,

$(N.1)$ $N(x, t) > 0$,

$(N.2)$ $N(x, t) = 1$ if and only if $x = 0$,

$(N.3)$ $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$, for all $\alpha \neq 0$,

$(N.4)$ $N(x, t) * N(y, s) \leq N(x + y, t + s)$,

$(N.5)$ $N(x, .): (0, \infty) \to [0,1]$ is continuous,

$(N.6)$ $\lim_{t\to\infty} N(x, t) = 1$.

$(X, N, *)$ is called fuzzy normed space.

Lemma(1.3.3) : [17] Let $(X, N,*)$ be a fuzzy normed space. Then:

(i) $N(x,.)$ is non-decreasing with respect to t for each $x \in X$.

(ii) $N(-x,t) = N(x,t)$ hence $N(x-y,t) = N(y-x,t)$.

Remark (1.3.4) : [8]

(1) For any $\alpha_1,\ \alpha_2 \in (0,1)$ with $\alpha_1 > \alpha_2$, there exists $\alpha_3 \in (0,1)$ such that $\alpha_1 * \alpha_3 \geq \alpha_2$.

(2) For any $\alpha_4 \in (0,1)$, there exists $\alpha_5 \in (0,1)$ such that $\alpha_5 * \alpha_5 \geq \alpha_4$.

Example (1.3.5) : [3] Let $(X, \|.\|)$ be a normed space. $a * b = a.b$ for all $a, b \in X$ and for all $x \in X, t > 0$

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & ,x \neq 0 \\ 1 & ,x = 0 \end{cases} \quad \dots\dots\dots (1.3.5)$$

Then $(X, N,*)$ is fuzzy normed space.

Solution: $(N.1)$ if $x = 0$ then $N(x,t) = 1 > 0$

if $x \neq 0$ then $N(x,t) = \frac{t}{t+\|x\|} > 0$

$(N.2)$If $x = 0$ then $N(x,t) = 1$

If $N(x,t) = 1 \Longrightarrow \frac{t}{t+\|x\|}=1 \quad \Longrightarrow \|x\| = 0 \Longrightarrow x = 0$

$(N.3)$ $N(\alpha x,t) = \frac{t}{t+\|\alpha x\|} = \frac{t}{t+|\alpha|\|x\|} = \frac{\frac{t}{|\alpha|}}{\frac{t}{|\alpha|}+\|x\|} = N\left(x, \frac{t}{|\alpha|}\right)$, for all $\alpha \neq 0$.

$(N.4)$$N(x+y,t+s) - N(x,t) * N(y,s) = \frac{t+s}{t+s+\|x+y\|} - \frac{t}{t+\|x\|} \cdot \frac{s}{s+\|y\|} \geq 0$.

$(N.5)$ $N(x,.)\colon (0,\infty) \to [0,1]$ is continuous.

$(N.6)$ $\lim_{t\to\infty} N(x,t) = 1$.

Therefore $(X, N,*)$ is a fuzzy normed space.

Example (1.3.6) : [17] Let $(X, \|.\|)$ be a normed space. For all $x \in X, t > 0$ $N(x,t) = \frac{t\|x\|}{t+1}$. Then $(X, N, *)$ is not fuzzy normed space.

Solution: Let $x = 0 \Longrightarrow N(x,t) = \frac{\|0\|}{t+1} \neq 1$.
Therefore $(X, N, *)$ is not a fuzzy normed space

Example (1.3.7): Let $(X, \|.\|)$ be a normed space . Defined

$$N(x,t) = \begin{cases} 0 & , t \leq \|x\| \\ 1 & , t > \|x\| \end{cases} \ldots\ldots.(1.3.7)$$

And $a *_m b = \min\{a, b\}$ for all $a, b \in X$ and for all $x \in X, t > 0$ then $(X, N, *)$ is a fuzzy normed space

Solution: $(N.1)$ and $(N.2)$ directly from definition (1.3.2).

$(N.3)$ $N(\alpha x, t) = 1$ for $t > \|\alpha x\|$, for all $\alpha \neq 0$

$$\Longrightarrow t > |\alpha|\|x\| \Longrightarrow \frac{t}{|\alpha|} > \|x\| \text{ then } N\left(x, \frac{t}{|\alpha|}\right) = 1$$

Therefore $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$, for all $\alpha \neq 0$ and same above when $t \leq \|x\|$

$(N.4)$ we must prove that $N(x + y, t + s) \geq \min\{N(x,t), N(y,s)\}$

For each $t, s > 0$

$N(x,t) = 1$ for , $t > \|x\|$

$N(y,s) = 1$ for, $s > \|y\|$

$\Longrightarrow t + s > \|x\| + \|y\| \geq \|x + y\| \Longrightarrow N(x + y, t + s) = 1$

Then $N(x + y, t + s) = \min\{N(x,t), N(y,s)\}$

Also same above when $t \leq \|x\|$

$(N.5)$ $N(x, .): (0, \infty) \to [0,1]$ is continuous.

$(N.6)$ $\lim_{t \to \infty} N(x,t) = 1$.

Therefore $(X, N, *)$ is a fuzzy normed space.

Theorem (1.3.8) :[21] Let $(X, N, *)$ be a fuzzy normed space, we further assume that,

$(N.7)$ $\alpha * \alpha = \alpha$ for all $\alpha \in [0,1]$,

$(N.8)$ $N(x,t) > 0$ for all $t > 0$,then $x = 0$.

Define $\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\}$. Then $\{\|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X . We call these norms as α-norms on X corresponding to fuzzy norm N on X.

Proof : Let $\alpha \in (0,1)$. To prove $\|x\|_\alpha$ is a norm on X, we will prove the followings:

(1) $\|x\|_\alpha \geq 0$ for all $x \in X$,
(2) $\|x\|_\alpha = 0$ if and only if $x = 0$,
(3) $\|\lambda x\|_\alpha = |\lambda|\|x\|_\alpha$,
(4) $\|x + y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$.
The prove of (1), (2)and (3) directly follows from the proof of the Theorem 2.1 in [4]. So, we now prove (4) :

$$\begin{aligned}
\|x\|_\alpha + \|y\|_\alpha &= \inf\{t > 0 : N(x,t) \geq \alpha\} + \inf\{s > 0 : N(y,s) \geq \alpha\} \\
&= \inf\{t + s > 0 : N(x,t) \geq \alpha , N(y,s) \geq \alpha\} \\
&= \inf\{t + s > 0 : N(x,t) * N(y,s) \geq \alpha * \alpha = \alpha\} \\
&\geq \inf\{t + s > 0 : N(x + y , t + s) \geq \alpha\} \\
&= \|x + y\|_\alpha, \text{ which proves (4).}
\end{aligned}$$

Let $0 < \alpha_1 < \alpha_2 < 1$.

$\|x\|_{\alpha_1} = \inf\{t > 0 : N(x,t) \geq \alpha_1\}$ and

$\|x\|_{\alpha_2} = \inf\{t > 0 : N(x,t) \geq \alpha_2\}$.

Since $\alpha_1 < \alpha_2$, $\{t > 0 : N(x,t) \geq \alpha_2\} \subset \{t > 0 : N(x,t) \geq \alpha_1\}$

$\Longrightarrow \inf\{t > 0 : N(x,t) \geq \alpha_2\} \geq \inf\{t > 0 : N(x,t) \geq \alpha_1\}$

$\Longrightarrow \|x\|_{\alpha_2} \geq \|x\|_{\alpha_1}$. Thus, we see that $\{\|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X.

1.4 Fuzzy Hilbert space

This section deals with the fuzzy Hilbert spaces and some of their properties

Definition (1.4.1) :[11] Let X be a vector space over the field F. An inner product on X is a function $\langle\, ,\rangle : X \times X \to F$ such that for all $x, y, z \in X$ and $\alpha, \beta \in F$ the following axioms are satisfied :

$(IP_1)\langle x, x\rangle \geq 0$,

$(IP_2)\langle x, x\rangle = 0 \iff x = 0$,

$(IP_3)\overline{\langle x, y\rangle} = \langle y, x\rangle$,

$(IP_4)\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

A pre-Hilbert (or inner product) space is a vector space with an inner product on it.

Example (1.4.2) :[11] Let $X = F^n$. Define $\langle\, ,\rangle : X \times X \to F$ by

$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ for $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in X$. Then X is a pre-Hilbert space.

Definition (1.4.3) : [11] A complete pre-Hilbert space is called Hilbert space.

Example (1.4.4) : [11] Let $X = \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ such that $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in X$. Then $(\mathbb{R}^n, \langle , \rangle)$ is a Hilbert space.

Definition (1.4.5) : [7] Let X be a real vector space, $*$ be a continuous t-norm on $I = [0,1]$. A function $H: X \times X \times \mathbb{R} \to [0,1]$ is called a fuzzy pre-Hilbert function if it satisfies the following axioms for every $x, y, z \in X$ and $s, t, r \in \mathbb{R}$:

Note : $h(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t \leq 0 \end{cases}$

(1) $H(x, x, 0) = 0$ and $H(x, x, t) > 0$ for each $t > 0$

(2) $H(x, x, t) \neq h(t)$ for some $t \in \mathbb{R}$ if and only if $x \neq 0$

(3) $H(x, y, t) = H(y, x, t)$

(4) For any real number α

$$H(\alpha x, y, t) = \begin{cases} H(x, y, \frac{t}{\alpha}) & , \alpha > 0 \\ h(t) & , \alpha = 0 \\ 1 - H(x, y, \frac{t}{-\alpha}) & , \alpha < 0 \end{cases}$$

(5) $H(x, x, t) * H(y, y, s) \leq H(x + y, x + y, t + s)$

(6) $sup_{s+r=t}\big(H(x, z, s) * H(y, z, r)\big) = H(x + y, z, t)$

(7) $H(x, y, .): \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$.

(8) $\lim_{t \to +\infty} H(x, y, t) = 1$.

$(X, H, *)$ is a fuzzy pre-Hilbert space.

Example (1.4.6) : [7] Let $(X, \langle\ ,\ \rangle)$ be an ordinary pre-Hilbert space. We define a function $H : X \times X \times \mathbb{R} \to [0,1]$ as follows :

$$H(\alpha x, y, t) = \begin{cases} \dfrac{t^{\frac{1}{2}}}{t^{\frac{1}{2}} + |\langle \alpha x, y \rangle|^{\frac{1}{2}}} & , \alpha \geq 0, t > 0 \\ 1 - \dfrac{t^{\frac{1}{2}}}{t^{\frac{1}{2}} + |\langle \alpha x, y \rangle|^{\frac{1}{2}}} & , \alpha < 0, t > 0 \\ 0 & , \quad t \leq 0 \end{cases}$$

Define $a * b = \min\{ a, b \}$ for all $a, b \in X$. This is a fuzzy pre-Hilbert and called the standard fuzzy pre-Hilbert induced by the pre-Hilbert $\langle\ .,.\ \rangle$.

Definition (1.4.7) : [18] A t-norm $* : [0,1] \times [0,1] \to [0,1]$ is called strong if it has the two following properties :

(1) For all $a, b \in (0,1)$, $a * b > 0$,

(2) For all $a, b, c, d \in [0,1]$ and $a > b$, $c > d$ we have $a * b > c * d$.

Theorem (1.4.8) : [7] Suppose that $(X, H, *)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X$,

$$\sup\{t \in \mathbb{R}, H(x, y, t) < 1\} < \infty.$$

Define $\langle\ .,.\ \rangle : X \times X \to \mathbb{R}$ by $\langle x, y \rangle = \sup\{ t \in \mathbb{R}, H(x, y, t) < 1 \}$.

Then $(X, \langle\ .,.\ \rangle)$ is a pre-Hilbert space.

Corollary (1.4.9) :[7] Let $(X, H, *)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X$, $\sup\{ \mathrm{t} \in \mathbb{R} , H(x, y, t) < 1\} < \infty$. If we define $\|x\| = (\sup\{ t \in \mathbb{R} , H(x, x, t) < 1\})^{\frac{1}{2}}$, then $(X, \|.\|)$ is a normed space.

CHAPTER TWO SPECTRAL THEORY OF LINEAR OPERATOR ON FUZZY NORMED SPACES

In this chapter , the focuss will be on discussing spectral theory of linear operator on fuzzy normed spaces . It consists of tow sections. Section one deals with Eigenvalue and Eigenvectors in fuzzy normed spaces and some of their properties . Section two deals with regular value ,resolvent,spectrum,the point spectrum , the continuous spectrum and the residual spectrum in fuzzy normed spaces and some of their properties.

2.1 Eigenvalue and Eigenvectors in fuzzy normed space

This section deals with Eigenvalue and Eigenvectors in fuzzy normed spaces and some of their properties.

Definition (2.1.1) : [13] A function $T : X \to Y$ is called an operator from X into Y if X and Y are linear spaces over the same field F.

Definition (2.1.2) : [13] A linear operator T is an operator such that:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ and for all $\alpha, \beta \in F$.

Definition (2.1.3) : [19] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces .A linear operator $T : (X, N_1,*) \to (Y, N_2,*)$ is said to be fuzzy bounded if and only if there exists r>0, such that for each $t > 0$

$$N_2(T(x), t) \geq N_1\left(x, \frac{t}{r}\right), \quad \forall x \in X.$$

Remark (2.1.4) :[21] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces over F, $FB(X, Y)$ is the space of all fuzzy bounded linear operator from X in to Y.

Definition (2.1.5) : Let $(X, N,*)$ be a fuzzy normed spaces over F and

$T \in L(X)$ then

(1)A scalar $\lambda \in F$ is called an eigenvalue of T, if there exists non zero

$x \in X$ such that $T(x) = \lambda x$

(2)A non zero vector $x \in X$ is called an eigenvector of T,if there exist

$\lambda \in F$ such that $T(x) = \lambda x$

Example (2.1.6) : Let $X=\mathbb{R}^2$ and $T:(X,N,*)\to(X,N,*)$ Define

By $T(x,y)=(-y,x)$ for all $(x,y)\in\mathbb{R}^2$ and $N:\mathbb{R}^2\times(0,\infty)\to[0,1]$

Define fuzzy norm in example(1.3.5) and T is linear operator has no

Eigenvalue.

Example (2.1.7) : Let $X=\mathbb{R}^2$ and $T:(X,N,*)\to(X,N,*)$ Define

By $T(x,y)=(x+2y,3x+2y)$ for all $(x,y)\in\mathbb{R}^2$ and

$N:\mathbb{R}^2\times(0,\infty)\to[0,1]$ Define fuzzy norm by equation(1.3.5) and T is linear operator have eigenvalues $\lambda=-1,\lambda=4$

Solution: Suppose $T(x,y)=\lambda(x,y)\Longrightarrow(x+2y,3x+2y)=\lambda(x,y)\Longrightarrow$ $(x+2y,3x+2y)=(\lambda x,\lambda y)\Longrightarrow x+2y=\lambda x,3x+2y=\lambda y\Longrightarrow$

$(1-\lambda)x+2y=0$, $3x+(2-\lambda)y=0\Longrightarrow\begin{vmatrix}1-\lambda & 2\\ 3 & 2-\lambda\end{vmatrix}=0\Longrightarrow$

$\lambda=-1,\lambda=4$

Example (2.1.8) : Let $X=\ell^2$and $T:(X,N,*)\to(X,N,*)$ Define

By $T(x_1,x_2,\dots)=(0,x_1,x_2,\dots)$ for all$(x_1,x_2,\dots)\in\ell^2$ and

$N:\ell^2\times(0,\infty)\to[0,1]$ Define fuzzy norm by equation(1.3.7)

and T is linear operator has no eigenvalue

Solution: Suppose $T(x_1,x_2,\dots)=\lambda(x_1,x_2,\dots)\Longrightarrow(0,x_1,x_2,\dots)=\lambda(x_1,x_2,\dots)$

$(0,x_1,x_2,\dots)=(\lambda x_1,\lambda x_2,\lambda x_3\dots)$ implies $0=\lambda x_1$, $x_1=\lambda x_2$, $x_2=\lambda x_3$,….

If $\lambda\neq0$, we divide by λ and conclude $x_1=x_2=\dots=0$ .If $\lambda=0$ we also

$x_1=x_2=\dots=0$. T is linear operator has no eigenvalue .

Theorem (2.1.9) : Let $(X,N,*)$ be a fuzzy finite dimensional normed spaces over F and $T\in L(X)$ if x one eigenvector of Tcorresponding to the eigenvalues λ and α is any non zero scalar then αx is also an eigenvector of T corresponding to the same eigenvalue λ

Proof: Since x is an eigenvector of T corresponding to the eigenvalue

λ then $x \neq 0$ and $T(x) = \lambda x$ since $x \neq 0$ and $\alpha \neq 0 \implies \alpha x \neq 0$

$T(\alpha x) = \alpha\, T(x) = \alpha(\lambda x) = (\alpha\lambda)x = (\lambda\alpha)x = \lambda(\alpha x)$

Therefore αx is an eigenvector of T corresponding to the eigenvalue λ

Remark(2.1.10): Corresponding to an eigenvalue λ there may correspond more than one eigenvectors.

Theorem (2.1.11) : Let $(X, N,*)$ be a fuzzy finite dimensional normed spaces over F and $T \in L(X)$ if x an eigenvector of T.Then x cannot correspond to more than one eigenvalues of T.

proof: Let be an eigenvector of T corresponding to two distinct eigenvalues λ_1 and λ_2 of T, $T(x) = \lambda_1 x$ and also $T(x) = \lambda_2 x$ .Therefore we have

$\lambda_1 x = \lambda_2 x \implies \lambda_1 x - \lambda_2 x = 0 \implies (\lambda_1 - \lambda_2) x = 0$

since $x \neq 0 \implies \lambda_1 - \lambda_2 = 0 \implies \lambda_1 = \lambda_2$

and α is any non zero scalar ,then αx is also an eigenvector of T corresponding to the same eigenvector λ

Definition (2.1.12) : [21] Let $(X, H,*)$ and $(Y, H,*)$ be fuzzy Hilbert spaces over F, and let $T \in FB(X, Y)$. A fuzzy Hilbert-adjoint operator T^* of T is the operator $T^*: (Y, H,*) \to (X, H,*)$ such that :

$\sup\{ t \in \mathbb{R}, H(T(x), y, t) < 1\} = \sup\{ t \in \mathbb{R}, H(x, T^*(y), t) < 1\}$ for all $x \in X$ and $y \in Y$.

Remark (2.1.13) : [21] We denoted $FB(X, X)$ by $FB(X)$.

Theorem (2.1.14) : [21] (Some Properties of fuzzy Hilbert-adjoint operator**)**

Let $(X, H,*)$ and $(Y, H,*)$ be a fuzzy Hilbert spaces over F, and let

$S, T \in FB(X, Y)$. Then we have :

(**a**)$\sup\{ t \in \mathbb{R}, H(T^*(y), x, t) < 1\} = \sup\{ t \in \mathbb{R}, H(y, T(x), t) < 1\}$ for all $x \in X$ and $y \in Y$

(b)$(T + S)^* = T^* + S^*$

(c)$(T^*)^* = T$

Theorem (2.1.15) : [21] Let $(X, H,*)$ be a fuzzy Hilbert space over F, and $T \in FB(X)$. Then $T = 0$ if and only if $\sup\{t \in \mathbb{R}, H(T(x), T(x), t) < 1\} = 0$ for all $x \in X$.

Definition (2.1.16) :[7] Let $(X, H,*)$ be a fuzzy pre-Hilbert space, where $*$ is a strong t-norm and for each $x, y \in X$, $\sup\{t \in \mathbb{R}, H(x, y, t) < 1\} < \infty$ and $\|x\| = (\sup\{ t \in \mathbb{R}, H(x, x, t) < 1\})^{\frac{1}{2}}$. We say that $(X, H,*)$ is a fuzzy Hilbert space if $(X, N,*)$ is complete normed space.

Definition (2.1.17) :[21] Let $(X, H,*)$ be a fuzzy Hilbert space over F and let $T \in B(X)$. T is said to be Normal if $T \circ T^* = T^* \circ T$.

Theorem (2.1.18) : Let T be a normal operator on a finite dimensional

Fuzzy Hilbert space X over F. Then

(1) $T - \lambda$I is normal

(2)Every eigenvector of T is also eigenvector for T^*

Proof:

(1) Since T is normal $\Longrightarrow T \circ T^* = T^* \circ T$

$(T - \lambda \mathrm{I})^* = T^* - \bar{\lambda}\,\mathrm{I}$ by Theorem (2.1.14) (b)

$(T - \lambda \mathrm{I}) \circ (T - \lambda \mathrm{I})^* = (T - \lambda \mathrm{I}) \circ (T^* - \bar{\lambda}\,\mathrm{I}) = T \circ T^* - \bar{\lambda} T - \lambda T^* + \lambda\bar{\lambda}$

$$(T-\lambda\mathrm{I})^{*}\circ(T-\lambda\mathrm{I})=(T^{*}-\bar{\lambda}\,\mathrm{I})\circ(T-\lambda\mathrm{I})=T^{*}\circ T-\bar{\lambda}T-\lambda T^{*}+\bar{\lambda}\lambda$$

$$=T\circ T^{*}-\bar{\lambda}T-\lambda T^{*}+\lambda\bar{\lambda}$$

$$\Longrightarrow (T-\lambda\mathrm{I})^{*}\circ(T-\lambda\mathrm{I})=(T-\lambda\mathrm{I})\circ(T-\lambda\mathrm{I})^{*}$$

Therefore $T-\lambda\mathrm{I}$ is normal

(2)Let x be an eigevector of T corresponding to eigenvalue λ

$\Longrightarrow T(x)=\lambda x$

$$\sup\{t\in\mathbb{R},\ H(T(x),T(x),t)<1\}=\sup\{t\in\mathbb{R},\ H(x,T^{*}(T(x)),t)<1\}$$
$$=\sup\{t\in\mathbb{R},\ H(x,T^{*}\circ T(x),t)<1\}$$
$$=\sup\{t\in\mathbb{R},\ H(x,T\circ T^{*}(x),t)<1\}$$
$$=\sup\{t\in\mathbb{R},\ H(x,T(T^{*}(x)),t)<1\}$$
$$=\sup\{t\in\mathbb{R},\ H(T^{*}(x),T^{*}(x),t)<1\}$$

Since $T-\lambda\mathrm{I}$ is normal, therefore $x\in X$, we have

$$\sup\{t\in\mathbb{R},\ H((T-\lambda\mathrm{I})(x),(T-\lambda\mathrm{I})(x),t)<1\}$$
$$=\sup\{t\in\mathbb{R},\ H((T-\lambda\mathrm{I})^{*}(x),(T-\lambda\mathrm{I})^{*}(x),t)<1\}$$

Since $T(x)=\lambda x \Longrightarrow T(x)=\lambda\mathrm{I}(x)\Longrightarrow T(x)-\lambda\mathrm{I}(x)=0\Longrightarrow(T-\lambda\mathrm{I})(x)=0$

then $T-\lambda\mathrm{I}=0$,then $\sup\{t\in\mathbb{R},\ H((T-\lambda\mathrm{I})(x),(T-\lambda\mathrm{I})(x),t)<1\}=0$

by theorem(2.1.15), $\sup\{t\in\mathbb{R},\ H((T-\lambda\mathrm{I})^{*}(x),(T-\lambda\mathrm{I})^{*}(x),t)<1\}=0$

$\Longrightarrow (T-\lambda\mathrm{I})^{*}=0$ then for each $x\in X\Longrightarrow (T-\lambda\mathrm{I})^{*}(x)=0$

$$\Longrightarrow T^{*}(x)-\bar{\lambda}\,\mathrm{I}(x)=0\Longrightarrow T^{*}(x)=\bar{\lambda}\,\mathrm{I}(x)\Longrightarrow T^{*}(x)=\bar{\lambda}x$$

Therefore x is eigenvector of T^{*} and corresponding eigenvalue is
$\bar{\lambda}$

Definition (2.1.19) :A subspace M of a fuzzy normed space X is said to be Invariant under a linear operator $T : (X, N,*) \to (X, N,*)$ if $T(\mathrm{M}) \subset \mathrm{M}$

Definition (2.1.20) : [7] Let $(X, H,*)$ be a fuzzy pre-Hilbert space. $x, y \in X$ is said to be fuzzy orthogonal if $H(x, y, t) = h(t)(\forall t \in \mathbb{R})$ and it is denoted by $x \perp y$.

Definition (2.1.21) : [9] Let $(X, H,*)$ be a fuzzy pre-Hilbert space. A subset B of X is called fuzzy orthogonal if $x \perp y$, for each $x, y \in B$.

Lemma (2.1.22) : [7] If $(X, H,*)$ be a fuzzy pre-Hilbert space , then $(X, H,*)$ is non decreasing with respect to t, for each $x, y \in X$.

Definition (2.1.23) : [9] If B is a subset of the fuzzy pre-Hilbert space $(X, H,*)$, then $B^{\perp} = \{ x \in X : x \perp y \quad \forall y \in B \}$.

Theorem (2.1.24) :[1]Let B be a non-empty subset of a fuzzy pre-Hilbert space $(X, H,*)$, then $B^{\perp}$ is closed fuzzy subspace of X.

Proof: Since $H(0, y, t) = h(t) \qquad \forall y \in B \Longrightarrow 0 \in B^{\perp}$ then $B^{\perp} \neq \emptyset$

Let $x, y \in B^{\perp}$ and $\alpha, \beta, r \in \mathbb{R}$

$H(x, z, t) = h(r) \quad \forall z \in B$ and

$H(y, z, t) = h(r) \quad \forall z \in B$

$\forall z \in B$ we have:

If α>0, β>0

$$H(\alpha x + \beta y, z, t) = sup_{s+t=r}\left(H\left(x, z, \frac{t}{\alpha}\right) * H\left(y, z, \frac{s}{\beta}\right)\right)$$

$$= h\left(\frac{t}{\alpha}\right) * \ h\left(\frac{s}{\beta}\right)$$

$$= h(t) * h(s) = h(r) \qquad \forall\, r \in \mathbb{R}$$

If $\alpha<0, \beta<0$

$$H(\alpha x + \beta y, z, t) = sup_{s+t=r}\left(1 - H\left(x, z, \frac{t}{-\alpha}\right) * 1 - H\left(y, z, \frac{s}{-\beta}\right)\right)$$

$$= h(r) \qquad \forall\, r \in \mathbb{R}$$

If $\alpha<0, \beta = 0$ or $\alpha = 0, \beta<0$,or $\alpha>0$, $\beta = 0, \alpha = 0, \beta>0$

$H(\alpha x + \beta y, z, t) = h(r) \qquad \forall\, r \in \mathbb{R}$

$\Longrightarrow \alpha x + \beta y \in B^{\perp}$

Therefore $B^{\perp}$ is a fuzzy subspace X.

Let $x \in \overline{B^{\perp}}$ $\exists$ $\{x_n\}$ in $B^{\perp}$ such that $x_n \longrightarrow x$

Let $y \in B \Longrightarrow H(x_n, y, t) = h(t)\ \forall\, n \in \mathbb{Z}^{+}$

And $t \in \mathbb{R}$ $(x_n \in B^{\perp}\ \forall\, n \in \mathbb{Z}^{+})$

Since $x_n \longrightarrow x \Longrightarrow H(x_n, y, t) \longrightarrow H(x, y, t)$

$\Longrightarrow H(x, y, t) = h(t)$ for all $y \in B$

$\Longrightarrow x \in B^{\perp} \Longrightarrow \overline{B^{\perp}} = B^{\perp}$

$\Longrightarrow B^{\perp}$ is closed fuzzy subspace

Theorem (2.1.25) : [1] Let $(X, H,*)$ be a fuzzy pre-Hilbert space . And $\boldsymbol{A} \subset X$;

(1)The relation of Orthogonality symmetric (i.e. if $x \perp y$ then $y \perp x$)

(2)If $x \perp y$ then $\boldsymbol{\alpha} x \perp y\ \forall\, t \in \mathbb{R}$

(3)Let $\boldsymbol{A} \subset B$ then $B^{\perp} \subset \boldsymbol{A}^{\perp}$

(4) $\boldsymbol{A} \subset \boldsymbol{A}^{\perp\perp}$

(5) $\boldsymbol{A} \subset B^{\perp} \leftrightarrow B \subset \boldsymbol{A}^{\perp}$

(6)If $x \perp \boldsymbol{x} \leftrightarrow \ x = \mathbf{0} \ \ \forall t \in \mathbb{R}$

(7) $X^{\perp} = \{\mathbf{0}\}$ for all $t \in \mathbb{R}$

(8) $\boldsymbol{A} \cap \boldsymbol{A}^{\perp} = \{\mathbf{0}\}$ for all $\boldsymbol{t} \in \mathbb{R}$

(9)For every vector $\boldsymbol{x} \in \boldsymbol{X}$,we have $\mathbf{0} \perp \boldsymbol{x} \ \forall \ \boldsymbol{x} \in \boldsymbol{X}$

Definition (2.1.26): Let M be a closed of a fuzzy Hilbert space X and $x \notin$ M said that projection of $x \in X$ onto M if there is $z \in$ M

$$N(x-z,t) \ = sup\left\{\frac{t}{t+\|x-y\|}: y \in \ \mathrm{M}, \mathrm{t}>0\right\} \text{,we write } y = P_{\mathrm{M}}(x)$$

Theorem (2.1.27): If M is subspace of a fuzzy Hilbert space X,for $x \in X$ there exist a unique $y \in$ M such that $x - y \perp$ M and $y = P_{\mathrm{M}}(x)$

Proof: Define $\langle .,. \rangle : X \times X \to \mathbb{R}$ by $\langle x, y\rangle = \sup\{ t \in \mathbb{R}, H(x,y,t) < 1 \}$. from theorem (1.4.8) ,we have $(X, \langle \ .,. \rangle)$ is a pre-Hilbert space. Also $\|x\| = (\sup\{ t \in \mathbb{R} \,, H(x,x,t) < 1\})^{\frac{1}{2}}$ from corollary (1.4.9) we have $(X, \|.\|)$ is a normed space .

Since X is a fuzzy Hilbert space then $(X, \|.\|)$ is complete normed space then X Hilbert space .By using [16] for $x \in X$ there exist a unique $y \in$ M such that $x - y \perp$ M and $y = P_{\mathrm{M}}(x)$

Then $\langle x - y, z\rangle = 0 \ \forall z \in$ M then $\sup\{ t \in \mathbb{R}, H(x,y,t) < 1 \} = 0$ there fore

$x - y \perp$ M $\forall z \in$ M in X fuzzy Hilbert space , since $y = P_{\mathrm{M}}(x)$ then by [16] there is $b \in$ M such that

$\|x - b\| = inf\{\|x-y\| : y \in$ M$\}$ then

$\|x-b\| \le \|x-y\|, y \in$ M $\Longrightarrow$ $t + \|x-b\| \le t + \|x-y\|$, t>0 $\Longrightarrow$

$$\frac{t}{t+\|x-y\|} \le \frac{t}{t+\|x-b\|}, \ y \in \ \mathrm{M}, \mathrm{t}>0 \ \text{ therefore}$$

$$N(x-b,t) \ = sup\left\{\frac{t}{t+\|x-y\|}: y \in \ \mathrm{M}, \mathrm{t}>0\right\}.$$

Theorem (2.1.28): If M is subspace of a fuzzy Hilbert space X,then $X = M \oplus M^{\perp}$, that is each $x \in X$ can be uniqully decomposed from $x = y + z$ with $y \in M$, $z \in M^{\perp}$.

Proof: For all $x \in X$ and M is subspace there exist y so that $x = x - y + y$ with

$x - y \in M^{\perp}$ and $y \in M$ such that $y = P_M(x)$ and $z = x - y \Rightarrow x = y + z \Rightarrow$

$X = M + M^{\perp}$ also since $M \cap M^{\perp} = \{0\}$ by theorem (2.1.25) ,there fore $X = M \oplus M^{\perp}$.

Theorem (2.1.29): If M is subspace of a fuzzy Hilbert space X, then M is fuzzy closed iff $M = M^{\perp\perp}$

Proof: Since $M \subset M^{\perp\perp}$ by theorem (2.1.25) ,we show that $M^{\perp\perp} \subset M$

Let $x \in M^{\perp\perp}$ then by theorem(2.1.28) $x = y + z$,where $y \in M$,$z \in M^{\perp}$ since

$M \subset M^{\perp\perp}$ and $M^{\perp\perp}$ is subspace $z = x - y \in M^{\perp\perp}$ but $z \in M^{\perp} \Rightarrow$

$z \in M^{\perp\perp} \cap M^{\perp}$

Since $M^{\perp\perp} \cap M^{\perp} = \{0\}$ then $z = 0$,thus $x = y \in M$ there fore $M^{\perp\perp} \subset M$ thus $M = M^{\perp\perp}$

Conversely suppose $M = M^{\perp\perp}$ since $(M^{\perp})^{\perp} = M^{\perp\perp}$ is close set then M is close set .

Theorem (2.1.30): Let M be a closed subspace of a fuzzy Hilbert space X over F, and let $T \in FB(X)$.Then M is invariant under T iff $M^{\perp}$ is invariant under T^*

Proof: Suppose M is invariant under T

Let $y \in M^{\perp}$ .To prove that $T^*(y) \in M^{\perp}$(i.e. $T^*(y) \perp M$)

Let $x \in M$,since M is invariant under $T \Rightarrow T(x) \in M$

Since $y \in M^{\perp} \Rightarrow \sup\{ t \in \mathbb{R} , H(T(x), y, t) < 1\} = 0 \Rightarrow$

$\sup\{ t \in \mathbb{R} , H(x, T^*(y), t) < 1\} = 0$ .Thus $T^*(y) \perp M$

Conversely suppose that $M^{\perp}$ is invariant under T^*.

Since $M^{\perp}$ is closed subspace of a fuzzy Hilbert space X by theorem (2.1.24) and since $M^{\perp}$ is invariant under T^*, therefore by first case $(M^{\perp})^{\perp}$ is invariant under $(T^*)^*$but $(M^{\perp})^{\perp} = M^{\perp\perp} = M$ and $(T^*)^* = T^{**} = T$ Therefore M is invariant under T.

Definition (2.1.31) : Let M be a closed subspace of a fuzzy Hilbert X over F and let $T \in FB(X)$ .We say that T is reduced by M if both M and $M^{\perp}$ are Invariant under T .If T is reduced by M , then some times we also say that M reduces T.

Theorem (2.1.32):A closed subspace M of a fuzzy Hilbert X over F reduces an operator T iff M is invariant under both T and T^*.

Proof: Suppose M reduces an operator T .Then by the definition of reducibility both M and $M^{\perp}$ are invariant under T by theorem (2.1.30) ,if $M^{\perp}$ is invariant under T .Then $(M^{\perp})^{\perp}$, i.e. M is invariant T^* .then M is invariant under both $T$and T^*.

Conversely suppose that M is invariant under both T and T^*

Since M is invariant under T^* ,therefore by theorem (2.1.30), $M^{\perp}$ is invariant

under $(T^*)^*$,i.e. T .Thus both M and $M^{\perp}$ are invariant under T .Therefore M reduces T.

Definition (2.1.33) : Let X be a fuzzy normed space over F ,$T \in FB(X)$ and let λ be eigenvalue of T. Then set consisting of all eigenvectors of T which correspond to eigenvalue λ together with the vector 0 is called eigenspace of T corresponding to the eigenvalue λ and is denoted by M_λ

(1)Since by definition an eigenvector is non zero vector,there fore the set M_λ necessary contains some non zero vector.

(2)Since by definition of M_λ a non zero vector x is in M_λ iff $T(x) = \lambda x$

Also it is given that the vector 0 is in M_λ the vector 0 defintly satisfies

The equation $T(x) = \lambda x$ therefor

$M_\lambda = \{x \in X: T(x) = \lambda x\} = \{x \in X: (T - \lambda \mathrm{I})(x) = 0\}$

Thus M_λ is null space (or kernel of linear operator $T-\lambda\mathrm{I}$ on X). Hence M_λ is a subspace of X.

(3)Let $x \in X$ since M_λ is a subspace of X and $\lambda \in F \Longrightarrow \lambda x \in \mathrm{M}_\lambda$ since

$x \in \mathrm{M}_\lambda \Longrightarrow T(x) = \lambda x \Longrightarrow T(x) \in \mathrm{M}_\lambda \Longrightarrow \mathrm{M}_\lambda$ is an invariant under T

from (1),(2) and (3) we have M_λ is non zero subspace of X invariant under of T.

(4)If $T \in FB(X)$ then M_λ is closed subspace of X , M_λ is called eigenspace of T ,corresponding to the eigenvalue λ.

Theorem (2.1.34): If T be a normal operator on n dimensional fuzzy Hilbert Space X over F ,then each eigenspace reduces T

Proof: Let x_i belong to M_i the eigenspace of T and corresponding eigenvalue

be λ_i, so that $T(x_i) = \lambda_i x_i$ since T is normal then by theorem(2.1.18)

eigenvalue for T^*(i.e. $T^*(x_i) = \overline{\lambda_i} x_i$) since M_i is a subspace $\Longrightarrow \overline{\lambda_i} x_i \in \mathrm{M}_i \Longrightarrow$

$T^*(x_i) \in \mathrm{M}_i \Longrightarrow \mathrm{M}_i$ is invariant under T^*,but M_i is invariant under T ,then by Theorem(2.2.32) M_i is reduces T .

2.2 Regular value ,resolvent set,spectrum, the point spectrum , the continuous spectrum and the residual spectrum in fuzzy normed spaces.

This section deals with regular value ,resolvent,spectrum,the point spectrum , the continuous spectrum and the residual spectrum in fuzzy normed spaces and some of their properties.

Definition (2.2.1):[10]Let$(X, N,*)$ be a fuzzy normed space.Then sequence $\{x_n\}$ in X is said to fuzzy converges to x in X if for each $\varepsilon \in (\mathbf{0}, \mathbf{1})$ and each t>0,there exist $n_0 \in \mathbb{Z}^+$ such that $N(x_n - x, t) > 1 - \varepsilon$, for all $n \geq n_0$(or eguivalently $\lim_{n\to\infty} N(x_n - x, t) = 1$)

Definition (2.2.2) :[4]A subset A of a fuzzy normed space $(X, N,*)$ is said to be closure of a subset $\overline{A}$ of X in case for any $x \in \overline{A}$ there exist a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for each t>0 , on the other hand a subset A of a fuzzy normed space $(X, N,*)$ is said to be dense case $\overline{A} = X$

Definition (2.2.3) : Let $(X, N,*)$ be a fuzzy normed space over the field $\mathbb{C}$ where $X \neq \{0\}$ and $N:X \times (0,\infty) \to [0,1]$ and $T: (X, N,*) \to (X, N,*)$ be a linear operator .A regular value λ of T is complex number such that

(1)$\mathrm{R}_\lambda(T)$ exist

(2) $\mathrm{R}_\lambda(T)$ is fuzzy bounded linear operator on range of $T_\lambda = T - \lambda\mathrm{I}$

(3) $\mathrm{R}_\lambda(T)$ is defined on a set which is dense in X

Where $\mathrm{R}_\lambda(T) = (T_\lambda)^{-1} = (T - \lambda\mathrm{I})^{-1}$ call resolvent operator of T and

The resolvent set $\rho(T)$ of T is the set of all regular value λ of T

Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in complex plane $\mathbb{C}$ is called spectrum of T ,A $\lambda \in \sigma(T)$ is called a spectral value of T.

Proposition(2.2.4): Let $(X, \|.\|)$ be a normed space and let N be the fuzzy norm defined by equation (1.3.5) for each $x \in X$ and $t \in (0,\infty)$ then closure of a subspace A of X with respect to $\|.\|$ is equal to the closure of A with respect N.

Proof: Suppose $\overline{A}$ is closure of A with respect to $\|.\|$ .Then for each $x \in \overline{A}$ there exist a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty}\|x_n - x\| = 0$ .Hence for each t>0 $\lim_{n\to\infty} N(x_n - x, t) = 1$ .Thus each element of $\overline{A}$ is closure of A with respect to N.

Conversely ,suppose $\overline{A}$ is the closure of A with respect to N .Then for each $x \in \overline{A}$ there exists sequence $\{x_n\}$ in A such that for each t>0 $\lim_{n\to\infty} N(x_n - x, t) = 1$ .Hence $\lim_{n\to\infty}\|x_n - x\| = 0$ .thus each element of

$\overline{A}$ the closure of with respect to $\|.\|$ then the closure of a subspace A of X With respect to $\|.\|$ is equal to the closure of A with respect to N.

Theorem(2.2.5):[13]Let X be a complex Banach space and $T : X \to X$ is bounded operator , and $\lambda \in \rho(T)$. Assume that (a) T is closed or(b) T is bounded then $\mathrm{R}_\lambda(T)$ is defined on the whole space X and is bounded.

Theorem(2.2.6): Let $(X, \|.\|)$ be a Banach space over the field $\mathbb{C}$ where $X \neq \{0\}$ $T : X \to X$ be a linear operator on X , N be the fuzzy norm defined by equation (`1.3.5) and let $\lambda \in \rho(T)$ with respect to $(X, N,*)$ if T is fuzzy bounded on X then $\mathrm{R}_\lambda(T)$ is fuzzy bounded on $(X, N,*)$.

Proof: Suppose T is fuzzy bounded on X then there exist $r > 0$ such that for each $x \in X$ and $t \in (0, \infty)$, $N(T(x), t) \geq N\left(x, \frac{t}{r}\right)$, Hence for each $x \in X$ and $\frac{t}{t+\|T(x)\|} \geq \frac{\frac{t}{r}}{\frac{t}{r}+\|x\|}$ .Therefore T is $\frac{t}{t+\|T(x)\|} \geq \frac{t}{t+\|rx\|} \Longrightarrow$there exist $r > 0$ such that $\|T(x)\| \leq r\|x\|$ fore each $x \in X$ there fore T is bounded linear operator in X moreover , since $\lambda \in \rho(T)$ with respect to $(X, N,*)$ Then $\mathrm{R}_\lambda(T)$ is exists and $\mathrm{R}_\lambda(T)$ is bounded on range T_λ and rang(T_λ) is dense in $(X, N,*)$ by proposition (2.2.4) rang(T_λ) is dense in $(X, \|.\|)$ that λ belong to resolvent set of bounded linear operator T by theorem (2.2.5) $\mathrm{R}_\lambda(T)$ is bounded linear operator in $(X, \|.\|)$ then there exist $r > 0$ Such that for each $x \in X$ $\|\mathrm{R}_\lambda(T)(x)\| \leq r\|x\|$ then for each t>0 and for each $x \in X$ $t + \|\mathrm{R}_\lambda(T)(x)\| \leq t + r\|x\| \Longrightarrow \frac{t}{t+\|\mathrm{R}_\lambda(T)(x)\|} \geq \frac{t}{t+\|rx\|} \Longrightarrow$

$\frac{t}{t+\|\mathrm{R}_\lambda(T)(x)\|} \geq \frac{\frac{t}{r}}{\frac{t}{r}+\|x\|}$ then there exist r >0 such that for each $x \in X$ and

$t \in (0,\infty)$, $N(\mathrm{R}_\lambda(T)(x),t) \geq N\left(x,\frac{t}{r}\right)$, there fore $\mathrm{R}_\lambda(T)$ is fuzzy bounded on $(X,N,*)$.

Proposition(2.2.7): Let $(X,\|.\|)$ be a normed space and let N be the fuzzy norm defined by equation (1.3.7) for each $x \in X$ and $t \in (0,\infty)$ then closure of a subspace A of X with respect to $\|.\|$ is equal to the closure of A with respect N.

Proof: Suppose $\overline{A}$ is closure of A with respect to $\|.\|$ .Then for each $x \in \overline{A}$ there

exist a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty}\|x_n - x\| = 0$ .That is for $\varepsilon \in (\mathbf{0},\mathbf{1})$ there exist a positive integer n_0 such that $\|x_n - x\| < \varepsilon$ for each $n \geq n_0$ .There fore $N(x_n - x,\varepsilon) = 1$ for each $n \geq n_0$. Thus $\lim_{n\to\infty} N(x_n - x,\varepsilon) = 1$ for each ε >0 there fore ,each element of $\overline{A}$ belong to the closure of A with respect to N .

Conversely,suppose $\overline{A}$ is closure of A with respect to N .Then for each $x \in \overline{A}$ there exist a sequence $\{x_n\}$ in A such that for each $t \in (0,\infty)$ $\lim_{n\to\infty} N(x_n - x,t) = 1$ .fix $\alpha \in (0,\infty)$,thus $\lim_{n\to\infty} N(x_n - x,t) = 1 > \alpha$ For each t>0 .That is for each t>0 ,there exist $n_0 \in \mathbb{Z}^+$,such that $N(x_n - x,t) > \alpha$ for each $n \geq n_0$. so $\|x_n - x\| < t$ for each $n \geq n_0$ .Hence $\lim_{n\to\infty}\|x_n - x\| = 0$ .Thus each element of $\overline{A}$ is closure of A with respect to $\|.\|$.

Then the closure of a subspace of X with respect to $\|.\|$ is equal to the closure of A with respect to N.

Theorem(2.2.8): Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$, where $X \neq \{0\}$ $T : X \rightarrow X$ be a linear operator , N be the fuzzy norm defined by equation (`1.3.7) and let $\lambda \in \rho(T)$ with respect to $(X, N,*)$ if T is fuzzy bounded on X then $\mathrm{R}_\lambda(T)$ is fuzzy bounded on $(X, N,*)$.

Proof: Suppose T is fuzzy bounded on X then there exist r_1>0 such that for each $x \in X$ and $t \in (0, \infty)$, $N(T(x), t) \geq N\left(x, \frac{t}{r_1}\right)$. Assume for the contrary there exist $x_1 \neq 0$ such that $\|T(x_1)\| > r_1\|x_1\|$ .Let $\|T(x_1)\| = t_{\boldsymbol{o}}$ Hence , $N(T(x_1), t_{\boldsymbol{o}})$=0 but $N(r_1 x_1, t_{\boldsymbol{o}})$=1 this is a contradiction .Then for each $x \in X$ there exist r_1>0 such that $\|T(x)\| \leq r_1 \|x\|$ .Then T is bounded linear operator in X. Moreover , since $\lambda \in \rho(T)$ with respect to $(X, N,*)$ Then $\mathrm{R}_\lambda(T)$ is exists and $\mathrm{R}_\lambda(T)$ is bounded on range T_λ and rang(T_λ) is dense in $(X, N,*)$ by proposition (2.2.7) rang(T_λ) is dense in $(X, \|.\|)$ thus λ belong to resolvent set of bounded linear operator T by theorem (2.2.5) $\mathrm{R}_\lambda(T)$ is bounded linear operator in $(X, \|.\|)$ then there exist r >0 Such that for each $x \in X$ $\|\mathrm{R}_\lambda(T)(x)\| \leq r\|x\|$ .Let $x \in X$ $t \in (0, \infty)$ then we have two case :

(1)If t>$\|\mathrm{R}_\lambda(T)(x)\|$ then $N(\mathrm{R}_\lambda(T)(x), t)$ =1 since $\|\mathrm{R}_\lambda(T)(x)\| \leq \|rx\|$ then either $t \leq \|rx\|$ or $\|rx\| < t$ .If $t \leq \|rx\|$ then $N(rx, t) = 0$ .Hence $N(\mathrm{R}_\lambda(T)(x), t)$ =1> $N(rx, t) = N\left(x, \frac{t}{r}\right) = 0$ .If $\|rx\| < t$ .Thus for each $x \in X$, $N(\mathrm{R}_\lambda(T)(x), t) = N(rx, t) = 1$

(2)If $\|R_\lambda(T)(x)\| \geq t$ then $N(R_\lambda(T)(x), t) = 0$,since $\|R_\lambda(T)(x)\| \leq \|rx\|$

$N(rx, t) = 0$ .Hence $N(R_\lambda(T)(x), t) = N\left(x, \frac{t}{r}\right) = 0$ .There fore $R_\lambda(T)$ is fuzzy bounded on $(X, N, *)$.

Theorem(2.2.9): Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ where $X \neq \{0\}$ $T : X \to X$ be a linear operator , N be the fuzzy norm defined by equation (`1.4.5) and let $\lambda \in \rho(T)$ with respect to $(X, N, *)$ if T is fuzzy bounded on X and $\lim_{n\to\infty} N(x_n, t) = 1$ for each $t \in (0, \infty)$,Then

$\lim_{n\to\infty} N(R_\lambda(T)(x_n), t) = 1$.

Proof: Since $\lim_{n\to\infty} N(x_n, t) = 1$ then for each $\varepsilon \in (\mathbf{0}, \mathbf{1})$ and for each t>0

There exist $n_0 \in \mathbb{Z}^+$ such that $N(x_n, t) > 1 - \varepsilon$, for all $n \geq n_0$

Since T is fuzzy bounded on $(X, N, *)$ and $\lambda \in \rho(T)$ with respect to $(X, N, *)$

Then by Theorem(2.2.6) $R_\lambda(T)$ is fuzzy bounded on $(X, N, *)$ .Hence

There exist r >0 such that for each $x \in X$ and $t \in (0, \infty)$

, $N(R_\lambda(T)(x), t) \geq N\left(x, \frac{t}{r}\right)$,there fore $N(R_\lambda(T)(x_n), t) \geq N\left(x_n, \frac{t}{r}\right)$

for all $n \in \mathbb{N}$ since that $N(x_n, t) > 1 - \varepsilon$, for all $n \geq n_0$, t>0

Put $t_{\boldsymbol{o}} = \frac{t}{r}$ >0 then $N(R_\lambda(T)(x_n), t) \geq N\left(x_n, \frac{t}{r}\right) = N(x_n, t_{\boldsymbol{o}}) > 1 - \varepsilon$

for all $n \geq n_0$, then $N(R_\lambda(T)(x_n), t) > 1 - \varepsilon$, for all $n \geq n_0$

there fore $\lim_{n\to\infty} N(R_\lambda(T)(x_n), t) = 1$.

Theorem(2.2.10): Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ where $X \neq \{0\}$ $T : X \to X$ be a linear operator , N be the fuzzy norm defined by equation (`1.3.7) and let $\lambda \in \rho(T)$ with respect to $(X, N, *)$ if T is fuzzy bounded on

X and $\lim_{n\to\infty} N(x_n, t) = 1$ for each $t \in (0, \infty)$,Then

$\lim_{n\to\infty} N(\mathrm{R}_\lambda(T)(x_n), t) = 1$.

Proof: Same is proof Theorem(2.2.9)

Example (2.2.11): Let $(X, N,*)$ be a fuzzy normed space over field $\mathbb{C}$ where $X \neq \{0\}$ .It is easy to check that $\rho(\mathrm{I}) = \mathbb{C} - \{1\}$ and $(0) = \mathbb{C} - \{0\}$, $\sigma(\mathrm{I})$ $=\{1\}$ $\sigma(0) = \{0\}$,where I is identity operator and 0 is the zero operator defined on X

Definition (2.2.12) :[18] Let $(X, N,*)$ be a fuzzy normed space . We define the open ball $B(x, r, t)$ and closed ball $B[x, r, t]$ with center $x \in X$ and radius $0 < r < 1$,as follows : For $t > 0$

$B(x, r, t) = \{ y \in X: N(x - y, t) > 1 - r\}$

$B[x, r, t] = \{ y \in X: N(x - y, t) \geq 1 - r\}$

Definition (2.2.13) :[15] Let $(X, N,*)$ be a fuzzy normed space . U subset of X Said to be open set ,if for all $x \in U$,$\exists\, r \in (0,1)$, $t \in (0, \infty)$ such that $B(x, r, t) \subset U$.

Theorem(2.2.14) :[13] Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ and $T : X \to X$ is bounded operator then $\sigma(T)$ is compact and lies in the disk given by $|\lambda| \leq \|T\|$ hence the resolvent set $\rho(T)$ of T is not empty.

Theorem(2.2.15) : Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ where $X \neq \{0\}$ and let $T: (X, N,*) \to (X, N,*)$ be fuzzy bounded linear operator on X , N

be the fuzzy norm defined by equation (`1.3.5).Then $\rho(T)$ is nonempty set.

Proof: Suppose T is fuzzy bounded on X then there exist $r > 0$ such that for each $x \in X$ and $t \in (0, \infty)$, $N(T(x), t) \geq N\left(x, \frac{t}{r}\right)$, Hence for each $x \in X$ and $\frac{t}{t+\|T(x)\|} \geq \frac{\frac{t}{r}}{\frac{t}{r}+\|x\|}$ .There fore T is $\frac{t}{t+\|T(x)\|} \geq \frac{t}{t+\|rx\|}$ ⟹there exist $r > 0$ such that $\|T(x)\| \leq r\|x\|$ fore each $x \in X$ there fore T is bounded linear operator in $(X, \|.\|)$ then by Theorem(2.2.14) the resolvent set of T is nonempty ,so there exist $\lambda \in \mathbb{C}$ such that $\mathrm{R}_\lambda(T)$ exists , $\mathrm{R}_\lambda(T)$ is bounded on range T_λ and range T_λ is dense in $(X, \|.\|)$ then by Theorem(2.2.5) $\mathrm{R}_\lambda(T)$ is bounded linear operator in $(X, \|.\|)$ then there exist $r > 0$
Such that for each $x \in X$ $\|\mathrm{R}_\lambda(T)(x)\| \leq r\|x\|$ then for each t>0 and for each $x \in X$ $t + \|\mathrm{R}_\lambda(T)(x)\| \leq t + r\|x\| \Longrightarrow \frac{t}{t+\|\mathrm{R}_\lambda(T)(x)\|} \geq \frac{t}{t+\|rx\|} \Longrightarrow$

$\frac{t}{t+\|\mathrm{R}_\lambda(T)(x)\|} \geq \frac{\frac{t}{r}}{\frac{t}{r}+\|x\|}$ then there exist r >0 such that for each $x \in X$ and $t \in (0, \infty)$, $N(\mathrm{R}_\lambda(T)(x), t) \geq N\left(x, \frac{t}{r}\right)$, there fore $\mathrm{R}_\lambda(T)$ is fuzzy bounded on $(X, N, *)$ since range T_λ is dense in $(X, \|.\|)$ then by proposition(2.2.4) range T_λ is dense in $(X, N, *)$ since $\mathrm{R}_\lambda(T)$ exists then $\rho(T)$ is nonempty set.

Theorem(2.2.16) :[13] Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ and T is bounded linear operator on X then $\rho(T)$ is open set and hence $\sigma(T)$ is closed set

Theorem(2.2.17) : Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ where

$X \neq \{0\}$ and let $T: (X, N,*) \to (X, N,*)$ be fuzzy bounded linear operator on X , N be the fuzzy norm defined by equation (`1.3.7).Then $\rho(T)$ is open set with respect fuzzy norm N and hence $\sigma(T)$ is closed set with respect fuzzy norm N

Proof: Suppose T is fuzzy bounded on X then there exist r_1>0 such that for each $x \in X$ and $t \in (0, \infty)$, $N(T(x), t) \geq N\left(x, \frac{t}{r_1}\right)$. Assume for the contrary there exist $x_1 \neq 0$ such that $\|T(x_1)\| > r_1\|x_1\|$ .Let $\|T(x_1)\| = t_o$ Hence , $N(T(x_1), t_o)$=0 but $N(r_1x_1, t_o)$=1 this is a contradiction .Then for each $x \in X$ there exist r_1>0 such that $\|T(x)\| \leq r_1 \|x\|$ .Then T is bounded linear operator in $(X, \|.\|)$ then by Theorem(2.2.16) $\rho(T)$ is open set then for each $\lambda \in \rho(T)$ there exist $t > 0$ such that $B(\lambda, t) \subset \rho(T)$

$B(\lambda, t) = \{c \in \mathbb{C}: |\lambda - c| < t\}$

To prove that $B(\lambda, t) = B(\lambda, r, t)$,such that $t > 0$ and $0 < r < 1$

Let $\lambda_1 \in B(\lambda, t)$ then $|\lambda - \lambda_1| < t$ then $N(\lambda - \lambda_1, t) = 1 > 1 - r$ there fore $\lambda_1 \in B(\lambda, r, t)$ then $B(\lambda, t) \subseteq B(\lambda, r, t)$, such that $t > 0$ and $0 < r < 1$

Let $\lambda_1 \in B(\lambda, r, t)$ then $N(\lambda - \lambda_1, t) > 1 - r$ then $|\lambda - \lambda_1| < t$ there fore $\lambda_1 \in B(\lambda, t)$ then $B(\lambda, r, t) \subseteq B(\lambda, t)$ thus $B(\lambda, t) = B(\lambda, r, t)$ such that $t > 0$ and $0 < r < 1$ thus $B(\lambda, r, t) \subset \rho(T)$ then $\rho(T)$ is open set with respect fuzzy norm N also since $\sigma(T) = \mathbb{C} - \rho(T)$ then $\sigma(T)$ is closed set with respect fuzzy norm N .

Theorem(2.2.18) : Let $(X, \|.\|)$ be Banach space over the field $\mathbb{C}$ where $X \neq \{0\}$ and let $T: (X, N,*) \to (X, N,*)$ be fuzzy bounded linear operator on X , N

be the fuzzy norm defined by equation (`1.3.5) and $\lambda, \beta \in \rho(T)$ then

(a)The resolvent R_λ of T satisfies resolvent equation

$R_\beta - R_\lambda = (\beta - \lambda) R_\beta R_\lambda$

(b) R_λ commutes with any $\phi \in FB(X)$ which commutes with T

(c) We have $R_\lambda R_\beta = R_\beta R_\lambda$

Proof: (a)Suppose T is fuzzy bounded on X then there exist $r > 0$ such that for each $x \in X$ and $t \in (0, \infty)$, $N(T(x), t) \geq N\left(x, \frac{t}{r}\right)$, Hence for each $x \in X$ and $\frac{t}{t + \|T(x)\|} \geq \frac{\frac{t}{r}}{\frac{t}{r} + \|x\|}$ .There fore T is $\frac{t}{t + \|T(x)\|} \geq \frac{t}{t + \|rx\|}$ ⟹there exist $r > 0$ such that $\|T(x)\| \leq r\|x\|$ fore each $x \in X$ there fore T is bounded linear operator in $(X, \|.\|)$ from theorem (2.2.4)and (2.2.5) the range of T_λ is all of X .Hence $I = T_\lambda R_\lambda$ where I the identity operator on X .Also $I = R_\beta T_\beta$

Consequently $R_\beta - R_\lambda = R_\beta (T_\lambda R_\lambda) - R_\lambda (R_\beta T_\beta) = R_\beta (T_\lambda - T_\beta) R_\lambda$

$$= R_\beta [T - \lambda I - (T - \beta I)] R_\lambda$$

$$= (\beta - \lambda) R_\beta R_\lambda$$

(b) By assumption , $\phi T = T\phi$ .Hence $\phi\, T_\lambda = T_\lambda\, \phi$ .Using $I = T_\lambda R_\lambda = R_\lambda T_\lambda$

We thus obtain

$R_\lambda\, \phi = R_\lambda\, \phi\, T_\lambda R_\lambda = R_\lambda\, T_\lambda\, \phi\, R_\lambda = \phi\, R_\lambda$

(c) R_β commutes with T by (b) .Hence R_λ commutes with R_β by (b) .

Definition (2.2.19) : Let $(X, N,*)$ be a fuzzy normed space over the field $\mathbb{C}$ where $X \neq \{0\}$ and $T:X \longrightarrow X$ be a linear operator the spectrum of T is partitioned into three disjoint sets as follows:-

(1) The point spectrum $\sigma_{\mathrm{p}}(T)$ is the set such that $\mathrm{R}_\lambda(T)$does not exists. $\lambda \in \rho_{\mathrm{p}}(T)$ is called an eigenvalue of T.

(2) The continuous spectrum $\sigma_{\mathrm{c}}(T)$ is the set such that $\mathrm{R}_\lambda(T)$ exists and satisfying the condition (3) but not the condition (2) in definition (2.2.3).

(3) The residual spectrum $\sigma_{\mathrm{r}}(T)$ is the set such that $\mathrm{R}_\lambda(T)$ exists (and may be fuzzy bounded or not) but does not satisfy the condition (3) in definition (2.2.3).

Examples(2.2.20):
(1) Let $(X, N,*)$ be a fuzzy normed space over the field $\mathbb{C}$ where $X \neq \{0\}$. Then $\sigma_{\mathrm{p}}(\mathrm{I})$ ={1}= σ (I) where I is the identity operator defined on X. On the other hand, $\sigma_{\mathrm{p}}(\mathrm{O})$ ={1}= σ(O) , where O is the zero operator defined on X .

(2) Let $X = \ell^2$, that is $\ell^2 = \{x = (x_1, x_2, \dots): \sum_{i=1}^{\infty} |x_i|^2 < \infty, x_i \in \mathbb{C}\}$

For $x \in \ell^2$, defined $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ .Let $N: \ell^2 \times (0,\infty) \to [0,1]$

Define fuzzy norm by equation(1.3.7).Consider $T\,(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ for

all $(x_1, x_2, \dots) \in \ell^2$ .We shall show that $0 \in \sigma_{\mathrm{r}}(T)$ .To do that , it is clear T is bounded linear operator with respect to $\|.\|$. Moreover by using the same last steps in proposition (2.2.7) one can get T is fuzzy bounded on ℓ^2 . On the other hand,

$T: \ell^2 \longrightarrow \ell^2$ is one to one . Then $T^{-1}: R(T) \longrightarrow \ell^2$ exists.Next,we show that

$R(T) = \{x \in \ell^2 : x = (0, x_1, x_2, \dots)\}$ is not dense in ℓ^2. To do this, let $x \in \ell^2$

such that $x = (4,0,0,\dots)$ and let $t = 0.3 > 0$ and $\{x_n\}$ be any sequence in

rang(T), that is,$x_n = (0, x_1^n, x_2^n, \dots)$.Since

$\|(0, x_1^n, x_2^n, \dots) - (4,0,0,\dots)\| = \|(-4, x_1^n, x_2^n, \dots)\|$

$= (16 + |x_1^n|^2 + |x_2^n|^2 + \dots)^{\frac{1}{2}}$, for each n then

$0.3 < \|(-4, x_1^n, x_2^n, \dots)\|$ for any choose for $x_1^n, x_2^n, \dots$ Hence

$N((0, x_1^n, x_2^n, \dots) - (4,0,0, \dots), 0.3) = N((-4, x_1^n, x_2^n, \dots), 0.3) = 0$ for each n.

So $\lim_{n\to\infty} N(x_n - x, 0.3) = 0$ Thus for any sequence $\{x_n\}$ in $R(T)$ there exists t=0.3>0 such that $\lim_{n\to\infty} N(x_n - x, 0.3) = 0$ so x not belong to the closure of $R(T)$. Hence $R(T)$ is not dense in ℓ^2 . Then $0 \in \sigma_{\mathrm{r}}(T)$.

Definition (2.2.21)[21] : Let X be a fuzzy Hilbert space over F, and let $T \in FB(X)$. T is said to be self-adjoint if $T^* = T$.

The fuzzy Hilber -adjoint operator T^* is defined by :

$$\sup\{t \in \mathbb{R}, H(T(x), y, t) < 1\} = \sup\{t \in \mathbb{R}, H(x, T^*(y), t) < 1\}.$$

If T is self-adjoint we have :

$$\sup\{t \in \mathbb{R}, H(T(x), y, t) < 1\} = \sup\{t \in \mathbb{R}, H(x, T(y), t) < 1\}.$$

Theorem (2.2.22):[22] Let X be a Hilbert space over F and $T \in L(X)$be a self-adjoint operator. Then $\sigma_{\mathrm{r}}(T) = \phi$.

Theorem (2.2.23): Let X be a fuzzy Hilbert space over F and $T \in FB(X)$be a self-adjoint operator. Then $\sigma_{\mathrm{r}}(T) = \phi$.

Proof: Suppose $\lambda \in \sigma_{\mathrm{r}}(T)$ then $\mathrm{R}_\lambda(T)$ exists.

Define $\langle .,. \rangle : X \times X \to \mathbb{R}$ by $\langle x, y \rangle = \sup\{ t \in \mathbb{R}, H(x, y, t) < 1 \}$. from theorem (1.4.8) ,we have $(X, \langle .,. \rangle)$ is a pre-Hilbert space. Also $\|x\| = (\sup\{ t \in \mathbb{R}, H(x, x, t) < 1\})^{\frac{1}{2}}$ from corollary (1.4.9) we have $(X, \|.\|)$ is a normed space

since X is a fuzzy Hilbert space then $(X, \|.\|)$ is complete normed space then X is Hilbert space since $T \in FB(X)$ be a self-adjoint operator.Then

$\langle T(x), y\rangle = \sup\{\, t \in \mathbb{R}, H(T(x), y, t) < 1\} = \sup\ \{\, t \in \mathbb{R}, H(x, T(y), t) < 1\}$

$$= \langle x, T(y)\rangle$$

Hence $T \in L(X)$be a self-adjoint operator and X is a Hilbert space.

From theorem(2.2.22)we have $\sigma_{\mathrm{r}}(T) = \phi$ .Then $\mathrm{R}_{\lambda}(T)$ not exists and this is contradiction.Hence $\sigma_{\mathrm{r}}(T) = \phi$ such that X be a fuzzy Hilbert space.

Examples(2.2.24):

(1) Let $X = \ell^2 and\ T : (X, N,*) \to (X, N,*)$ Define

By $T\ (x_1, x_2,...)= (0, x_1, x_2,...)$ for all $(x_1, x_2,...) \in \ell^2$ and

$N: \ell^2 \times (0, \infty) \to [0,1]$ Define fuzzy norm by equation(1.3.7)

Suppose $T(x_1, x_2,...)= \lambda\ (x_1, x_2,...) \Longrightarrow (0, x_1, x_2,...)= \lambda\ (x_1, x_2,...)$

$(0, x_1, x_2, ...)= (\lambda x_1, \lambda x_2, \lambda x_3 ...)$ implies $0 = \lambda x_1$, $x_1= \lambda x_2$, $x_2= \lambda x_3$,….

If $\lambda \neq 0$, we divide by λ and conclude $x_1= x_2=\ldots=0$ .If $\lambda = 0$ we also

$x_1= x_2=\ldots=0$. T is linear operator has no eigenvalue . Consequently

$\sigma_{\mathrm{p}}(T) = \phi$.

(2) Let $X = \ell^2 and\ T : (X, N,*) \to (X, N,*)$ Define

By $T\ (x_1, x_2,...)= (x_2, x_3, x_4,...)$ for all $(x_1, x_2,...) \in \ell^2$ and Define fuzzy norm by equation(1.3.7).

Suppose $T(x_1, x_2,...)= \lambda\ (x_1, x_2, x_3 ...) \Longrightarrow (x_2, x_3, x_4,...)= (\lambda x_1, \lambda x_2, \lambda x_3,...)$ is

equivalent to $\lambda x_1=x_2$, $\lambda x_2=x_3$, $\lambda x_3 = x_4$, ...

consequently , $x = (x_k)_{k=1}^{\infty}$ with $x_k = \lambda^{k-1} x_1$ for all $k \geq 2$ .This sequence

belongs to ℓ^2 if and only if $\sum_{k=1}^{\infty} |x_k|^2 = \sum_{k=1}^{\infty} |x_1||\lambda|^{2k}$ converges or equivalently $|\lambda| < 1$ then $N(\lambda, 1) = 1 \Longrightarrow N(\lambda, 1) = 1 > 1 - r$,

$\forall 0 < r < 1 \Longrightarrow \sigma_{\mathrm{p}}(T) = \{ \lambda \in \mathbb{C}: N(\lambda, 1) > 1 - r \} = B(0, r, 1).$

CHAPTER THREE SPECTRAL PROPERTIES OF FUZZY COMPACT LINEAR OPERATOR ON FUZZY NORMED SPACES

This chapter deals with the focuss will be on discussing properties of fuzzy compact linear operator on fuzzy normed spaces. It consists of three sections. Section one deals with Compact set in fuzzy normed space and some of their properties. Section two deals with fuzzy Compact linear operator on fuzzy normed space and some of their properties. Section three we consider spectral properties of fuzzy compact linear operator $T : X \to X$ on fuzzy normed spaces X. For this purpose we shall again use the operator $T_{\lambda} = T - \lambda \mathrm{I}$ and λ spectral value .

3.1 Compact set in fuzzy normed space

This section deals with Compact set in fuzzy normed space and some of their properties.

Definition (3.1.1) : [4] Let $(X, N,*)$ be a fuzzy normed linear space.

A subset B of X is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B .

Lemma (3.1.2) : [2] Let $(X, N,*)$ be a fuzzy normed space satisfying the condition $(N.8)$ and $\{x_1, x_2, \dots, x_n\}$ be a finite set of linearly independent vectors of X. Then for each $\alpha \in (0,1)$ there exists a constant $C_\alpha > 0$ such that for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|_\alpha \geq C_\alpha \sum_{i=1}^{n} |\alpha_i|$$

Where $\|.\|_\alpha$ is defined in the Theorem (1.3.8).

Definition (3.1.3) : [20] Let $(X, N,*)$ be a fuzzy normed linear space and $B \subset X$. B is said to be fuzzy bounded if for each r, $0 < r < 1$, $\exists\, t > 0$ such that $N(x,t) > 1 - r$, $\forall\, x \in B$.

Theorem (3.1.4) : Let $(X, N,*)$ fuzzy normed linear space $(X, N,*)$ satisfying the conditions $(N.7)$ a subset B of X is compact then B is closed and fuzzy bounded in $(X, N,*)$.

Proof : $\Longrightarrow$ Suppose that B is compact we have to show that B is closed

and bounded .Let $x \in \overline{B}$ .Then there exist sequence $\{x_n\}$ in B such that

$\lim_{n\to\infty} N(x_n - x, t) = 1$ since B is compact ,there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$converges to a point in B.Again $\{x_n\} \longrightarrow x$ so $\{x_{n_k}\} \longrightarrow x$ and hence $x \in B$ then $\overline{B} = B$ there fore B is closed . If possible suppose that B is not bounded then $\exists\, r$,0< r <1 such that for each positive integer n , $\exists\, x_n \in B$ such that $N(x_n, n) \leq 1 - r$ .since B is compact there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to element $x \in B$ thus $\lim_{k\to\infty} N\left(x_{n_k} - x, t\right) = 1$ $\forall\, t$>0 .Also $N\left(x_{n_k}, n_k\right) \leq 1 - r$

Now $1 - r \geq N\left(x_{n_k}, n_k\right) = N\left(x_{n_k} - x + x, n_k - k + k\right)$ where k>0 $\Longrightarrow$

$1 - r \geq N\left(x_{n_k} - x, k\right) * N(x, n_k - k)$

$\Longrightarrow 1 - r \geq \lim_{k\to\infty} N\left(x_{n_k} - x, k\right) * \lim_{k\to\infty} N(x, n_k - k) \Longrightarrow$

$1 - r \geq 1 * 1 = 1$ by $(N.7)$ and $(N.5) \Longrightarrow$ $r \leq 0$ which is contradiction
Hence B is bounded

Theorem (3.1.5) : In a finite dimensional fuzzy normed linear space $(X, N, *)$ satisfying the conditions $(N.7)$ *and* $(N.8)$ a subset B of X is compact if and only if B is closed and fuzzy bounded in $(X, N, *)$.

Proof : $\Longrightarrow$ First we suppose that B is compact we have to show that B is closed and bounded .Let $x \in \overline{B}$ .Then there exist sequence $\{x_n\}$ in B such that

$\lim_{n\to\infty} N(x_n - x, t) = 1$ since B is compact ,there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$converges to a point in B.Again $\{x_n\} \longrightarrow x$ so $\{x_{n_k}\} \longrightarrow x$ and hence $x \in B$ then $\overline{B} = B$ there fore B is closed . If possible suppose that B is not bounded then $\exists\, r$,0< r <1 such that for each positive integer n , $\exists\, x_n \in B$

such that $N(x_n, n) \leq 1 - r$ .since B is compact there exist a subsequence

$\{x_{n_k}\}$ of $\{x_n\}$ converging to element $x \in B$ thus $\lim_{k\to\infty} N(x_{n_k} - x, t) = 1$

$\forall\ t>0$ .Also $N(x_{n_k}, n_k) \leq 1 - r$

Now $1 - r \geq N(x_{n_k}, n_k) = N(x_{n_k} - x + x, n_k - k + k)$ where $k>0 \implies$

$1 - r \geq N(x_{n_k} - x, k) * N(x, n_k - k)$

$\implies 1 - r \geq \lim_{k\to\infty} N(x_{n_k} - x, k) * \lim_{k\to\infty} N(x, n_k - k) \implies$

$1 - r \geq 1 * 1 = 1$ by $(N.7)$ and $(N.5) \implies r \leq 0$ which is contradiction
Hence B is bounded

$\Leftarrow$ part (2) : In this part, we suppose that B is closed and fuzzy bounded in the finite dimensional fuzzy normed linear space $(X, N, *)$. To show B is compact, consider $\{x_n\}$ an arbitrary sequence in B. Since X finite dimensional,

let dim $X = n$ and $\{e_1, e_2, \dots, e_n\}$ be a basis of X. So for each x_k,
$\exists\ \beta_1^k, \beta_2^k, \dots, \beta_n^k \in B$ such that

$$x_k = \beta_1^k e_1 + \beta_2^k e_2 + \cdots + \beta_n^k e_n \ , \ k = 1,2, \dots$$

Since B is fuzzy bounded, $\{x_k\}$ is also fuzzy bounded. So $\exists\ t_0 > 0$ and r_0 where $0 < r_0 < 1$ such that

$$N(x_k, t_0) > 1 - r_0 = \alpha_0 \ \forall k = 1,2, \dots \qquad (2.1)$$

Let $\|x\|_\alpha = \wedge \{t : N(x,t) \geq \alpha\}, \alpha \in (0,1)$. So by (2.1) we have

$$\|x\|_{\alpha_0} \leq t_0 \qquad \dots \quad (2.2)$$

Since $\{e_1, e_2, \dots, e_n\}$ is linearly independent, by Lemma (2.3.1), $\exists\ c > 0$ such that $\forall k = 1,2, \dots$,

$$\|x_k\|_{\alpha_0} = \left\|\sum_{i=1}^n \beta_i^k e_i\right\|_{\alpha_0} > c \sum_{i=1}^n |\beta_i^k| \qquad \dots \quad (2.3)$$

From (2.2) and (2.3) we have $\sum_{i=1}^{n}\left|\beta_i^k\right| \le \frac{t_0}{c}$ for $k = 1,2,\ldots$

$\Longrightarrow$ For each i, $\left|\beta_i^k\right| \le \sum_{i=1}^{n}\left|\beta_i^k\right| \le \frac{t_0}{c}$ for $k = 1,2,\ldots$

$\Longrightarrow \{\beta_i^k\}$ is fuzzy bounded sequence, for each $i = 1,2,\ldots,n$

$\Longrightarrow \{\beta_i^k\}$ has a fuzzy convergent subsequence say $\{\beta_i^{k_l}\}$.

$\Longrightarrow \{\beta_1^{k_l}\}, \{\beta_2^{k_l}\}, \ldots, \{\beta_n^{k_l}\}$ all are fuzzy convergent.

Let $x_{k_l} = \beta_1^{k_l}e_1 + \beta_2^{k_l}e_2 + \cdots + \beta_n^{k_l}e_n$ and

$\beta_1 = \lim_{n\to\infty}\beta_1^{k_l}$, $\beta_2 = \lim_{n\to\infty}\beta_2^{k_l}$, … , $\beta_n = \lim_{n\to\infty}\beta_n^{k_l}$ and

$x = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$.

Now $\forall\, t > 0$, we have

$$\begin{aligned} N\left(x_{k_l} - x, t\right) &= N\left(\sum_{i=1}^{n}\beta_i^{k_l}e_i - \sum_{i=1}^{n}\beta_i e_i, t\right) \\ &= N\left(\sum_{i=1}^{n}\left(\beta_i^{k_l} - \beta_i\right)e_i, t\right) \\ &\ge N\left(\left(\beta_1^{k_l} - \beta_1\right)e_1, \frac{t}{n}\right) * \cdots * N\left(\left(\beta_n^{k_l} - \beta_n\right)e_n, \frac{t}{n}\right) \\ &= N\left(e_1, \frac{t}{n\left|\beta_1^{k_l} - \beta_1\right|}\right) * \cdots * N\left(e_n, \frac{t}{n\left|\beta_n^{k_l} - \beta_n\right|}\right). \end{aligned}$$

Since $\lim_{l\to\infty}\frac{t}{n\left|\beta_i^{k_l} - \beta_i\right|} = \infty$, we see that $\lim_{l\to\infty}N\left(e_i, \frac{t}{n\left|\beta_i^{k_l} - \beta_i\right|}\right) = 1$

$\Longrightarrow \lim_{l\to\infty}N\left(x_{k_l} - x, t\right) \ge 1 * \cdots * 1 = 1 \quad \forall t > 0$

$\Longrightarrow \lim_{l\to\infty}N\left(x_{k_l} - x, t\right) = 1, \quad \forall t > 0$ … (2.4)

Thus from (2.4) we see that

$\lim_{l\to\infty} x_{k_l} = x \implies x \in B$ [since B is closed].

$\implies B$ is compact.

Theorem (3.1.6) :[21] (Riesz Lemma) Let V be closed proper subspace of a fuzzy normed linear space $(X, N,*)$ and let λ be a real number such that $0 < \lambda < 1$. Then there exists a vector $x_\lambda \in X$ such that $N(x_\lambda, 1) > 0$ and $N(x_\lambda - x, \lambda) = 0$ for all $x \in V$.

Proof : Since V is proper subspace of X, $\exists\, \upsilon \in X - V$.

Denote $d = \wedge_{x\in V} \wedge \{ t > 0 : N(\upsilon - x, t) > 0 \}$.

We claim that $d > 0$, i.e. $\wedge_{x\in V} \wedge \{ t > 0 : N(\upsilon - x, t) > 0 \} = 0$

$\implies$ for a given $\varepsilon > 0$, $\exists\, x(\varepsilon) \in Y$ such that

$\wedge \{ t > 0 : N(\upsilon - x, t) > 0 \} < \varepsilon \implies N(\upsilon - x, \varepsilon) > 0$.

Choose $\alpha \in (0,1)$ such that $N(\upsilon - x, \varepsilon) > 1 - \alpha$. i.e. $y \in B(\upsilon, 1 - \alpha, \varepsilon)$.

Since $\varepsilon > 0$ is arbitrary, it follows that υ is in the closure of V.

Since V is closed, it implies that $\upsilon \in V$ which is a contradiction.

Thus $d > 0$. We now take $\lambda \in (0,1)$. So $\frac{d}{\lambda} > d$. Thus for some $x_0 \in V$, we have $d \le \wedge \{ t > 0 : N(\upsilon - x_0, t) > 0\} < K' < \frac{d}{\lambda}$ $\quad \dots \quad$ (2.5)

Let $x_\lambda = \frac{\upsilon - x_0}{k'}$. Now $(x_\lambda, 1) = N(\frac{\upsilon - x_0}{k'}, 1)$.

$$\text{i.e.} \quad N(x_\lambda, 1) = N(\upsilon - x_0, k') \quad \dots \quad (2.6)$$

Now $\wedge \{ t > 0 : N(\upsilon - x_0, t) > 0 \} < k' \implies N(\upsilon - x_0, k') > 0$.

From (2.6) we have $N(x_\lambda, 1) > 0$.

Now for $x \in \upsilon$,

$$\wedge \{ t > 0 : N(x_\lambda - x, t) > 0\} = \wedge \{t > 0 : N(\upsilon - x_0 - k'x, k't) > 0 \}$$

$$= \frac{1}{k'} \wedge \{s > 0 : N(\upsilon - x_0 - k'x, s) > 0 \}.$$

i.e. $\wedge\{t > 0 : N(x_\lambda - x, t) > 0\} \geq \frac{d}{k'}$ ($since\ x_0 + k'x \in V$)

$\Longrightarrow \wedge\{t > 0 : N(x_\lambda - x, t) > 0\} > \lambda$ by (2.5)

i.e. $N(x_\lambda - x, \lambda) \leq 0 \Longrightarrow N(x_\lambda - x, \lambda) = 0, \ \forall x \in V.$

Theorem (3.1.7) :[21] Let $(X, N, *)$ be a fuzzy normed linear space and $x \neq 0$. If suppose that $A = \{x \in X : N(x, 1) > 0\}$ is compact, then X is finite dimensional.

Proof : If possible suppose that dim $X = \infty$. Take $x_1 \in X$ such that $N(x_1, 1) > 0$. Suppose V_1 is the subspace of X generated by x_1.

Since dim $V_1 = 1$, it is a closed and proper subset of X.

Thus by the Lemma (3.1.6),

$\exists\ x_2 \in X\ such\ that\ N(x_2, 1) > 0\ \ and\ N(x_2 - x_1, \frac{1}{2}) = 0$. The elements x_1, x_2 generate a two dimensional proper closed subspace of X.

By the Lemma (3.1.6), $\exists\ x_3 \in X$ with $N(x_3, 1) > 0$ such that

$$N\left(x_3 - x_1, \frac{1}{2}\right) = 0\ ,\ \ N\left(x_3 - x_2, \frac{1}{2}\right) = 0.$$

Proceeding in the same way, we obtain a sequence $\{x_n\}$ of elements $x_n \in A$ such that $N(x_n, 1) > 0\ \ and\ N\left(x_n - x_m, \frac{1}{2}\right) = 0\ \ (m \neq n)$.

It follows that neither the sequence $\{x_n\}$ nor its any subsequence converges. This contradicts the compactness of A. Hence dim X is finite dimensional.

3.2 Fuzzy compact linear operator on fuzzy normed space

This section deals with fuzzy Compact linear operator on fuzzy normed space and some of their properties.

Definition(3.2.1): Let X and Y be a fuzzy normed spaces with norm N .An operator $T : X \to Y$ is called fuzzy compact linear operator if linear and if for every fuzzy bounded sub set B of X that $\overline{T(B)}$ is compact in Y.

Definition (3.2.2) :[21] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces over the same field F. The operator $T: (X, N_1,*) \to (Y, N_2,*)$ is said to be fuzzy continuous at $x_0 \in X$ if for every $\varepsilon \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$:

$$N_1(x - x_0 , s) > 1 - \delta \implies N_2(T(x) - T(x_0), t) > 1 - \varepsilon.$$

Theorem (3.2.3) : [19] Let $T : (X, N_1,*) \to (Y, N_2,*)$ be a linear operator. Then T is fuzzy bounded if and only if T is fuzzy continuous .

Theorem (3.2.4):[17] Let X, Y be fuzzy normed spaces and let $f : X \to Y$ be a linear function. If f is a fuzzy continuous at 0 then it is fuzzy continuous at every point.

Lemma (3.2.5): Let X, Y be fuzzy normed spaces and space $(Y, N,*)$ satisfying the conditions $(N.7)$ .Then every fuzzy compact linear operator $T : X \to Y$ is fuzzy continuous and hence fuzzy bounded.

Proof : Let B is fuzzy bounded sub set of X and $x \in B$ then $x \in X$

Let T is not fuzzy continuous at 0 ,then $\exists\ \varepsilon \in (0,1)$ and $t > 0, \forall\ \delta \in (0,1)$ and $s > 0$ such that

$$N_1(x-0\,,s)>1-\delta \;\Rightarrow\; N_2(T(x)-T(0),t)\;\leq 1-\varepsilon$$

$\Rightarrow N_2(T(x),t)\;\leq 1-\varepsilon$ since $x\in B \Rightarrow T(x)\in T(B\;)$

Since $T : X \to Y$ fuzzy compact linear operator we have $\overline{T(B)}$ is compact in $(Y,N,*)$ from theorem (3.1.4) we have $\overline{T(B)}$ is bounded in Y .since $T(B\;)\subseteq \overline{T(B)}$ then $T(x)\in \overline{T(B)}$ since $N_2(T(x),t)\;\leq 1-\varepsilon$ and $\varepsilon\in(0,1)$

There fore $\overline{T(B)}$ is not bounded which is contradiction then T is fuzzy continuous at 0 from theorem(3.2.4) we have T is fuzzy continuous at every point there fore T is fuzzy continuous , also from theorem(3.2.3) we have T is fuzzy bounded.

Theorem (3.2.6): Let X,Y be fuzzy normed spaces and $T : X \to Y$ is linear operator .Then T is fuzzy compact linear operator if and only if it maps every fuzzy bounded sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a fuzzy convergent subsequence.

Proof : If T is fuzzy compact linear operator and $\{x_n\}$ is fuzzy bounded ,then

the closure of $\{T(x_n)\}$ in Y is compact and from definition (3.1.1) shows that $\{T(x_n)\}$ contains a fuzzy convergent subsequence.

Conversely,assume that every fuzzy bounded sequence $\{x_n\}$ contains a

subsequence $\{x_{n_k}\}$ such that $\{T(x_{n_k})\}$ fuzzy converges in Y.Consider any

fuzzy bounded subset $B\subset X$,and let $\{y_n\}$ be any sequence in $T(B\;)$ .Then

$y_n=T(x_n)$ for some $x_n\in B$, and $\{x_n\}$ is fuzzy bounded since B is fuzzy

bounded . By assumption , $\{T(x_n)\}$ contains a fuzzy convergent subsequence.

Hence $\overline{T(B)}$ by definition (3.1.1) because $\{y_n\}$ in $T(B\;)$ was arbitrary. By

definition , this shows that T is fuzzy compact linear operator.

Theorem (3.2.7)[17]: Let $\{x_n\}$, $\{y_n\}$ be a sequences in fuzzy normed space X and for all $\alpha_1\in(0,1)$ there exist $\alpha\in(0,1)$ such that $\alpha*\alpha\geq\alpha_1$

(1)Every sequence in X has a unique fuzzy convergence.

(2)If $x_n \longrightarrow x$ then $cx_n \longrightarrow cx, c \in F - \{0\}$, (F is field)

(3)If $x_n \longrightarrow x$, $y_n \longrightarrow y$ then $x_n + y_n \longrightarrow x + y$

Theorem (3.2.8): Let X and Y be a fuzzy normed spaces and for all $\alpha_1 \in (0,1)$ there exist $\alpha \in (0,1)$ such that $\alpha * \alpha \geq \alpha_1$ and $T_j : X \rightarrow Y$ is fuzzy compact linear operator where $j = 1,2$ .Then $T_1 + T_2$ is fuzzy compact linear operator and also cT_j is fuzzy compact linear operator ,where c any scalar $c \in F - \{0\}$, (F is field and $j = 1,2$).

Proof : Let $\{x_n\}$ fuzzy bounded sequence in fuzzy normed space X .Since $T_j : X \rightarrow Y$ is fuzzy compact linear operator where $j = 1,2$ .Then from theorem (3.2.6) we have $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{T_1(x_{n_k})\}$ and $\{T_2(x_{n_k})\}$ are fuzzy converges in Y ,then from theorem (3.2.7) we have $\{T_1(x_{n_k}) + T_2(x_{n_k})\}$ is fuzzy converges in $Y \Longrightarrow \{(T_1 + T_2)(x_{n_k})\}$ is fuzzy converges in Y ,there fore from theorem (3.2.6) we have $T_1 + T_2$ is fuzzy compact linear operator .

Also since $\{T_j(x_{n_k})\}$ is fuzzy converges in Y where $j = 1,2$.Then by theorem (3.2.7) $\{cT_j(x_{n_k})\}$ is fuzzy converges in Y where where c any scalar $c \in F - \{0\}$, (F is field).Then from theorem (3.2.6) we have cT_j is fuzzy compact linear operator , where c any scalar $c \in F - \{0\}$, (F is field and $j = 1,2$).

Theorem (3.2.9): Let X and Y be a fuzzy normed spaces and space $(Y, N, *)$ satisfying the conditions $(N.7) and$ $(N.8)$ and $T : X \rightarrow Y$ is linear operator .Then if T fuzzy bounded and Y is finite dimensional ,the operator T is fuzzy compact.

Proof : Let $\{x_n\}$ be any fuzzy bounded sequence in fuzzy normed space X .

Then $\forall\ 0 < r < 1, \exists\ t>0$ such that $N(x_n, t) > 1 - r\ , \forall\ n$

Also since T fuzzy bounded then there exist $r_1 > 0$ such that for each

$t_1 > 0, N(T(x), t_1) \geq N\left(x, \frac{t_1}{r_1}\right),\ \ \forall x \in X$. Since $N(x_n, t) > 1 - r\ , \forall\ n$

$1 - r < N(x_n, t) = N\left(\frac{r_1}{r_1} x_n, t\right) = N\left(\frac{x_n}{r_1}, \frac{t}{r_1}\right), \forall\ n$

Put $y_n = \frac{x_n}{r_1} \Longrightarrow y_n \in X\ , \forall\ n \Longrightarrow N\left(y_n, \frac{t}{r_1}\right) > 1 - r\ , \forall\ n$

Since $N(T(x), t_1) \geq N\left(x, \frac{t_1}{r}\right),\ \ \forall x \in X, t_1 > 0 \Longrightarrow N(T(y_n), t) \geq N\left(y_n, \frac{t}{r_1}\right)$.

Since $N\left(y_n, \frac{t}{r_1}\right) > 1 - r\ , \forall\ n \Longrightarrow N(T(y_n), t) > 1 - r\ , \forall\ n$

$1 - r < N(T(y_n), t) = N\left(T\left(\frac{x_n}{r_1}\right), t\right) = N\left(\frac{1}{r_1} T(x_n), t\right) = N(T(x_n), r_1 t)$
Put $t_2 = r_1 t \Longrightarrow t_2 > 0$

Then $\forall\ 0 < r < 1, \exists\ t_2>0$ such that $N(T(x_n), t_2) > 1 - r\ , \forall\ n$,there fore

$\{T(x_n)\}$ is fuzzy bounded in Y since Y is finite dimensional then from theorem

(3.1.5) we have $\{T(x_n)\}$ is compact .It follows that $\{T(x_n)\}$ has a fuzzy

convergent subsequence. Since $\{x_n\}$ was an arbitrary fuzzy bounded sequence in X ,the operator T is fuzzy compact by theorem (3.2.5).

3.3 Spectral Properties Of Fuzzy Compact Linear Operator On Fuzzy Normed Spaces

In this section we consider spectral properties of fuzzy compact linear operator $T : X \to X$ on fuzzy normed spaces X. For this purpose we shall again use the operator $T_\lambda = T - \lambda I$ and λ spectral value .

Theorem (3.3.1): Let $T : X \to X$ be a fuzzy compact linear operator on a fuzzy normed spaces X . Then for every $\lambda \neq 0$ and λ eigenvalue then null space (eigenspace) $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional .

Proof: We show that $A = \{ x \in X : N(x, 1) > 0\}$ is compact in $\mathcal{N}(T_\lambda)$ and then apply theorem(3.1.7).

Let $\{x_n\}$ is fuzzy bounded such that $\forall\ 0 < r < 1, N(x_n, 1) > 1 - r$, $\forall\ n$

Since $N(x_n, 1) > 0$, $\forall\ n$ then $\{x_n\} \subset A$,since $\{x_n\}$ is fuzzy bounded and

$T : X \to X$ is fuzzy compact operator from theorem (3.2.6),then $\{T(x_n)\}$

has fuzzy convergent subsequence $\{T(x_{n_k})\}$. Now $x_n \in A \subset \mathcal{N}(T_\lambda)$ implies

$T_\lambda(x_n) = T(x_n) - \lambda \mathrm{I}(x_n) = 0$,so that $x_n = \frac{1}{\lambda} T(x_n)$ because $\lambda \neq 0$.

Consequently, $\{x_{n_k}\} = \{\frac{1}{\lambda} T(x_{n_k})\}$ from theorem (3.2.6) we have $\{x_{n_k}\}$ is fuzzy

converges .Let y point converges (e.i. $\{x_{n_k}\} \longrightarrow y$) .Since $y \in X$ and X fuzzy

normed space we have $N(y, 1) > 0$,so that $y \in A$ .Hence A is compact by

definition(3.1.1) because $\{x_n\}$ was arbitrary and $\{x_n\} \subset A$ .This proves $\mathcal{N}(T_\lambda)$

is finite dimensional by theorem(3.1.7).

Lemma (3.3.2): Let $T: X \to X$ be a fuzzy compact linear operator and $S: X \to X$ be a fuzzy bounded linear operator on a fuzzy normed spaces X .
Then TS and ST are fuzzy compact linear operator.

Proof: Let $B \subset X$ be any fuzzy bounded set .Since S is fuzzy bounded linear operator there fore $\exists\ r$>0 $\ni$ $\forall t$>0 such that

$$N(S(x), t) \geq N\left(x, \frac{t}{r}\right), \quad \forall x \in X .$$

Since B is fuzzy bounded set then $\forall\ 0 < r_1 < 1, \exists t_1$>0 such that

$N(x_1, t_1) > 1 - r_1$, $\forall\ x_1 \in B$.

$1 - r_1 < N(x_1, t_1) = N\left(\frac{r}{r} x_1, t_1\right) = N\left(\frac{x_1}{r}, \frac{t_1}{r}\right)$, $\forall\ x_1 \in B$.

Put $y = \frac{x_1}{r} \Longrightarrow y \in X$ .Since S is fuzzy bounded linear operator there fore $\exists\, r{>}0 \ni \forall t{>}0$ such that

$$N(S(x),t) \geq N\left(x,\frac{t}{r}\right), \quad \forall x \in X .$$

Then $N(S(y),t_1) \geq N\left(y,\frac{t_1}{r}\right) = N\left(\frac{x_1}{r},\frac{t_1}{r}\right) > 1 - r_1 \Rightarrow N(S(y),t_1) > 1 - r_1$

$1 - r_1 < N(S(y),t_1) = N\left(S\left(\frac{x_1}{r}\right),t_1\right) = N\left(\frac{1}{r}S(x_1),t_1\right) = N(S(x_1),rt_1)$

Put $t_2 = rt_1 \Longrightarrow t_2{>}0$ .Let $z = S(x_1)$ .Hence $\forall\ 0 < r_1 < 1$, $\exists t_2{>}0$ such that $N(z,t_2) > 1 - r_1$,$\forall z \in S(B)$ .Then $S(B)$ is fuzzy bounded set .Since T is fuzzy compact operator then $\overline{T(S(B))}$ is compact in X.Since

$T(S(B)) = TS(B)$ then $\overline{TS(B)}$ is compact in X there fore TS is fuzzy compact linear operator by definition(3.2.1).

We prove that ST is fuzzy compact linear operator .Let $\{x_n\}$ be any fuzzy bounded sequence in X. Since T is fuzzy compact linear operator then by theorem (3.2.8) $\{T(x_n)\}$ has convergent subsequence $\{T(x_{n_k})\}$,since S is fuzzy bounded then $r{>}0 \ni \forall t{>}0$, such that

$$N(S(x),t) \geq N\left(x,\frac{t}{r}\right), \quad \forall x \in X .$$

Since $\{T(x_{n_k})\}$ is fuzzy converges $y \in X \Longrightarrow \{T(x_{n_k})\} \longrightarrow y \Longrightarrow$

$\forall \epsilon \in (0,1), \forall t_0{>}0$, $\exists n_0 \in \mathbb{Z}^+$ such that $N\left(T(x_{n_k}) - y, t_0\right) > 1 - \epsilon$,$\forall n \geq n_0$

Since $N(S(x),t) \geq N\left(x,\frac{t}{r}\right), \ \forall x \in X$, $T(x_{n_k}) - y \in X$ .Hence

$$N\left(S\left(T(x_{n_k}) - y\right),t_0\right) \geq N\left(T(x_{n_k}) - y,\frac{t_0}{r}\right)$$

Put $t_2 = \frac{t_0}{r} \Longrightarrow t_2{>}0$. Then $N\left(T\left(x_{n_k}\right) - y, t_2\right) > 1 - \epsilon$,$\forall n \geq n_0 \Longrightarrow$

$N\big(S\big(T(x_{n_k}) - y\big), t_0\big) > 1 - \epsilon$,$\forall n \geq n_0$ $N\big(ST\big(x_{n_k}\big)) - S(y), t_0\big) > 1 - \epsilon$

$\forall n \geq n_0$ .Hence $\{ST(x_{n_k})\} \longrightarrow S(y)$ .Hence $\{ST(x_n)\}$ has fuzzy convergent

sequence .There fore ST is fuzzy compact operator by theorem(3.2.8).

Theorem(3.3.3)(Null spaces): In theorem (3.3.1)

$$\dim(\mathcal{N}(T_\lambda{}^n)) < \infty \,, n = 1.2\ldots.$$

Proof: $T_\lambda{}^n = (T - \lambda I)^{\boldsymbol{n}} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} T^{\boldsymbol{k}} \, (-\boldsymbol{\lambda})^{\boldsymbol{n-k}} = (-\lambda)^n I + T \sum_{\boldsymbol{k=0}}^{\boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{k}} T^{\boldsymbol{k-1}}$
This can be written

$T_\lambda{}^n = w - \beta \mathrm{I}$, $\beta = -(-\lambda)^n$

Where $w = TS = ST$ and S denotes the sum on the right. T is fuzzy compact

and S is fuzzy bounded since T is bounded by theorem (3.2.5). Hence w fuzzy compact by lemma (3.3.2), so that we obtain

$$\dim(\mathcal{N}(T_\lambda{}^n)) < \infty \,, n = 1.2\ldots.$$

By Appyling theorem (3.3.1).

Theorem(3.3.4):[13]Let $T: X \rightarrow X$ be a compact linear operator on a normed space X .Then for every $\lambda \neq 0$ the range of $T_\lambda = T - \lambda$ I is closed.

Theorem(3.3.5): Let $T: X \rightarrow X$ be a fuzzy compact linear operator on a fuzzy normed space X where norm defined by equation(1.3.7) .Then for every $\lambda \neq 0$ the range of $T_\lambda = T - \lambda$ I is closed in X .

Proof: Let $B = \{x \in X: \|x\| < 2\}$ then B is bounded in X with respect $\|.\| \Longrightarrow$ $N(x, 2)$=1.

Let $\alpha \in (0,1) \Longrightarrow 0 < \alpha < 1 \Longrightarrow 1 - \alpha < 1 \Longrightarrow N(x, 2) > 1 - \alpha$ then

$N(x, 2) > 1 - \alpha$,$\forall x \in B$ there fore B is fuzzy bounded in X.

Since T is fuzzy compact linear operator from definition(3.2.1) we have $\overline{T(B)}$ is compact in fuzzy normed space X .

To prove $\overline{T(B)}$ is compact in X with to respect $\|.\|$.

Let $\{y_n\}$ is sequence in $\overline{T(B)}$. Since $\overline{T(B)}$ is compact in $(X, N,*)$ then $\overline{T(B)}$ has subsequence $\{y_{n_k}\}$ converging to element of $\overline{T(B)}$ (i.e. $\{y_{n_k}\} \longrightarrow y$, $y \in \overline{T(B)}$)

$\Longrightarrow \lim_{k\to\infty} N\left(y_{n_k} - y, t\right) = 1$ for each t>0 . Fix $\beta \in (0,1)$ thus

$\lim_{k\to\infty} N\left(y_{n_k} - y, t\right) = 1 > \beta$ for each t>0 .That is for each t>0 there exist $n_0 \in \mathbb{Z}^+$ such that $N\left(y_{n_k} - y, t\right) > \beta$ for each $n \geq n_0$ .Hence

$\lim_{k\to\infty}\left\|y_{n_k} - y\right\| = 0$ .There fore $\overline{T(B)}$ is compact in X with to respct $\|.\|$.

Hence T is compact linear operator in X with to respect $\|.\|$. From theorem (3.3.4) we have range of $T_\lambda = T - \lambda$ I is closed in X with to respect $\|.\|$ .Also from theorem(2.2.7) We have range of $T_\lambda = T - \lambda$ I is closed in X with to respct N.

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المستخلص

في هذه الرسالة نتناول بعض خصائص لنظرية الاطياف للمؤثر الخطي المعرف على الفضاء المعياري الضبابي الذي يعتبر توسيع لنظرية الاطياف للمؤثر الخطي المعرف على الفضاء المعياري .حيث تقدم الدراسة بعض التعاريف والمبرهنات الاساسية في الفضاء المعياري الضبابي والفضاء الضبابي والفضاء هلبرت الابتدائي الضبابي الذي يكونان توسيعا للفضائين المعياري وهلبرت الابتدائي وكذلك قدمنا تعاريف التي تتعلق بنظرية الطيف للمؤثر الخطي المعرف على الفضاء المعياري الضبابي ومن هذه التعاريف متجه الذاتي وقيمة الذاتية والطيف والطيف النقطي والطيف المستمر والطيف المتبقي و قيمة المنتظمة ومجموعة المحللة وبعض المبرهنات المتعلقة فيهم اما في الفصل الاخير نتناول بعض التعاريف حول المجموعة المتراصة في الفضاء المعياري الضبابي وبعض المبرهنات المتعلقة فيه وكما قدمنا التعاريف والمبرهنات حول المؤثر الخطي المرصوص ضبابياً على الفضاء المعياري الضبابي الذي يعتبر توسيع للمؤثر الخطي المرصوص المعرف على الفضاء المعياري وندرس ايضا خصاص الطيف للمؤثر الخطي المرصوص ضبابياًالمعرف على المؤثر المعياري الضبابي .

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رسالة تقدم بها

عباس محمد عباس

الى

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كجزءاً من متطلبات نيل درجة الماجستير علوم في الرياضيات

بأشراف

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1437هـ **2016م**