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I-LINDELOF SPACES

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

وَقُلِ اعْمَلُوا فَسَيَرَى اللَّهُ
عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ

صدق الله العلي العظيم

سورة التوبة (١٠٥)

شكر وتقدير

الحمد لله لما وفقني إليه والصلاة والسلام على سيد المرسلين

نبينا محمد (صلى الله عليه وعلى آله الأطهار)

اتقدم بالشكر الجزيل الى كل من ساندني بالبحث واهص بالشكر

للدكتور المشرف :- محمد محزير حسين

لما قدمه لي من عون ومساعدة . . .

الاهداء

الى العراق بما يسموا الصوى فينا
اهدي هذا العمل المتواضع الى الامام
صاحب العصر والزمان (عجل الله فرجه)
الى الذين قرن الله بهم بعبادته.. وطاعتهم برضوانه
الى من اثاروا لي طريق المعرفة بعد طريق الهداية
الى كل من احبني واحببته
اقدم جهدي المتواضع ثمره يانعه الى كل من لهم فضل
رعايتي والدي و والدي ليس جزاء بل
المحترفة ببعض ما لهم علي من حقوق

I-LINDELOF SPACES

We define a space (X, T) to be *I*-Lindelof if every cover $\mathcal{A}$ of X by regular closed subsets of the space (X, T) contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup \{\text{int}(A) : A \in \mathcal{A}'\}$. We provide several characterizations of *I*-Lindelof spaces and relate them to some other previously known classes of spaces, for example, rc-Lindelof, nearly Lindelof, and so forth. Our study here of *I*-Lindelof spaces also deals with operations on *I*-Lindelof spaces and, in its last part, investigates images and inverse images of *I*-Lindelof spaces under some considered types of functions.

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1. Definitions and characterizations. In [2], a topological space (X, T) is called *I*-compact if every cover $\mathcal{A}$ of the space by regular closed subsets contains a finite subfamily $\{A_1, A_2, \dots, A_n\}$ such that $X = \bigcup_{k=1}^n \text{int}(A_k)$. Recall that a subset A of (X, T) is regular closed (regular open, resp.) if $A = \text{cl}(\text{int}(A))$ ($\text{int}(\text{cl}(A))$, resp.). We let $\text{RC}(X, T)$ ($\text{RO}(X, T)$, resp.) denote the family of all regular closed (all regular open, resp.) subsets of a space (X, T) . A study that contains some properties of *I*-compact spaces appeared in [10]. In the present work, we study the class of *I*-Lindelof spaces.

DEFINITION 1.1. A space (X, T) is called *I*-Lindelof if every cover $\mathcal{A}$ of the space (X, T) by regular closed subsets contains a countable subfamily $\{A_n : n \in \mathbb{N}\}$ such that $X = \bigcup_{n \in \mathbb{N}} \text{int}(A_n)$.

To obtain characterizations of *I*-Lindelof spaces, we need the definitions of some classes of generalized open sets.

DEFINITION 1.2. A subset G of a space (X, T) is called semiopen (preopen, semi-preopen, resp.) if $G \subseteq \text{cl}(\text{int}(G))$ ($G \subseteq \text{int}(\text{cl}(G))$, $G \subseteq \text{cl}(\text{int}(\text{cl}(G)))$, resp.). $\text{SO}(X, T)$ ($\text{SPO}(X, T)$, resp.) is used to denote the family of all semiopen (all semi-preopen, resp.) subsets of a space (X, T) . The complement of a semiopen subset (semi-preopen subset, resp.) is called semiclosed (semi-preclosed, resp.). It is clear that a subset G is semiopen if and only if $U \subseteq G \subseteq \text{cl}(U)$, for some open set U . A subset G is called regular semiopen if there exists a regular open set W such that $W \subseteq G \subseteq \text{cl}(W)$.

The following diagram relates some of these classes of sets:

$$\text{regular closed} \Rightarrow \text{regular semiopen} \Rightarrow \text{semiopen} \Rightarrow \text{semi-preopen}. \quad (1.1)$$

It is well known that if G is a semi-preopen set, then $\text{cl}(G)$ is regular closed (see [6]). The next result gives several characterizations of *I*-Lindelof spaces and its proof is now clear.

THEOREM 1.3. *The following statements are equivalent for a space (X, T) .*

- (a) (X, T) is I -Lindelof.
- (b) Every cover $\mathcal{A}$ of the space (X, T) by semi-preopen subsets contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup \{\text{int}(\text{cl}(A)) : A \in \mathcal{A}'\}$.
- (c) Every cover $\mathcal{A}$ of the space (X, T) by semiopen subsets contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup \{\text{int}(\text{cl}(A)) : A \in \mathcal{A}'\}$.
- (d) Every cover $\mathcal{A}$ of the space (X, T) by regular semiopen subsets contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup \{\text{int}(\text{cl}(A)) : A \in \mathcal{A}'\}$.

Next we give another characterization of I -Lindelof spaces using the fact that a subset G is regular closed if and only if its complement is regular open.

THEOREM 1.4. *A space (X, T) is I -Lindelof if and only if every family $\mathcal{U}$ of regular open subsets of (X, T) with $\bigcap \{U : U \in \mathcal{U}\} = \emptyset$ contains a countable subfamily $\mathcal{U}'$ such that $\bigcap \{\text{cl}(U) : U \in \mathcal{U}'\} = \emptyset$.*

PROOF. To prove necessity, let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a family of regular open subsets of (X, T) such that $\bigcap \{U_\alpha : \alpha \in A\} = \emptyset$. Then the family $\{X - U_\alpha : \alpha \in A\}$ forms a cover of the I -Lindelof space (X, T) by regular closed subsets and therefore A contains a countable subset A' such that $X = \bigcup \{\text{int}(X - U_\alpha) : \alpha \in A'\}$. Then

$$\begin{aligned} \emptyset &= X - \bigcup \{\text{int}(X - U_\alpha) : \alpha \in A'\} \\ &= \bigcap \{X - \text{int}(X - U_\alpha) : \alpha \in A'\} = \bigcap \{\text{cl}(U_\alpha) : \alpha \in A'\}. \end{aligned} \quad (1.2)$$

To prove sufficiency, let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be a cover of the space (X, T) by regular closed subsets. Then $\{X - G_\alpha : \alpha \in A\}$ is a family of regular open subsets of (X, T) with $\bigcap \{X - G_\alpha : \alpha \in A'\} = \emptyset$. By assumption, there exists a countable subset A' of A such that $\bigcap \{\text{cl}(X - G_\alpha) : \alpha \in A\} = \emptyset$. So $X = X - \bigcap \{\text{cl}(X - G_\alpha) : \alpha \in A'\} = \bigcup \{X - \text{cl}(X - G_\alpha) : \alpha \in A'\} = \bigcup \{\text{int}(G_\alpha) : \alpha \in A'\}$. This proves that (X, T) is I -Lindelof. $\square$

In [7], a space (X, T) is called rc-Lindelof if every cover $\mathcal{A}$ of the space (X, T) by regular closed subsets contains a countable subcover for X . It is clear, by definitions, that every I -Lindelof space is rc-Lindelof. However, the converse is not true as we show in Example 1.7.

Recall that a space (X, T) is extremally disconnected (e.d.) if $\text{cl}(U)$ is open for each open $U \in T$. It is easy to show that a space (X, T) is e.d. if and only if, given any two regular open subsets U and V with $U \cap V = \emptyset$, $\text{cl}(U) \cap \text{cl}(V) = \emptyset$.

PROPOSITION 1.5. *Every I -Lindelof space (X, T) is e.d.*

PROOF. Suppose that (X, T) is not e.d. Then we find $U, V \in \text{RO}(X, T)$ such that $U \cap V = \emptyset$ but $\text{cl}(U) \cap \text{cl}(V) \neq \emptyset$, say $t \in \text{cl}(U) \cap \text{cl}(V)$. Now, the family $\{X - U, X - V\}$ forms a cover of the I -Lindelof space (X, T) by regular closed subsets. Thus $X = \text{int}(X - U) \cup \text{int}(X - V)$. Assume $t \in \text{int}(X - U)$. But $t \in \text{cl}(U)$ and therefore $\emptyset \neq \text{int}(X - U) \cap U \subseteq (X - U) \cap U$, a contradiction. The proof is now complete. $\square$

THEOREM 1.6. *A space (X, T) is I -Lindelof if and only if it is an e.d. rc-Lindelof space.*

PROOF. As necessity is clear, we prove only sufficiency. We let $\mathcal{A}$ be a cover of (X, T) by regular closed subsets. If $A \in \mathcal{A}$, then A is regular closed and can be written as $A = \text{cl}(U)$ for some $U \in T$. Since (X, T) is e.d., the set $A = \text{cl}(U)$ is open. Now, since (X, T) is rc-Lindelof, the cover $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{A : A \in \mathcal{A}'\} = \bigcup\{\text{int}(A) : A \in \mathcal{A}'\}$ because $A = \text{int}(A)$ for each $A \in \mathcal{A}$. This proves that (X, T) is I -Lindelof as required. $\square$

EXAMPLE 1.7. We construct an rc-Lindelof space which is not I -Lindelof. We let X be a countable infinite set and we fix a point $t \in X$. We provide X with the topology $T = \{U \subseteq X : t \notin U\} \cup \{U \subseteq X : t \in U \text{ and } X - U \text{ is finite}\}$. It is immediate that (X, T) is rc-Lindelof. However, (X, T) is not e.d. and therefore, by Theorem 1.6, is not I -Lindelof. To see that (X, T) is not e.d., we write $X = A \cup B$, where A and B are disjoint infinite subsets. Assume that $t \in A$. Then B is an open subset of (X, T) and $\text{cl}(B) = B \cup \{t\}$. But $\text{cl}(B)$ is not open and hence (X, T) is not e.d.

DEFINITION 1.8. A space (X, T) is called:

- (a) nearly Lindelof if every open cover $\mathcal{U}$ of (X, T) contains a countable subfamily $\mathcal{U}'$ such that $X = \bigcup\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}'\}$ (see [3]);
- (b) countably nearly compact if every countable open cover $\mathcal{U}$ of (X, T) contains a finite subfamily $\mathcal{U}'$ such that $X = \bigcup\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}'\}$.

It is clear that a space (X, T) is I -compact if and only if it is I -Lindelof and countably nearly compact.

THEOREM 1.9. A space (X, T) is I -Lindelof if and only if it is an e.d. nearly Lindelof space.

PROOF. To prove necessity, we see that (X, T) is, by Proposition 1.5, e.d. Now, let $\mathcal{U}$ be an open cover of (X, T) . Then $\{\text{cl}(U) : U \in \mathcal{U}\}$ is a cover of the I -Lindelof space (X, T) by regular closed subsets. So $\mathcal{U}$ contains a countable subfamily $\mathcal{U}'$ such that $X = \bigcup\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}'\}$. This proves that (X, T) is nearly Lindelof. Next, to prove sufficiency, we let $\mathcal{A}$ be a cover of (X, T) by regular closed subsets. Since (X, T) is e.d., then each $A \in \mathcal{A}$ is open. So $\mathcal{A}$ is an open cover of the nearly Lindelof space (X, T) and therefore $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup\{\text{int}(\text{cl}(A)) : A \in \mathcal{A}'\} = \bigcup\{\text{int}(A) : A \in \mathcal{A}'\}$ and we conclude that (X, T) is I -Lindelof. $\square$

THEOREM 1.10. Let (X, T) be e.d. Then the following statements are equivalent:

- (a) (X, T) is I -Lindelof;
- (b) (X, T) is rc-Lindelof;
- (c) (X, T) is nearly Lindelof.

Recall that the family of all regular open subsets of a space (X, T) is a base for a topology T_s on X , weaker than T . The space (X, T_s) is called the semiregularization of (X, T) (see [7]). A property P of topological spaces is called a semiregular property if a space (X, T) has property P if and only if (X, T_s) has property P .

We will prove that I -Lindelofness is a semiregular property. First, we need the following result.

PROPOSITION 1.11 [8, Proposition 2.2]. *Given a space (X, T) , let $G \in \text{SO}(X, T)$. Then $\text{cl}_T(G) = \text{cl}_{T_s}(G)$.*

THEOREM 1.12. *The property of being an I -Lindelof space is a semiregular property.*

PROOF. First, the property of being an e.d. space is a semiregular property (see [7, page 99]). Now let (X, T) be an I -Lindelof space. Then (X, T) is, by Proposition 1.5, e.d. and hence (X, T_s) is also e.d. So $\text{RC}(X, T) = \text{RO}(X, T)$ and $\text{RC}(X, T_s) = \text{RO}(X, T_s)$. To show that (X, T_s) is rc-Lindelof, let $\mathcal{A}$ be a cover of (X, T) by regular closed subsets. Then each $A \in \mathcal{A}$ is T_s -open and $\mathcal{A} \subseteq T_s \subseteq T$. Thus $\mathcal{A}$ contains a countable subfamily $\mathcal{A}'$ such that $X = \bigcup \{\text{cl}_T(A) : A \in \mathcal{A}'\} = (\text{Proposition 1.11}) \bigcup \{\text{cl}_{T_s}(A) : A \in \mathcal{A}'\} = \bigcup \{A : A \in \mathcal{A}'\}$ and therefore (X, T_s) is rc-Lindelof and hence I -Lindelof. Conversely, let (X, T_s) be I -Lindelof. Then both (X, T) and (X, T_s) are e.d. We show that (X, T) is rc-Lindelof. We let $\mathcal{A}$ be a cover of (X, T) by regular closed subsets, that is, $\mathcal{A} \subseteq \text{RC}(X, T) = \text{RO}(X, T) \subseteq T_s$. Since (X, T_s) is rc-Lindelof, there exists a countable subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that $X = \bigcup \{\text{cl}_{T_s}(A) : A \in \mathcal{A}'\} = (\text{Proposition 1.11}) \bigcup \{\text{cl}_T(A) : A \in \mathcal{A}'\} = \bigcup \{A : A \in \mathcal{A}'\}$. This shows that (X, T) is rc-Lindelof and the proof is complete. $\square$

2. Operations on I -Lindelof spaces. We note that the property of being an I -Lindelof space is not hereditary. Consider the discrete space N of all natural numbers and let βN be its Stone-Ćech compactification. Then βN is an rc-compact Hausdorff space (see [7, page 102]) and therefore βN is e.d. (see [11]). So βN is an I -Lindelof space. However, the subspace $\beta N - N$ is not I -Lindelof as it is not e.d. (see [7, page 102]). Here, (X, T) is called rc-compact or S -closed if every cover of X by regular closed subsets contains a finite subcover (see [7]).

Recall that a subset A of a space (X, T) is called preopen if $A \subseteq \text{int}(\text{cl}(A))$. We let $\text{PO}(X, T)$ denote the family of all preopen subsets of (X, T) .

PROPOSITION 2.1 [4, Corollary 2.12]. *Let (X, T) be rc-Lindelof and let $U \in \text{RO}(X, T)$. Then the subspace $(U, T|_U)$ is rc-Lindelof.*

PROPOSITION 2.2 [8, Proposition 4.2]. *The property of being an e.d. space is hereditary with respect to preopen subspaces.*

REMARK 2.3. It is well known that a space (X, T) is e.d. if and only if $\text{RC}(X, T) = \text{RO}(X, T)$ if and only if $\text{SO}(X, T) \subseteq \text{PO}(X, T)$. Thus if (X, T) is e.d., then

$$\text{RO}(X, T) = \text{RC}(X, T) \subseteq \text{SO}(X, T) \subseteq \text{PO}(X, T). \quad (2.1)$$

In view of Propositions 2.1, 2.2, and Remark 2.3, the proof of the following result is now clear.

THEOREM 2.4. *Every regular open (and hence every regular closed) subspace of an I -Lindelof space is I -Lindelof.*

THEOREM 2.5. *If a space (X, T) is a countable union of open I -Lindelof subspaces, then it is I -Lindelof.*

PROOF. Assume that $X = \bigcup \{U_n : n \in N\}$, where $(U_n, T|_{U_n})$ is an I -Lindelof subspace for each $n \in N$. Let $\mathcal{A}$ be a cover of the space (X, T) by regular closed subsets. For each $n \in N$, the family $\{A \cap U_n : A \in \mathcal{A}\}$ is a cover of U_n by regular closed subsets of the I -Lindelof subspace $(U_n, T|_{U_n})$ (see [4, Lemma 2.5]). So we find a countable subfamily $\mathcal{A}_n$ of $\mathcal{A}$ such that $U_n = \bigcup \{\text{int}_{U_n}(A \cap U_n) : A \in \mathcal{A}_n\}$. Put $\mathcal{B} = \bigcup \{\mathcal{A}_n : n \in N\}$. Then $\mathcal{B}$ is a countable subfamily of $\mathcal{A}$ such that $X = \bigcup \{U_n : n \in N\} = \bigcup_{n \in N} \bigcup \{\text{int}_{U_n}(A \cap U_n) : A \in \mathcal{A}_n\} = \bigcup_{n \in N} \bigcup \{\text{int}_X(A \cap U_n) : A \in \mathcal{A}_n\} \subseteq \bigcup \{\text{int}_X(A) : A \in \mathcal{B}\} \subseteq X$, that is, $X = \bigcup \{\text{int}(A) : A \in \mathcal{B}\}$. Therefore (X, T) is I -Lindelof. $\square$

If $\{(X_\alpha, T_\alpha) : \alpha \in A\}$ is a family of spaces, we let $\oplus_{\alpha \in A} X_\alpha$ denote their topological sum. Now we have, as a consequence of Theorem 2.5, the following result.

THEOREM 2.6. *The topological sum $\oplus_{\alpha \in A} X_\alpha$ of a family $\{(X_\alpha, T_\alpha) : \alpha \in A\}$ is I -Lindelof if and only if (X_α, T_α) is I -Lindelof for each $\alpha \in A$ and that A is a countable set.*

PROOF. It is clear that sufficiency is a direct consequence of Theorem 2.5. To prove necessity, we note that (X_α, T_α) is a clopen (and hence regular open) subspace of the I -Lindelof space $\oplus_{\alpha \in A} X_\alpha$ and therefore (X_α, T_α) is, by Theorem 2.5, I -Lindelof for each $\alpha \in A$. Moreover, the family $\{X_\alpha : \alpha \in A\}$ forms a cover of the rc-Lindelof space $\oplus_{\alpha \in A} X_\alpha$ by mutually disjoint regular closed subsets and therefore must contain a countable subfamily whose union is $\oplus_{\alpha \in A} X_\alpha$. Thus A must be a countable set. $\square$

We now turn to products of I -Lindelof spaces. As noted earlier, the space βN is I -Lindelof while $\beta N \times \beta N$ is not even e.d. However, we have the next special case.

THEOREM 2.7. *Let (X, T) be a compact space and (Y, M) an I -Lindelof space. If the product $X \times Y$ is e.d., then it is I -Lindelof.*

PROOF. By [1, Theorem 2.4], the space $X \times Y$ is rc-Lindelof. Since it is, by assumption, e.d., then it is, by Theorem 1.6, I -Lindelof. $\square$

3. Images and inverse images of I -Lindelof spaces. Let $f : (X, T) \rightarrow (Y, M)$. Recall that f is semicontinuous (see [9]) if $f^{-1}(V) \in \text{SO}(X, T)$ whenever $V \in M$, and f is almost open (see [8, page 86]) if $f^{-1}(\text{cl}(V)) \subseteq \text{cl}(f^{-1}(V))$ for each $V \in M$. Finally, f is preopen (see [8, page 86]) if $f(U)$ is a preopen subset of (Y, M) for each $U \in T$. It is mentioned in [8] that preopenness and almost openness coincide. Accordingly, we have the following result.

THEOREM 3.1. *Let $f : (X, T) \rightarrow (Y, M)$ be semicontinuous almost open and let (X, T) be I -Lindelof. Then (Y, M) is I -Lindelof.*

PROOF. First we have, by [8, Proposition 4.4], that (Y, M) is e.d. Next, by [1, Theorem 3.4], we have that (Y, M) is rc-Lindelof. Then, by Theorem 1.6, (Y, M) is I -Lindelof. $\square$

COROLLARY 3.2. *Every open continuous image of an I -Lindelof space is I -Lindelof.*

COROLLARY 3.3. *If a product space $\prod_{\alpha \in I} X_\alpha$ is I -Lindelof, then (X_α, T_α) is I -Lindelof, for each $\alpha \in I$.*

We recall that a function $f : (X, T) \rightarrow (Y, M)$ is irresolute if $f^{-1}(S) \in \text{SO}(X, T)$ for each $S \in \text{SO}(Y, M)$. Each irresolute is semicontinuous (see [1, Lemma 3.8]).

COROLLARY 3.4. *Every preopen irresolute image of an I -Lindelof space is I -Lindelof.*

We turn now to the inverse image of I -Lindelof spaces under certain class of functions. Recall that A is a semi-preclosed subset of a space (X, T) if its complement is semi-preopen.

DEFINITION 3.5. A function $f : (X, T) \rightarrow (Y, M)$ is called (weakly) semi-preclosed if $f(A)$ is a semi-preclosed subset of (Y, M) for each (regular) closed subset A of (X, T) .

The easy proof of the next result is omitted.

LEMMA 3.6. *A function $f : (X, T) \rightarrow (Y, M)$ is (weakly) semi-preclosed if and only if, for every $y \in Y$ and for each $(U \in \text{RO}(X, T)) \ U \in T$ with $f^{-1}(y) \subseteq U$, there exists $W \in \text{SPO}(Y, M)$ such that $y \in W$ and $f^{-1}(W) \subseteq U$.*

COROLLARY 3.7. *Let $f : (X, T) \rightarrow (Y, M)$ be weakly semi-preclosed. If $B \subseteq Y$ and $f^{-1}(B) \subseteq U$, with $U \in \text{RO}(X, T)$, then there exists $W \in \text{SPO}(Y, M)$ such that $B \subseteq W$ and $f^{-1}(W) \subseteq U$.*

We recall that a space (X, T) is km -perfect (see [5]) if, for each $U \in \text{RO}(X, T)$ and each point $x \in X - U$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of (X, T) such that $\bigcup \{U_n : n \in \mathbb{N}\} \subseteq U \subseteq \bigcup \{\text{cl}(U_n) : n \in \mathbb{N}\}$ and $x \notin \bigcup \{\text{cl}(U_n) : n \in \mathbb{N}\}$.

It is easy to see that every e.d. space is km -perfect. The converse, however, is not true as the space constructed in Example 1.7 is easily seen to be km -perfect but not e.d.

LEMMA 3.8. *If (X, T) is a km -perfect P -space ($\equiv$ the countable union of closed subsets is closed), then (X, T) is e.d.*

PROOF. We show that $\text{cl}(U)$ is open for each $U \in T$. Note that $\text{int}(\text{cl}(U))$ is regular open and if $x \notin \text{int}(\text{cl}(U))$, then, since (X, T) is km -perfect, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets such that $\bigcup \{U_n : n \in \mathbb{N}\} \subseteq \text{int}(\text{cl}(U)) \subseteq \bigcup \{\text{cl}(U_n) : n \in \mathbb{N}\}$ and $x \notin \bigcup \{\text{cl}(U_n) : n \in \mathbb{N}\}$. Since (X, T) is a P -space, then $\bigcup \{\text{cl}(U_n) : n \in \mathbb{N}\}$ is closed and contains $\text{int}(\text{cl}(U))$ and so it contains $\text{cl}(\text{int}(\text{cl}(U)))$. Thus $x \notin \text{cl}(\text{int}(\text{cl}(U)))$ and we obtain that $\text{cl}(\text{int}(\text{cl}(U))) = \text{int}(\text{cl}(U))$. But $U \subseteq \text{int}(\text{cl}(U))$ and therefore $\text{cl}(U) \subseteq \text{int}(\text{cl}(U)) = \text{cl}(\text{int}(\text{cl}(U))) \subseteq \text{cl}(U)$, that is, $\text{cl}(U) = \text{int}(\text{cl}(U))$, which shows that $\text{cl}(U)$ is open. $\square$

DEFINITION 3.9. A subset A of a space (X, T) is called an rc-Lindelof set (see [4]) if each cover of A by regular closed subsets of (X, T) contains a countable subcover of A .

We now state our final result which deals with an inverse image of an I -Lindelof space.

THEOREM 3.10. *Let (X, T) be a km -perfect P -space. Let $f : (X, T) \rightarrow (Y, M)$ be weakly semi-preclosed almost open with $f^{-1}(y)$ an rc-Lindelof set for each $y \in Y$. If (Y, M) is I -Lindelof, then so is (X, T) .*

PROOF. It is clear, by Lemma 3.8, that (X, T) is e.d. and therefore we only show that (X, T) is rc-Lindelof (Theorem 1.6). We let $\mathcal{A}$ be a cover of X by regular closed subsets of the space (X, T) . For each $y \in Y$, $\mathcal{A}$ forms a cover of the rc-Lindelof subset $f^{-1}(y)$ so we find a countable subfamily $\mathcal{A}_y$ of $\mathcal{A}$ such that $f^{-1}(y) \subseteq \bigcup \{A : A \in \mathcal{A}_y\} = G_y$. Then G_y is open, because (X, T) is e.d. and therefore $\text{RC}(X, T) = \text{RO}(X, T)$. But $f^{-1}(y) \subseteq G_y$, then we find, by Lemma 3.6, a subset $V_y \in \text{SPO}(X, T)$ such that $y \in V_y$ and $f^{-1}(V_y) \subseteq G_y$. Now, the family $\{V_y : y \in Y\}$ forms a cover of Y by semi-preopen subsets of the rc-Lindelof space (Y, M) . By [1, Theorem 1.9], it contains a countable subfamily $\{V_{y_n} : n \in N\}$ such that $Y = \bigcup \{\text{cl}(V_{y_n}) : n \in N\}$. We put $\mathcal{A}' = \bigcup \{\mathcal{A}_{y_n} : n \in N\}$. Then $\mathcal{A}'$ is countable and $\mathcal{A}'$ is a cover of X . To see this, let $x \in X$ and let $y = f(x)$. Choose $k \in N$ such that $y \in \text{cl}(V_{y_k})$. Then $x \in f^{-1}(\text{cl}(V_{y_k})) \subseteq (f \text{ is almost open}) \text{cl}(f^{-1}(V_{y_k})) \subseteq \text{cl}(G_{y_k}) = G_{y_k}$ (because (X, T) is a P -space and G_{y_k} is a countable union of closed subsets). We have $x \in G_{y_k} = \bigcup \{A : A \in \mathcal{A}_{y_k}\} \subseteq \bigcup \{A : A \in \mathcal{A}'\}$. The proof is now complete. $\square$

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