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Spectral Properties of Fuzzy Compact Linear Operator on Fuzzy Normed Spaces

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Abstract: ***In this paper we give some definitions and spectral properties of fuzzy compact linear operator on fuzzy normed spaces also we introduce definitions compact in fuzzy normed spaces and fuzzy compact linear operator on fuzzy normed spaces and we prove theorems related to compact in fuzzy normed fuzzy compact linear operator on fuzzy normed spaces. For this purpose we shall use the operator $T - \lambda I$ on fuzzy normed spaces and λ is value spectral and show relationship to fuzzy compact linear operator on fuzzy normed spaces and operator $T - \lambda I$ on fuzzy normed spaces and λ is value spectral.***

Keywords: fuzzy normed space, set fuzzy bounded, fuzzy compact operator, spectral properties of fuzzy compact linear operators on fuzzy normed spaces

1. Introduction

The theory of fuzzy sets was introduced by L. A. Zadeh [1] in 1965.Aftar the pioneer work of Zadeh, many researchers have extended this concept in various branches, many other mathematicians have studied fuzzynormed space from several points of view [2].T. Bag and Samanta [8] in 2003 have definition compact set in fuzzy normed space.

2. Preliminaries

Definition (2.1): [3] Let $*$ be a binary operation on the set I, i.e., $*: I \times I \to I$ is a function. Then $*$ is said to be t-norm (triangular-norm) on the set I if the following axioms are satisfied:

(1) $a * 1 = a$, for all $a \in I$.
(2) $*$ is commutative (i.e. $a * b = b * a$, for all $a, b \in I$).
(3) $*$ is monotone (i.e. if $b, c \in I$ such that $b \le c$, then $a * b \le a * c$, for all $a \in I$).
(4) $*$ is associative (i.e. $a * (b * c) = (a * b) * c$, for all $a, b, c \in I$).

If, in addition, $*$ is continuous then $*$ is called a continuous t-norm.

The following theorem introduces the characteristics of the t-norm:

Definition (2.2): [2] Let X be a vector space over F,$*$ be a continuous t-norm on I, a function $N: X \times (0, \infty) \to [0,1]$ is called fuzzy norm if it satisfies the following conditions: for all $x, y \in X$ and $t, s > 0$,
$(N.1)$ $N(x,t) > 0$,
$(N.2)$ $N(x,t) = 1$ if and only if $x = 0$,
$(N.3)$ $N(\alpha x, t) = N\left(x, \frac{t}{|\alpha|}\right)$, for all $\alpha \ne 0$,
$(N.4)$ $N(x,t) * N(y,s) \le N(x+y, t+s)$,
$(N.5)$ $N(x,.): (0,\infty) \to [0,1]$ is continuous,
$(N.6)$ $\lim_{t\to\infty} N(x,t) = 1$.
$(X, N, *)$ is called fuzzy normed space

Theorem (2.3): [4] Let $(X, N, *)$ be a fuzzy normed space. Then:

(i) $N(x,.)$ is non-decreasing with respect to t for each $x \in X$.
(ii) $N(-x,t) = N(x,t)$ hence $N(x-y,t) = N(y-x,t)$.

Remark (2.4): [5]

(1) For any $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 > \alpha_2$, there exists $\alpha_3 \in (0,1)$ such that $\alpha_1 * \alpha_3 \ge \alpha_2$.
(2) For any $\alpha_4 \in (0,1)$, there exists $\alpha_5 \in (0,1)$ such that $\alpha_5 * \alpha_5 \ge \alpha_4$.

Example (2.5): [6] Let $(X, \|.\|)$ be a normed space.$a * b = a.b$ for all $a, b \in X$ and for all $x \in X, t > 0$

$$N(x,t) = \begin{cases} \frac{t}{t + \|x\|}, x \ne 0 \\ 1, x = 0 \end{cases} \quad \ldots\ldots (2.5)$$

Then $(X, N, *)$is fuzzy normed space.

Theorem (2.6): [7] Let $(X, N, *)$ be a fuzzy normed space, we further assume that,

$(N.7)$ $\alpha * \alpha = \alpha$ for all $\alpha \in [0,1]$,
$(N.8)$ $N(x,t) > 0$ for all $t > 0$ then $x = 0$.

Define $\|x\|_\alpha = \inf\{t > 0: N(x,t) \ge \alpha\}$. Then $\{\|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. We call these norms as α-norms on X corresponding to fuzzy norm N on X.

Proof: Let $\alpha \in (0,1)$. To prove $\|x\|_\alpha$ is a norm on X, we will prove the followings:

(1) $\|x\|_\alpha \ge 0$ for all $x \in X$,
(2)$\|x\|_\alpha = 0$ if and only if $x = 0$,
(3) $\|\lambda x\|_\alpha = |\lambda|\|x\|_\alpha$,
(4) $\|x + y\|_\alpha \le \|x\|_\alpha + \|y\|_\alpha$.

The prove of (1), (2)and (3) directly follows from the proof of the Theorem 2.1 in [8]. So, we now prove (4):

$$\|x\|_\alpha + \|y\|_\alpha = \inf\{\, t > 0 : N(x,t) \ge \alpha \,\} + \inf\{\, s > 0 : N(y,s) \ge \alpha \,\}$$
$$= \inf\{\, t + s > 0 : N(x,t) \ge \alpha, N(y,s) \ge \alpha \,\}$$
$$= \inf\{\, t + s > 0 : N(x,t) * N(y,s) \ge \alpha * \alpha = \alpha \,\}$$
$$\ge \inf\{\, t + s > 0 : N(x+y, t+s) \ge \alpha \,\}$$

$= \|x+y\|_\alpha$, which proves (4).

Let $0 < \alpha_1 < \alpha_2 < 1$.

$\|x\|_{\alpha_1} = \inf\{t > 0 : N(x,t) \ge \alpha_1\}$and $\|x\|_{\alpha_2} = \inf\{\, t > 0 : N(x,t) \ge \alpha_2 \,\}$.Since $\alpha_1 < \alpha_2$, $\{\, t > 0 : N(x,t) \ge \alpha_2\} \subset \{\, t > 0 : N(x,t) \ge \alpha_1 \,\}$

$$\Longrightarrow \inf\{\, t > 0 : N(x,t) \ge \alpha_2 \,\} \ge \inf\{\, t > 0 : N(x,t) \ge \alpha_1 \,\}$$

$\Longrightarrow \|x\|_{\alpha_2} \ge \|x\|_{\alpha_1}$.

Thus, we see that $\{\, \|x\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X.

3. Compact set in fuzzy normed space

This section deals with Compact set in fuzzy normed space and some of their properties.

Definition (3.1): [8] Let $(X, N, *)$ be a fuzzy normed linear space. A subset B of X is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B.

Lemma (3.2): [9] Let $(X, N, *)$ be a fuzzy normed space satisfying the condition $(N.8)$ and$\{x_1, x_2, \dots, x_n\}$ be a finite set of linearly independent vectors of X. Then for each $\alpha \in (0,1)$ there exists a constant $C_\alpha > 0$ such that for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n$,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|_\alpha \ge C_\alpha \sum_{i=1}^{n} |\alpha_i|$$

Where $\|.\|_\alpha$ is defined in the Theorem (2.6).

Definition (3.3): [10] Let $(X, N, *)$ be a fuzzy normed linear space and $B \subset X$. B is said to be fuzzy bounded if for each $r, 0 < r < 1, \exists\, t > 0$ such that $N(x,t) > 1 - r, \forall\, x \in B$.

Theorem (3.4): Let$(X, N, *)$ fuzzy normed linear space $(X, N, *)$ satisfying the conditions $(N.7)$a subset B of X is compact then B is closed and fuzzy bounded in $(X, N, *)$.

Proof: $\Longrightarrow$ Suppose that B is compact we have to show that B is closed and bounded. Let $x \in \overline{B}$.Then there exist sequence $\{x_n\}$ in B such thatlim$_{n\to\infty} N(x_n - x, t) = 1$ since B is compact, there exist a subsequence $\{x_{n_k}\}$of $\{x_n\}$converges to a point in B.Again $\{x_n\} \longrightarrow x$ so $\{x_{n_k}\} \longrightarrow x$ and hence$x \in B$ then $\overline{B} = B$ there fore B is closed. If possible suppose that B is not bounded then $\exists\, r$, 0< r <1 such that for each positive integer n, $\exists\, x_n \in B$ such that $N(x_n, n) \le 1 - r$.since B is compact there exist a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ converging to $x \in B$ thus lim$_{k\to\infty} N(x_{n_k} - x, t) = 1 \forall\, t$>0.Also $N(x_{n_k}, n_k) \le 1 - r$

Now

$1 - r \ge N(x_{n_k}, n_k) = N(x_{n_k} - x + x, n_k - k + k)$

where k>0 $\Longrightarrow$

$$1 - r \ge N(x_{n_k} - x, k) * N(x, n_k - k)$$

$\Longrightarrow 1 - r \ge \lim_{k\to\infty} N(x_{n_k} - x, k) * \lim_{k\to\infty} N(x, n_k - k)$ $\Longrightarrow$

$1 - r \ge 1 * 1 = 1$ by $(N.7)$ and $(N.5) \Longrightarrow$ $r \le 0$ which is contradiction Hence B is bounded.

Theorem (3.5): In a finite dimensional fuzzy normed linear space $(X, N, *)$ satisfying the conditions $(N.7)$ *and* $(N.8)$ a subset B of X is compact if and only if B is closed and fuzzy bounded in $(X, N, *)$.

Proof: $\Longrightarrow$First we suppose that B is compact we have to show that B is closed and bounded. Let $x \in \overline{B}$.Then there exist sequence $\{x_n\}$ in B such that lim$_{n\to\infty} N(x_n - x, t) =$ 1 since B is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$converges to a point in B.Again $\{x_n\} \longrightarrow x$ so $\{x_{n_k}\} \longrightarrow x$ and hence$x \in B$ then $\overline{B} = B$ there fore B is closed. If possible suppose that B is not bounded then $\exists\, r$, 0< r <1 such that for each positive integer n, $\exists\, x_n \in B$such that $N(x_n, n) \le 1 - r$.since B is compact there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to element $x \in B$ thus lim$_{k\to\infty} N(x_{n_k} - x, t) = 1$ $\forall\, t$>0.

Also $N(x_{n_k}, n_k) \le 1 - r$

Now

$1 - r \ge N(x_{n_k}, n_k) = N(x_{n_k} - x + x, n_k - k + k)$

where

k>0 $\Longrightarrow$

$$1 - r \ge N(x_{n_k} - x, k) * N(x, n_k - k)$$

$\Longrightarrow 1 - r \ge \lim_{k\to\infty} N(x_{n_k} - x, k) * \lim_{k\to\infty} N(x, n_k - k)$ $\Longrightarrow$

$1 - r \ge 1 * 1 = 1$ by $(N.7)$ and $(N.5) \Longrightarrow$ $r \le 0$ which is contradiction

Hence B is bounded

$\Longleftarrow$ part (2): In this part, we suppose that B is closed and fuzzy bounded in the finite dimensional fuzzy normed linear space $(X, N, *)$. To show B is compact, consider $\{x_n\}$ an arbitrary sequence in B. Since X finite dimensional,

let dim $X = n$ and $\{e_1, e_2, \dots, e_n\}$ be a basis of X. So for each $x_k, \exists\, \beta_1^k, \beta_2^k, \dots, \beta_n^k \in B$ such that

$$x_k = \beta_1^k e_1 + \beta_2^k e_2 + \cdots + \beta_n^k e_n, k = 1, 2, \dots$$

Since B is fuzzy bounded, $\{x_k\}$ is also fuzzy bounded. So $\exists t_0 > 0$ and r_0 where $0 < r_0 < 1$ such that
$N(x_k, t_0) > 1 - r_0 = \alpha_0 \; \forall k = 1, 2, \ldots$ (2.1)
Let $\|x\|_\alpha = \wedge\{t: N(x,t) \geq \alpha\}, \alpha \in (0,1)$. So by (2.1) we have
$\|x\|_{\alpha_0} \leq t_0 \ldots$ (2.2)
Since $\{e_1, e_2, \ldots, e_n\}$ is linearly independent, by Lemma (3.2), $\exists\, c > 0$ such that $\forall k = 1, 2, \ldots,$
$\|x_k\|_{\alpha_0} = \left\|\sum_{i=1}^n \beta_i^k e_i\right\|_{\alpha_0} > c\sum_{i=1}^n |\beta_i^k| \ldots$ (2.3)
From (2.2) and (2.3) we have $\sum_{i=1}^n |\beta_i^k| \leq \frac{t_0}{c}$ for $k = 1, 2, \ldots$

$\Longrightarrow$ For each i, $|\beta_i^k| \leq \sum_{i=1}^n |\beta_i^k| \leq \frac{t_0}{c}$ for $k = 1, 2, \ldots$
$\Longrightarrow \{\beta_i^k\}$ is fuzzy bounded sequence, for each $i = 1, 2, \ldots, n \Longrightarrow \{\beta_i^k\}$ has a fuzzy convergent subsequence say $\{\beta_i^{k_l}\}$. $\Longrightarrow \{\beta_1^{k_l}\}, \{\beta_2^{k_l}\}, \ldots, \{\beta_n^{k_l}\}$ all are fuzzy convergent. Let $x_{k_l} = \beta_1^{k_l} e_1 + \beta_2^{k_l} e_2 + \cdots + \beta_n^{k_l} e_n$ and $\beta_1 = \lim_{n\to\infty} \beta_1^{k_l}$, $\beta_2 = \lim_{n\to\infty} \beta_2^{k_l}$, $\ldots$, $\beta_n = \lim_{n\to\infty} \beta_n^{k_l}$ and $x = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n$. Now $\forall\, t > 0$,
we have $N(x_{k_l} - x, t) = N(\sum_{i=1}^n \beta_i^{k_l} e_i - \sum_{i=1}^n \beta_i e_i, t)$

$$= N\left(\sum_{i=1}^n (\beta_i^{k_l} - \beta_i)\, e_i\, , t\right)$$

$$\geq N\left((\beta_1^{k_l} - \beta_1)e_1, \frac{t}{n}\right) * \cdots * N\left((\beta_n^{k_l} - \beta_n)e_n, \frac{t}{n}\right)$$

$$= N\left(e_1, \frac{t}{n|\beta_1^{k_l} - \beta_1|}\right) * \cdots * N\left(e_n, \frac{t}{n|\beta_n^{k_l} - \beta_n|}\right).$$

Since $\lim_{l\to\infty} \frac{t}{n|\beta_i^{k_l} - \beta_i|} = \infty$, we see that $\lim_{l\to\infty}\left(e_i, \frac{t}{n|\beta_i^{k_l} - \beta_i|}\right) = 1$

$$\Longrightarrow \lim_{l\to\infty}(x_{k_l} - x, t) \geq 1 * \cdots * 1 = 1 \;\forall t > 0$$

$\Longrightarrow \lim_{l\to\infty}(x_{k_l} - x, t) = 1, \forall t > 0$ $\qquad \ldots \qquad$ (2.4)

Thus from (2.4) we see that
$\lim_{l\to\infty} x_{k_l} = x \Longrightarrow x \in B$ [since B is closed]. $\Longrightarrow B$ is compact.

Theorem (3.6): [7](RieszLemma) Let Vbe closed proper subspace of a fuzzy normed linear space $(X, N, *)$ and let λ be a real number such that $0 < \lambda < 1$. Then there exists a vector $x_\lambda \in X$ such that $N(x_\lambda, 1) > 0$ and $N(x_\lambda - x, \lambda) = 0$ for all $x \in V$.

Proof: Since V is proper subspace of X, $\exists\, v \in X - V$.
Denote $d = \wedge_{x\in V} \wedge\{\, t > 0: N(v - x, t\,) > 0\,\}$.
We claim that $d > 0$, i.e. $\wedge_{x\in V} \wedge\{\, t > 0: N(v - \; x, t\,) > 0 = 0$
$\Longrightarrow$ for a given $\varepsilon > 0, \exists\, x(\varepsilon) \in Y$ such that $\wedge\{\, t > 0: N(v - x, t > 0 < \varepsilon \Longrightarrow N(v - x, \varepsilon > 0$.Choose $\alpha \in 0, 1$ such that $N(v - x, \varepsilon) > 1 - \alpha$. i.e. $y \in B(v, 1 - \alpha, \varepsilon)$.

Since $\varepsilon > 0$ is arbitrary, it follows that v is in the closure of V.
Since V is closed, it implies that $v \in V$ which is a contradiction.
Thus $d > 0$. We now take $\lambda \in (0,1)$. So$\frac{d}{\lambda} > d$. Thus for some$x_0 \in V$, we
have $d \leq \wedge\{\, t > 0: N(v - x_0\, , t) > 0\} < K' < \frac{d}{\lambda} \ldots$ (2.5)
Let $x_\lambda = \frac{v - x_0}{k'}$. Now $(x_\lambda\, , 1) = N(\frac{v - x_0}{k'}, 1)$.
i.e. $N(x_\lambda, 1) = N(v - x_0\, , k') \ldots$ (2.6)
Now $\wedge\{\, t > 0: N(\, v - x_0, t\,) > 0\,\} < k' \Longrightarrow N(v - x0\, , k' > 0$.
From (2.6) we have $N(x_\lambda\, , 1) > 0$.Now for $x \in v$,
$\wedge\{\, t > 0: N(x_\lambda - x, t) > 0\}$
$= \wedge\{t > 0: N(v - x_0 - k'x, k't) > 0\,\}$
$$= \frac{1}{k'} \wedge \{s > 0: N(v - x_0 - k'x, s\,) > 0\,\}.$$

i.e. $\wedge\{t > 0: N(x_\lambda - x, t\,) > 0\,\} \geq \frac{d}{k'}(\, since\; x_0 + k'x \in V$
$\Longrightarrow \wedge\{\, t > 0: N(x_\lambda - x, t) > 0\,\} > \lambda$ by (2.5)
i.e. $N(x_\lambda - x, \lambda) \leq 0 \Longrightarrow N(x_\lambda - x, \lambda) = 0, \forall\, x \in V$.

Theorem (3.7): [7] Let $(X, N, *)$ be a fuzzy normed linear space and $x \neq 0$. If suppose that $A = \{\, x \in X: N(x, 1) > 0$ is compact, then X is finite dimensional.

Proof: If possible suppose that dim $X = \infty$. Take$x_1 \in X$ such that $N(x_1, 1) > 0$. Suppose V_1 is the subspace of X generated by x_1.
Since dim$V_1 = 1$, it is a closed and proper subset of X. Thus by the Lemma (3.6), $\exists\, x_2 \in X$ such that $N(x_2\, , 1) > 0$ and $N(x_2 - x_1\, , \frac{1}{2}) = 0$. The elements x_1, x_2 generate a two dimensional proper closed subspace of X.By the Lemma (3.6), $\exists\, x_3 \in X$ with $N(x_3, 1) > 0$ such that
$$N\left(x_3 - x_1, \frac{1}{2}\right) = 0, N\left(x_3 - x_2, \frac{1}{2}\right) = 0.$$
Proceeding in the same way, we obtain a sequence $\{x_n\}$ of elements $x_n \in A$ such that $N(x_n, 1\,) > 0$ and $N(x_n - xm, 12 = 0\; m \neq n$.

It follows that neither the sequence $\{x_n\}$ nor its any subsequence.
converges. This contradicts the compactness of A. Hence dim X is finite dimensional.

4. Fuzzy compact linear operator on fuzzy normed space

This section deals with fuzzy Compact linear operator on fuzzy normed space and some of their properties.

Definition (4.1): Let X and Y be a fuzzy normed spaces with norm N.An operator $T: X \to Y$ is called fuzzy

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compact linear operator if linear and if for every fuzzy bounded sub set B of X that $\overline{T(B)}$ is compact in Y.

Definition (4.2): [7] Let $(X, N_1,*)$ and $(Y, N_2,*)$ be a fuzzy normed spaces over the same field F. The operator $T: (X, N_1,*) \to (Y, N_2,*)$ is said to be fuzzy continuous at $x_0 \in X$ if for every $\varepsilon \in (0,1)$ and all $t > 0$ there exist $\delta \in (0,1)$ and $s > 0$ such that for all $x \in X$:
$N_1(x - x_0, s) > 1 - \delta \implies N_2(T(x) - T(x_0), t) > 1 - \varepsilon$.

Theorem (4.3): [11] Let $T: (X, N_1,*) \to (Y, N_2,*)$ be a linear operator. Then T is fuzzy bounded if and only if T is fuzzy continuous.

Theorem (4.4): [4] Let X, Y be fuzzy normed spaces and let $f: X \to Y$ be a linear function. If f is a fuzzy continuous at 0 then it is fuzzy continuous at every point.

Lemma (4.5): Let X, Y be fuzzy normed spaces and space $(Y, N,*)$satisfying the conditions $(N.7)$.Then every fuzzy compact linear operator $T: X \to Y$ is fuzzy continuous and hence fuzzy bounded.

Proof: Let B is fuzzy bounded subset of X and $x \in B$ then $x \in X$. Let T is not fuzzy continuous at 0, then $\exists \varepsilon \in (0,1)$ and $t > 0, \forall \delta \in (0,1)$ and $s > 0$
such that
$N_1(x - 0, s) > 1 - \delta \implies N_2(T(x) - T(0), t) \le 1 - \varepsilon$
$\implies N_2(T(x), t) \le 1 - \varepsilon$ since $x \in B \implies T(x) \in T(B)$
Since $T: X \to Y$ fuzzy compact linear operator we have $\overline{T(B)}$ is compact in $(Y, N,*)$ from theorem (3.4) we have $\overline{T(B)}$ is bounded in Y.

Since
$T(B) \subseteq \overline{T(B)}$ then $T(x) \in \overline{T(B)}$ since $N_2(T(x), t) \le 1 - \varepsilon$and $\varepsilon \in (0,1)$

There fore $\overline{T(B)}$ is not bounded which is contradiction then T is fuzzy continuous at 0 from theorem (4.4) we have T is fuzzy continuous at every point therefore T is fuzzy continuous, also from theorem (4.3) we have T is fuzzy bounded.

Theorem (4.6): Let X, Y be fuzzy normed spaces and $T: X \to Y$ is linear operator. Then T is fuzzy compact linear operator if and only if it maps every fuzzy bounded sequence $\{x_n\}$ in X onto a sequence $\{T(x_n)\}$ in Y which has a fuzzy convergent subsequence.

Proof: If T is fuzzy compact linear operator and $\{x_n\}$ is fuzzy bounded, then the closure of $\{T(x_n)\}$ in Y is compact and from definition (3.1) shows that $\{T(x_n)\}$ contains a fuzzy convergent subsequence. Conversely, assume that every fuzzy bounded sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{T(x_{n_k})\}$ fuzzy converges in Y.Consider any fuzzy bounded subset $B \subset X$, and let $\{y_n\}$ be any sequence in $T(B)$.Then $y_n = T(x_n)$ for some $x_n \in B$, and $\{x_n\}$ is fuzzy bounded since B is fuzzy bounded. By assumption, $\{T(x_n)\}$ contains a fuzzy convergent subsequence. Hence $\overline{T(B)}$ by definition (3.1) because $\{y_n\}$ in $T(B)$ was arbitrary. By definition, this shows that T is fuzzy compact linear operator.

Theorem (4.7) [4]: Let $\{x_n\}$, $\{y_n\}$ be a sequences in fuzzy normed space X and for all $\alpha_1 \in (0,1)$ there exist $\alpha \in (0,1)$ such that $\alpha * \alpha \ge \alpha_1$

(1)Every sequence in X has a unique fuzzy convergence.
(2)If$x_n \longrightarrow x$ then $cx_n \longrightarrow cx, c \in F - \{0\}$, ($F$is field)
(3)If $x_n \longrightarrow x, y_n \longrightarrow y$ then $x_n + y_n \longrightarrow x + y$

Theorem (4.8): Let X and Y be a fuzzy normed spaces and for all $\alpha_1 \in (0,1)$ there exist $\alpha \in (0,1)$ such that $\alpha * \alpha \ge \alpha_1$ and $T_j: X \to Y$is fuzzy compact linear operator where $j = 1, 2$.Then $T_1 + T_2$is fuzzy compact linear operator and also $cT_j$is fuzzy compact linear operator, where c any scalar $c \in F - \{0\}$, ($F$is field and $j = 1, 2$).

Proof: Let $\{x_n\}$fuzzy bounded sequence in fuzzy normed space X.Since

$T_j: X \to Y$is fuzzy compact linear operator where $j = 1, 2$.Then from Theorem (4.6) we have $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ such that $\{T_1(x_{n_k})\}$ and $\{T_2(x_{n_k})\}$ are fuzzy converges in Y, then from theorem (4.7) we have $\{T_1(x_{n_k}) + T_2(x_{n_k})\}$ is fuzzy converges in $Y \implies \{(T_1 + T_2)(x_{n_k})\}$ is fuzzy converges in Y, therefore from theorem (4.6) we have $T_1 + T_2$is fuzzy compact linear operator.

Also since $\{T_j(x_{n_k})\}$ is fuzzy converges in Y where $j = 1, 2$. Then by theorem (4.7) $\{cT_j(x_{n_k})\}$ is fuzzy converges in Y where where c any scalar$c \in F - \{0\}$, ($F$is field).Then from theorem (4.6) we have $cT_j$is fuzzy compact linear operator, where c any scalar $c \in F - \{0\}$, ($F$is field and $j = 1, 2$).

Theorem (4.9): Let X and Y be a fuzzy normed spaces and space
$(Y, N,*)$ satisfying the conditions $(N.7)$ *and* $(N.8)$and $T: X \to Y$ is linear operator. Then if Tfuzzy bounded and Y is finite dimensional, the operator T is fuzzy compact.

Proof: Let $\{x_n\}$ be any fuzzy bounded sequence in fuzzy normed space X.

Then $\forall 0 < r < 1, \exists\, t{>}0$ such that $N(x_n, t) > 1 - r, \forall n$. Also since Tfuzzy bounded then there exist $r_1 > 0$ such that for each

$t_1 > 0, N(T(x), t_1) \ge N\left(x, \frac{t_1}{r_1}\right), \forall x \in X$. Since $(x_n, t) > 1 - r, \forall n$
$1 - r < N(x_n, t) = N\left(\frac{r_1}{r_1} x_n, t\right) = N\left(\frac{x_n}{r_1}, \frac{t}{r_1}\right), \forall n$
Put $y_n = \frac{x_n}{r_1} \implies y_n \in X, \forall n \implies N\left(y_n, \frac{t}{r_1}\right) > 1 - r, \forall n$
Since $N(T(x), t_1) \ge N\left(x, \frac{t_1}{r}\right), \forall x \in X, t_1 > 0 \implies N(T(y_n), t) \ge N\left(y_n, \frac{t}{r_1}\right)$.
Since $N\left(y_n, \frac{t}{r_1}\right) > 1 - r, \quad \forall n \implies N(T(y_n), t) > 1 - r, \forall n$

$$1-r< N(T(y_n),t) = N\left(T\left(\frac{x_n}{r_1}\right),t\right) = N\left(\frac{1}{r_1}T(x_n),t\right) = N(T(x_n),r_1 t)$$

Put $t_2 = r_1 t \Longrightarrow t_2 > 0$

Then $\forall 0 < r < 1, \exists\, t_2>0$ such that $N(T(x_n),t_2) > 1-r$, $\forall\, n$, therefore $\{T(x_n)\}$ is fuzzy bounded in Y since Y is finite dimensional then from theorem (3.5) we have $\{T(x_n)\}$ is compact. It follows that $\{T(x_n)\}$ has a fuzzy convergent subsequence. Since $\{x_n\}$ was an arbitrary fuzzy bounded sequence in X, the operator T is fuzzy compact by theorem (4.6).

5. Spectral Properties of Fuzzy Compact Linear Operator on Fuzzy Normed Spaces

In this section we consider spectral properties of fuzzy compact linear operator $T: X \to X$ on fuzzy normed spaces X. For this purpose we shall again use the operator $T_\lambda = T - \lambda I$ and λ spectral value.

Theorem (5.1): Let $T: X \to X$ be a fuzzy compact linear operator on a fuzzy normed spaces X. Then for every $\lambda \neq 0$ and λ eigenvalue then null space (eigenspace) $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional.

Proof: We show that $A = \{x \in X: N(x,1) > 0\}$ is compact in $\mathcal{N}(T_\lambda)$ and then apply theorem (3.7). Let $\{x_n\}$ is fuzzy bounded such that $\forall 0 < r < 1, N(x_n,1) > 1-r$, $\forall\, n$

Since $N(x_n,1) > 0$, $\forall\, n$ then $\{x_n\} \subset A$, since $\{x_n\}$ is fuzzy bounded and $T: X \to X$ is fuzzy compact operator from theorem (4.6), then $\{T(x_n)\}$ has fuzzy convergent subsequence $\{T(x_{n_k})\}$. Now $x_n \in A \subset \mathcal{N}(T_\lambda)$ implies $T_\lambda(x_n) = T(x_n) - \lambda I(x_n) = 0$, so that $x_n = \frac{1}{\lambda}T(x_n)$ because $\lambda \neq 0$. Consequently, $\{x_{n_k}\} = \{\frac{1}{\lambda}T(x_{n_k})\}$ from theorem (4.7) we have $\{x_{n_k}\}$ is fuzzy converges. Let y point converges (i.e. $\{x_{n_k}\} \longrightarrow y$).Since $y \in X$ and X fuzzy normed space we have $N(y,1) > 0$, so that $y \in A$.Hence A is compact by Definition(3.1) because $\{x_n\}$ was arbitrary and $\{x_n\} \subset A$.This proves $\mathcal{N}(T_\lambda)$ is finite dimensional by theorem(3.7).

Lemma (5.2): Let $T: X \to X$ be a fuzzy compact linear operator and $S: X \to X$ be a fuzzy compact linear operator on a fuzzy normed spaces X.Then TS and ST are fuzzy compact linear operator.

Proof: Let $B \subset X$ be any fuzzy bounded set. Since S is fuzzy bounded linear operator therefore $\exists\, r>0 \ni \forall t>0$ such that

$$N(S(x),t_1) \geq N\left(x,\frac{t}{r}\right), \forall x \in X.$$

Since B is fuzzy bounded set then $\forall 0 < r_1 < 1, \exists t_1>0$ such that

$N(x_1,t_1) > 1-r_1$, $\forall\, x_1 \in B. \Longrightarrow 1-r_1 < N(x_1,t_1) = N\left(\frac{r}{r}x_1,t_1\right) = N\left(\frac{x_1}{r},\frac{t_1}{r}\right)$, $\forall\, x_1 \in B$.

Put $y = \frac{x_1}{r} \Longrightarrow y \in X$.Since S is fuzzy bounded linear operator therefore $\exists\, r>0 \ni \forall t>0$ such that

$$N(S(x),t_1) \geq N\left(x,\frac{t}{r}\right), \forall x \in X.$$

Then $N(S(y),t_1) \geq N\left(y,\frac{t_1}{r}\right) = N\left(\frac{x_1}{r},\frac{t_1}{r}\right) > 1-r_1 \Longrightarrow N(S(y),t_1) > 1-r_1$

$$1-r_1 < N(S(y),t_1) = N\left(S\left(\frac{x_1}{r}\right),t_1\right) = N\left(\frac{1}{r}S(x_1),t_1\right) = N(S(x_1),rt_1)$$

Put $t_2 = rt_1 \Longrightarrow t_2>0$.Let $z = S(x_1)$.Hence $\forall\, 0 < r_1 < 1, \exists t_2>0$ such that

$N(z,t_2) > 1-r_1$, $\forall z \in S(B)$.Then $S(B)$ is fuzzy bounded set. Since T is fuzzy compact operator then $\overline{T(S(B))}$ is compact in X.Since $T(S(B)) = TS(B)$ then $\overline{TS(B)}$ is compact in X therefore TS is fuzzy compact linear operator by definition (4.1).We prove that ST is fuzzy compact linear operator. Let $\{x_n\}$ be any fuzzy Bounded sequence in X. Since T is fuzzy compact linear operator then by Theorem (4.6) $\{T(x_n)\}$ has convergent subsequence $\{T(x_{n_k})\}$, since S is fuzzy bounded then $r>0 \ni \forall t>0$, such that: $N(S(x),t) \geq N\left(x,\frac{t}{r}\right), \forall x \in X$.Since $\{T(x_{n_k})\}$ is fuzzy converges $y \in X \Longrightarrow \{T(x_{n_k})\} \longrightarrow y \Longrightarrow$

$\forall \epsilon \in (0,1), \forall t_0>0, \exists n_0 \in \mathbb{Z}^+$ such that $N(T(x_{n_k}) - y, t0)>1-\epsilon$, $\forall n \geq n0$ Since $NSx, t \geq Nx, tr, \forall x \in X$, $T(x_{n_k}) - y \in X$.Hence

$$N(S(T(x_{n_k})-y),t_0) \geq N\left(T(x_{n_k})-y,\frac{t_0}{r}\right)$$

Put $t_2 = \frac{t_0}{r} \Longrightarrow t_2>0$. Then $N(T(x_{n_k})-y,t_2) > 1-\epsilon$, $\forall n \geq n_0 \Longrightarrow$
$N(S(T(x_{n_k})-y),t_0) > 1-\epsilon$, $\forall n \geq n_0 N(ST(x_{n_k})) - S(y), t0>1-\epsilon$
$\forall n \geq n_0$.Hence $\{ST(x_{n_k})\} \longrightarrow S(y)$.Hence $\{ST(x_n)\}$ has fuzzy convergent Sequence. Therefore ST is fuzzy compact operator by theorem (4.6).

Theorem (5.3) (Null spaces): In theorem (5.1)

$\dim(\mathcal{N}(T_\lambda{}^n)) < \infty, n = 1.2\ldots.$

Proof: $T_\lambda{}^n = (T-\lambda I)^{\boldsymbol{n}} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}T^{\boldsymbol{k}}(-\boldsymbol{\lambda})^{\boldsymbol{n-k}} = (-\lambda)^n I + T\sum_{\boldsymbol{k=0}}^{\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{k}}T^{\boldsymbol{k-1}}$

This can be written

$T_\lambda{}^n = w - \beta I, \beta = -(-\lambda)^n$

Where $w = TS = ST$ and S denotes the sum on the right. T is fuzzy compact and S is fuzzy bounded since T is bounded by theorem (4.5). Hence w fuzzy compact by lemma (5.2), so that we obtain

$\dim(\mathcal{N}(T_\lambda{}^n)) < \infty, n = 1.2\ldots.$

By applying theorem (5.1)

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