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University of AL\_ Qadisiya  
College of Computer science and information Teachnolog  
Department of Mathematics***

***On some inequities and stability resultes related to the  
differential equation***

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{ ولسوف يعطيك ربك فترضى }

صدق الله العظيم

# (( الاهداء ))

الى من ضحوا بدمائهم الطاهرة من اجل الاسلام ....

اهل البيت (( عليهم السلام ))

الى كل من ضحى من اجل الحق والعلم .....

العلماء

الى من اوحى الله بهما برا" واحسانا" .....

ابي واممي

الى الذي لم يبخل علي بعلمه وارشاده ....

استاذي الفاضل

الى من رافقني سنوات الدراسة .....

اصدقائي وزملائي

والى من يقدر الجهود العلمي اهدي ثمرة جهودي

## ﴿ شكر وتقدير ﴾

الحمد والشكر لله العلي القدير والصلاة والسلام على سيدنا محمد وآل بيته الطيبين الطاهرين  
الى يوم الدين . . . . . بعد ان اشكر الله سبحانه وتعالى في تيسير امرنا في اتمام بحثنا اتقدم  
بخالص شكري وتقديري الى استاذي الدكتور (( قصي حاتم الكفاري ))  
الذي قدم لي النصائح والمتابعة العلمية الصادقة التي كان لها الاثر الواضح في اتمام هذا البحث  
ادعو من الله عز وجل التوفيق والتقدم في مسيرته العلمية . . .

## ***Abstract:***

In this paper, we investigate the stability of  $n$ th order linear ordinary differential non-homogeneous equation with initial conditions in the Hyers-Ulam-Rassias sense.

*Keywords:* Differential equation, Differential inequality, Hyers-Ulam-Rassias stability, Initial Value Problem, Integral Equation.

*2010 MSC:* 34A30, 34A40, 45D05.

# 1. Introduction

In 1940, S.M. Ulam while he was giving talk at Wisconsin University, he proposed the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping? for details see [18]. A year later, D.H. Hyers in [4] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Let  $E_1$  and  $E_2$  be two real Banach spaces and  $f: X_1 \rightarrow X_2$  be a mapping. If there exist an  $c \geq 0$  such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all  $x, y \in X_1$ , then there exist a unique additive mapping  $g: X_1 \rightarrow X_2$  with the property

$$\|f(x) - g(x)\| \leq \varepsilon$$

$\forall x \in X_1$ . A generalized solution to Ulam's problem for approximately linear mappings was proved by Th.M. Rassias in 1978 [13]. He considered a mapping  $f: E_1 \rightarrow E_2$  such that  $t \rightarrow f(tx)$  is continuous in  $t$  for each fixed  $x$ . Assume that there exists  $\theta \geq 0$  and  $0 \leq p < 1$  such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)$$

for any  $x, y \in E_1$ . After Hyers result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers result in various directions, see ([2], [5]).

Soon-Mo Jung [17], investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. Miura et al [11], proved the Hyers-Ulam stability of the first-order linear

differential equations of the form  $y'(t) + g(t)y(t) = 0$ , where  $g(t)$  is a continuous function, while Jung [14], proved the Hyers-Ulam stability of differential equations of the form  $\phi(t)y'(t) = y(t)$ . Furthermore, the result

of Hyers-Ulam stability for first-order linear differential equations has been generalized in ([15], [16], [19]).

Javadian, E. Sorouri, G.H. Kim and M. Eshaghi Gordji [6], investigated generalized Hyers-Ulam stability

of the second order linear differential equations of the form  $y'' + P(x)y' + q(x)y = f(x)$  with some conditions. Maher Nazmi Qorawani [10], investigated Hyers-Ulam stability of second order linear differential equations

of the form  $z'' + p(x)z' + (q(x) - \alpha(x))z = 0$  and nonlinear differential equations of the form  $z'' + p(x)z' + q(x)z = h(x) |z|^\beta e^{\frac{\beta-1}{2} \int p(x)dx} \operatorname{sgn} z$  with initial conditions. Li

and Yan [8], investigated the Hyers-Ulam Stability of nonhomogeneous second order Linear Differential Equations of the form  $y'' + p(x)y' + q(x)y + r(x) = 0$

under some special conditions. Pasc Gavruta, Jung, Li [3], investigated the Hyers-Ulam stability for second order linear differential equations with boundary conditions of the form  $y'' + \beta(x)y(x) = 0$ . Jinghao

Huang, Qusuay H. Alqifiary, and Yongjin Li [7], proved the generalized superstability of  $n$ th order linear differential equations with initial conditions of the form  $y^{(n)}(x) + \beta(x)y(x) = 0$ . Recently, M.I. Modebei,

O.O. Olaiya, I. Otaide [12], investigated generalized Hyers-Ulam stability of second order linear ordinary differential equation  $y'' + \beta(x)y = f(x)$  with initial condition.

In this paper, we investigate the Hyers-Ulam-Rassias Stability of  $n$ th order linear ordinary differential equations with initial conditions

$$y^{(n)} + \beta(x)y(x) = f(x)$$

$$y(a) = y'(a) = y''(a) = \dots = y^{(n-1)}(a) = 0,$$

where  $y \in C^n[a, b]$ ,  $\beta \in C[a, b]$  and  $f: [a, b] \rightarrow \mathbb{R}$  continuous.

Let  $(X, \|\cdot\|)$  be a real or complex Banach space with  $a, b \in \mathbb{R}$  where  $-\infty < a < b < \infty$ ,  $c$  be a positive real number. Let  $y: (a, b) \rightarrow X$  be a continuous function. We consider the following differential equation

$$y^{(n)}(t) = \sum_{k=0}^{n-1} P_k y^{(k)}(t), \quad t \in I \tag{1.1}$$

and the following differential inequality

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \epsilon, \quad t \in I$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varphi(t), \tag{1.2}$$

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} P_k y^{(k)}(t) \right| \leq \varphi(t), \tag{1.3}$$

## 2. Preliminaries

**2.1.** The equation (1.1) is said to have the Hyers-Ulam stability for any  $c > 0$ , there exist a real number  $K > 0$  such that for each approximate solution  $y \in C^n(I, X)$  of (1.2) there exist a solution  $y_0 \in C^n(I, X)$  of (1.1) with

$$|y - y_0| \leq K\varepsilon \quad \forall t \in I. \quad (2.1)$$

**2.2.** The equation (1.1) is said to have the Hyers-Ulam-Rassias stability if there exist  $\theta\phi \in C(R^+, R^+)$ , such that for each approximate solution  $y \in C^n(I, X)$  of (1.3) there exist a solution  $y_0 \in C^n(I, X)$  of (1.1) with

$$|y - y_0| \leq \theta\phi(t) \quad \forall t \in I. \quad (2.2)$$

**2.3.** The equation  $y^{(n)}(x) + \beta(x)y(x) = 0$  has the Hyers-Ulam stability with initial conditions  $y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0$ , if there exists a positive constant  $K$  with the following property: For every  $c > 0$ ,  $y \in C^n[a, b]$ , if

$$|y^{(n)}(x) + \beta(x)y(x)| \leq \varepsilon, \quad (2.3)$$

and  $y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0$ , then there exists some  $z \in C^n[a, b]$  satisfying  $z^{(n)} + \beta(x)z(x) = 0$  and  $z(a) = z'(a) = \dots = z^{(n-1)}(a) = 0$ , such that:

$$|y(x) - z(x)| \leq K\varepsilon.$$

We need the following Lemma to prove our main results.

### Lemma 2.4 (Generalized Replacement Lemma)

Suppose that  $g : [a, b] \rightarrow \mathbb{R}$  is a continuous. Then:

$$\int_a^{s_{n-1}} \int_a^{s_{n-2}} \dots \int_a^{s_2} \int_a^{s_1} \int_a^x g(s) ds ds_1 ds_2 \dots ds_{n-1} = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} g(s) ds, \forall x \in [a, b]$$

### Theorem 2.5 :

If  $\max |\beta(x)| < \frac{n!}{(b-a)^n}$  Then

$$y^{(n)}(x) + \beta(x)y(x) = 0 \quad (2.4)$$

has the Hyers-Ulam stability with initial conditions

$$y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0 \quad (2.5)$$

where  $y \in C^n[a, b]$ ,  $\beta \in C[a, b]$  and  $f: [a, b] \rightarrow \mathbb{R}$  continuous.

**Proof:** For every  $\epsilon > 0$ , By using the Taylor formula, we have:

$$y(x) = y(a) + y'(x-a) + \dots + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Thus

$$\begin{aligned} |y(x)| &= \left| \frac{y^{(n)}(\xi)}{n!}(x-a)^n \right| \\ &\leq \max |y^{(n)}(x)| \frac{(b-a)^n}{n!} \quad \forall x \in [a, b], \end{aligned}$$

then

$$\max |y(x)| \leq \frac{(b-a)^n}{n!} \left[ \max |y^{(n)}(x) - \beta(x)y(x) + \beta(x)y(x)| \right]$$

Now using (2.4), we obtain

$$\begin{aligned} \max |y(x)| &\leq \frac{(b-a)^n}{n!} \left[ \max |y^{(n)}(x) - \beta(x)y(x)| + \max |\beta(x)| \max |y(x)| \right] \\ &\leq \frac{(b-a)^n}{n!} \epsilon + \frac{(b-a)^n}{n!} \max |\beta(x)| \max |y(x)|. \end{aligned}$$

Let  $\eta = ((b-a)^n \max |\beta(x)|) / n!$ ,  $K = (b-a)^n / (n!(1-\eta))$ . It is easy to see that  $z_0(x) = 0$  is a solution of  $y^{(n)}(x) - \beta(x)y = 0$  with initial conditions (2.5).

$$|y - z_0| \leq K\epsilon.$$

Hence (2.4) has the Hyers-Ulam-Rassias stability with initial conditions (2.5).

### **3. Main Results**

#### **3.1 Hyers-Ulam-Rassias Stability of the IVP**

In this section, we shall prove the Generalized Hyers-Ulam-Rassias Stability of the IVP

$$y^{(n)} + \beta(x)y(x) = f(x) \quad (3.1)$$

$$y(a) = y^r(a) = y^{rr}(a) = \dots = y^{(n-1)}(a) = 0, \quad (3.2)$$

where  $y \in C^n[a, b]$ ,  $\beta \in C[a, b]$  and  $f : [a, b] \rightarrow \mathbb{R}$  continuous.

**Theorem 3.1.1** Suppose  $|\beta(x)| < M$  where  $\overline{M} = (b - a)n^{n!}$ ,  $\phi : [a, b] \rightarrow [0, \infty)$  in an increasing function. The equation (3.1) has the Hyers-Ulam-Rassias stability if for  $\theta\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$  and for each approximate solution  $y \in C^n[a, b]$  of (3.1) satisfying.

$$|y^{(n)} - \beta(x)y(x) - f(x)| \leq \phi(x) \quad (3.3)$$

there exist a solution  $z_0 \in C^n[a, b]$  of (3.1) with condition (3.2) such that:

$$|y(x) - z_0(x)| \leq \theta\phi(x). \quad (3.4)$$

**Proof.** From (3.3) we have that:

$$-\phi(x) \leq y^{(n)} - \beta(x)y(x) - f(x) \leq \phi(x).$$

Integrating from  $a$  to  $x$ , and applying condition (3.2) we have

$$-\int_a^x \varphi(s)ds \leq y^{(n-1)}(x) - \int_a^x \beta(s)y(s)ds - \int_a^x f(s)ds \leq \int_a^x \varphi(s)ds.$$

On further integration and also applying condition (3.2) we have

$$\begin{aligned} -\int_a^{s_1} \int_a^x \varphi(s)dsds_1 &\leq y^{(n-2)}(x) - \int_a^{s_1} \int_a^x \beta(s)y(s)dsds_1 \\ &\quad - \int_a^{s_1} \int_a^x f(s)dsds_1 \leq \int_a^{s_1} \int_a^x \varphi(s)dsds_1. \end{aligned}$$

Continuing the process finally we can get,

$$\begin{aligned} -\int_a^{s_{n-1}} \int_a^{s_{n-2}} \dots \int_a^{s_2} \int_a^{s_1} \int_a^x \varphi(s)dsds_1 \dots ds_{n-1} \\ \leq y(x) - \int_a^{s_{n-1}} \int_a^{s_{n-2}} \dots \int_a^{s_2} \int_a^{s_1} \int_a^x \beta(s)y(s)dsds_1 \dots ds_{n-1} \\ - \int_a^{s_{n-1}} \int_a^{s_{n-2}} \dots \int_a^{s_2} \int_a^{s_1} \int_a^x f(s)dsds_1 \dots ds_{n-1} \\ \leq \int_a^{s_{n-1}} \int_a^{s_{n-2}} \dots \int_a^{s_2} \int_a^{s_1} \int_a^x \varphi(s)dsds_1 \dots ds_{n-1}. \end{aligned}$$

Now applying Lemma (2.1.4), we obtain

$$\begin{aligned} -\int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds \leq y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \beta(s)y(s)ds - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)ds \\ \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds. \end{aligned}$$

Hence we have

$$\left| y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) \right| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds. \quad (3.5)$$

If we choose  $Z_0(x)$  such that it solves equation (3.1) with (3.2) such that:

$$z_0(x) = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)z_0(s)ds + f(s)ds),$$

thus we estimate

$$\begin{aligned} |y(x) - z_0(x)| &\leq \left| y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) \right| \\ &\quad + \left| \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)z_0(s)ds + f(s)ds) \right| \\ |y(x) - z_0(x)| &\leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds + \int_a^x \left| \frac{(x-s)^{n-1}}{(n-1)!} \beta(s)[y(s) - z_0(s)] \right| ds. \end{aligned}$$

Now applying (3.5) and Theorem 2.1.5, we get

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds + |\beta(s)| \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} |y(s) - z_0(s)| ds$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds + M \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} |y(s) - z_0(s)| ds.$$

Applying Gronwall's inequality, we have

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp \left\{ M \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} ds \right\}$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp \left\{ M \left[ \frac{(x-a)^n}{n!} \right] \right\}$$

$$|y(x) - z_0(x)| \leq c \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds$$

With

$$c = \exp \left\{ \left[ \frac{x-a}{b-a} \right]^n \right\}$$

### **3.2 Hyers–Ulam–Rassias stability of linear differential equation of first order**

*The aim of this section is to improve the results by proving the Hyers–Ulam–Rassias stability of the following nonhomogeneous linear differential equation of first order:*

$$y'(t) + g(t)y(t) + h(t) = 0. \quad (3.6)$$

*We assume that  $X$  is a complex Banach space and  $I = (a, b)$  is an arbitrary interval. We moreover assume that  $g : I \rightarrow \mathbb{C}$ ,  $h : I \rightarrow X$  and  $\phi : I \rightarrow [0, \infty)$  are functions such that for arbitrary  $c \in I$ ,  $g(t)$  and  $\exp \int_a^t g(u)du \cdot h(t)$  integrable on  $(a, c)$ , and further such that  $\phi(t) \exp \{R \int_a^t g(u)du\}$  is integrable on  $I$ , where we use  $R(\omega)$  to denote the real part of complex numbers  $\omega$ .*

*We prove in [Theorem 3.2.1](#) that if a continuously differentiable function  $y : I \rightarrow X$  satisfies the differential inequality*

$$\|y'(t) + g(t)y(t) + h(t)\| \leq \phi(t) \quad (3.7)$$

*for all  $t \in I$ , then there exists a unique solution  $y_0(t)$  of the differential equation (3.6) such that:*

$$\|y(t) - y_0(t)\| \leq \exp \left\{ -\Re \left( \int_a^t g(u)du \right) \right\} \int_t^b \phi(v) \exp \left\{ \Re \left( \int_a^v g(u)du \right) \right\} dv$$

In the following theorem, we will prove the Hyers–Ulam–Rassias stability of the nonhomogeneous linear differential equation (1). More precisely, we solve the

differential inequality (2) for the class of functions  $y : I \rightarrow X$ .

**Theorem 3.2.1** Let  $X$  be a complex Banach space and let  $I = (a, b)$  be an open interval, where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$

are arbitrarily given with  $a < b$ . Assume that  $g : I \rightarrow \mathbb{C}$  and  $h : I \rightarrow X$  are continuous functions such that  $g(t)$  and  $\exp\left\{\int_a^t g(u)du\right\}$  the real part of complex numbers  $\omega$ .

$\int_a^t g(u)du\}h(t)$  are integrable on  $(a, c)$  for each  $c \in I$ . Moreover, suppose  $\phi : I \rightarrow [0, \infty)$  is a function such that  $\phi(t) \exp\left\{\Re\left(\int_a^t g(u)du\right)\right\}$  is integrable on  $I$ . If a continuously differentiable function  $y : I \rightarrow X$  satisfies the

differential inequality (2) for all  $t \in I$ , then there exists a unique  $x \in X$  such that:

$$\begin{aligned} & \left\| y(t) - \exp\left\{-\int_a^t g(u)du\right\} \left(x - \int_a^t \exp\left\{\int_a^v g(u)du\right\} h(v)dv\right) \right\| \\ & \leq \exp\left\{-\Re\left(\int_a^t g(u)du\right)\right\} \int_t^b \phi(v) \exp\left\{\Re\left(\int_a^v g(u)du\right)\right\} dv \end{aligned}$$

**Proof.** For simplicity, we use the following notation:

$$z(t) := \exp\left\{\int_a^t g(u)du\right\} y(t) + \int_a^t \exp\left\{\int_a^v g(u)du\right\} h(v)dv$$

for each  $t \in I$ . By making use of this notation and by (3.7), we get

$$\begin{aligned} \|z(t) - z(s)\| &= \left\| \exp\left\{\int_a^t g(u)du\right\} y(t) - \exp\left\{\int_a^s g(u)du\right\} y(s) \right. \\ &\quad \left. + \int_s^t \exp\left\{\int_a^v g(u)du\right\} h(v)dv \right\| \\ &= \left\| \int_s^t \frac{d}{dv} \left[ \exp\left\{\int_a^v g(u)du\right\} y(v) \right] dv \right. \\ &\quad \left. + \int_s^t \exp\left\{\int_a^v g(u)du\right\} h(v)dv \right\| \\ &= \left\| \int_s^t \exp\left\{\int_a^v g(u)du\right\} \{y'(v) + g(v)y(v) + h(v)\} dv \right\| \\ &\leq \left| \int_s^t \exp\left\{\Re\left(\int_a^v g(u)du\right)\right\} \phi(v) dv \right| \end{aligned} \tag{3.9}$$

for any  $s, t \in I$ .

Since  $\phi(t) \exp\{\Re \int_a^t g(u) du\}$  is assumed to be integrable on  $I$ , we may select  $t_0 \in I$ , for any given  $\varepsilon > 0$ , such that  $s, t \geq t_0$  implies  $\|z(t) - z(s)\| < \varepsilon$ . That is,  $\{z(s)\}_{s \in I}$  is a Cauchy net and hence there exists an  $x \in X$  such that  $z(s)$  converges to  $x$  as  $s \rightarrow b$ , since  $X$  is complete.

Finally, it follows from (3.9) and the above argument that for any  $t \in I$

$$\begin{aligned} & \left\| y(t) - \exp \left\{ - \int_a^t g(u) du \right\} \left( x - \int_a^t \exp \left\{ \int_a^v g(u) du \right\} h(v) dv \right) \right\| \\ &= \left\| \exp \left\{ - \int_a^t g(u) du \right\} (z(t) - x) \right\| \\ &\leq \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \|z(t) - z(s)\| \\ &\quad + \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \|z(s) - x\| \\ &\leq \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \left| \int_s^t \varphi(v) \exp \left\{ \Re \left( \int_a^v g(u) du \right) \right\} dv \right| \\ &\quad + \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \|z(s) - x\| \\ &\rightarrow \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \int_t^b \varphi(v) \exp \left\{ \Re \left( \int_a^v g(u) du \right) \right\} dv \end{aligned}$$

as  $s \rightarrow b$ , since  $z(s) \rightarrow x$  as  $s \rightarrow b$ .

It now remains to prove the uniqueness of  $x$ . Assume that  $x_1 \in X$  also satisfies the inequality (3.8) in place of  $x$ . Then, we have

$$\begin{aligned} & \left\| \exp \left\{ - \int_a^t g(u) du \right\} (x_1 - x) \right\| \\ & \leq 2 \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \int_t^b \varphi(v) \exp \left\{ \Re \left( \int_a^v g(u) du \right) \right\} dv \end{aligned}$$

for any  $t \in I$ . It follows from the integrability hypotheses that

$$\|x_1 - x\| \leq 2 \int_t^b \varphi(v) \exp \left\{ \Re \left( \int_a^v g(u) du \right) \right\} dv \rightarrow 0$$

as  $t \rightarrow b$ . This implies the uniqueness of  $x$



**Remark 3.2.2.** We may now remark that:

$$y(t) = \exp \left\{ - \int_a^t g(u) du \right\} \left( x - \int_a^t \exp \left\{ \int_a^v g(u) du \right\} h(v) dv \right)$$

is the general solution of the differential equation (3.6), where  $x$  is an arbitrary element of  $X$

**Corollary 3.2.3:** Let  $X$  be a complex Banach space and let  $I = (a, b)$  be an open interval, where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$

are arbitrarily given with  $a < b$ . Assume that  $g : I \rightarrow \mathbb{C}$  and  $h : I \rightarrow X$  are continuous functions such that  $g(t)$

and  $\exp\{\Re \int_a^t g(u) du\}$  are integrable on  $(c, b)$  for every  $c \in I$ . Moreover, suppose  $\phi : I \rightarrow [0, \infty)$  is a function.

Such that  $Q(t) \exp\{\Re \int_a^t g(u) du\}$  is integrable on  $I$ . If a continuously differentiable function  $y : I \rightarrow X$  satisfies the differential inequality (3.7) for all  $t \in I$ , then there exists a unique  $x \in X$  Such that

$$\begin{aligned} & \left\| y(t) - \exp \left\{ - \int_b^t g(u) du \right\} \left( x - \int_b^t \exp \left\{ \int_b^v g(u) du \right\} h(v) dv \right) \right\| \\ & \leq \exp \left\{ - \Re \left( \int_b^t g(u) du \right) \right\} \int_a^t \phi(v) \exp \left\{ \Re \left( \int_b^v g(u) du \right) \right\} dv \end{aligned} \quad (3.10)$$

for any  $t \in I$

**Proof.** Let  $J = (-b, -a)$  and define functions  $g_1 : J \rightarrow \mathbb{C}$ ,  $h_1 : J \rightarrow X$ ,  $y_1 : J \rightarrow X$ ,

and  $\phi_1 : J \rightarrow [0, \infty)$  by  $g_1(t) = g_1(-t)$ ,  $h_1(t) = h_1(-t)$ ,  $\phi_1(t) = \phi_1(-t)$ , respectively.

Using these definitions, We may transform the inequality (3.7) into

$$\|y_1'(t) - g_1(t)y_1(t) - h_1(t)\| \leq \phi_1(t)$$

for each  $t \in J$ .

Since  $g(t)$  is integrable on  $(c, b)$  with  $c \in I$ , so is  $-g_1(t)$  on  $(-b, -c)$  with  $-c \in J$ . Similarly, other integrability hypotheses imply that for any  $-c \in J$ ,

$$\exp \left\{ \int_{-b}^t (-g_1(u)) du \right\} (-h_1(t)) \quad \text{and} \quad \varphi_1(t) \exp \left\{ \Re \left( \int_{-b}^t (-g_1(u)) du \right) \right\}$$

are integrable on  $(-b, -c)$  and on  $J$ , respectively. Hence, we can now use Theorem (3.2.1) to conclude that there exists a unique  $x \in X$  such that:

$$\begin{aligned} & \left\| y_1(t) - \exp \left\{ \int_{-b}^t g_1(u) du \right\} \left( x + \int_{-b}^t \exp \left\{ - \int_{-b}^v g_1(u) du \right\} h_1(v) dv \right) \right\| \\ & \leq \exp \left\{ \Re \left( \int_{-b}^t g_1(u) du \right) \right\} \int_t^{-a} \varphi_1(v) \exp \left\{ - \Re \left( \int_{-b}^v g_1(u) du \right) \right\} dv \end{aligned}$$

for any  $t \in J$ . Indeed, we can verify by some tedious substitutions that this inequality is equivalent to that in (3.10).

**Remark 3.2.4.** We remark that for any element  $x$  of  $X$ ,

$$y(t) = \exp \left\{ - \int_b^t g(u) du \right\} \left( x - \int_b^t \exp \left\{ \int_b^v g(u) du \right\} h(v) dv \right)$$

is a solution of the differential equation (3.6).

**Remark 3.2.5.** If we replace  $C$  by  $R$  in the proofs of both Theorem 3.2.1 and Corollary 3.2.2, we can see that Theorem 3.2.1 and Corollary 3.2.2 are also true for a real Banach space  $X$ . In both cases, we naturally assume that  $g$  is a real-valued continuous function.

### Example

If we set  $h(1) \equiv 0$  and  $\phi(t) \equiv \varepsilon$  in Theorem 1, we obtain the following result: Let  $X$  be a complex Banach space and let  $I = (a, b)$  be an open interval, where  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ . Assume that  $g : I \rightarrow C$  is a continuous and integrable function on  $(a, c)$  for each  $c \in I$  such that  $\exp\{\Re \int_a^t g(u) du\}$  is integrable on  $I$ . If a continuously differentiable function  $y : I \rightarrow X$  satisfies the differential inequality

$$\|y'(t) + g(t)y(t)\| \leq \varepsilon$$

for all  $t \in I$ , then there exists a unique  $x \in X$  such that:

$$\begin{aligned} & \left\| y(t) - \exp \left\{ - \int_a^t g(u) du \right\} x \right\| \\ & \leq \varepsilon \exp \left\{ - \Re \left( \int_a^t g(u) du \right) \right\} \int_t^b \exp \left\{ \Re \left( \int_a^v g(u) du \right) \right\} dv \end{aligned}$$

for each  $t \in I$

### **3.3 Hyers–Ulam–Rassias stability of differential equation with second order**

Now, the main results of this section are given in the following theorems.

**Theorem 3.3.1** *If a twice continuously differentiable function  $y : I \rightarrow R$  satisfies the differential inequality*

$$|y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon \quad (3.11)$$

*for all  $t \in I$  and for some  $\varepsilon > 0$ , and the Riccati equation  $u'(x) + p(x)u(x) - u^2(x) = q(x)$  has a particular solution, then there exists a solution  $v : I \rightarrow R$  of (1.5) such that*

$$|y(x) - v(x)| \leq K\varepsilon, \quad (3.12)$$

*where  $K > 0$  is a constant.*

**Proof:** Let  $\varepsilon > 0$  and  $y : I \rightarrow R$  be a continuously differentiable function such that

$$|y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon. \quad (3.13)$$

We will show that there exists a constant  $K$  independent of  $\varepsilon$  and  $v$  such that  $|y - v| < K\varepsilon$  for some  $v \in C^2(I)$  satisfying  $v'' + p(x)v' + q(x)v + r(x) = 0$ .

Assume that  $c(x)$  is a particular solution of Riccati equation  $u'(x) + p(x)u(x) - u^2(x) = q(x)$ ; if we set

$$g(x) = y'(x) + c(x)y(x), \quad d(x) = p(x) - c(x), \quad (3.14)$$

then

$$g'(x) = y''(x) + c(x)y'(x) + c'(x)y(x), \quad (3.15)$$

thus

$$\begin{aligned} |g'(x) + d(x)g(x) + r(x)| &= \\ |y''(x) + (c(x) + d(x))y'(x) + (c'(x) + d(x)c(x))y(x) + r(x)| &= \\ = |y'' + p(x)y' + q(x)y + r(x)| &\leq \varepsilon \end{aligned} \quad (3.16)$$

Using the similar technique in [14], we can prove

$$\begin{aligned}
 w(x) &= \exp \left\{ \int_a^x (-d(s)) ds \right\} \left[ (g(b) - \varepsilon) \exp \left\{ - \int_a^b (-d(x)) ds \right\} \right. \\
 &\quad \left. - \int_x^b -r(s) \exp \left\{ - \int_a^s (-d(t)) dt \right\} ds \right] \\
 &= \exp \left\{ \int_a^x (-d(s)) ds \right\} \left[ (g(b) - \varepsilon) \exp \left\{ \int_a^b (d(x)) ds \right\} + \int_x^b r(s) \exp \left\{ \int_a^s (d(t)) dt \right\} ds \right]
 \end{aligned}$$

satisfying

$$w'(x) + d(x)w(x) + r(x) = 0, \quad (3.17)$$

and there exists an  $L > 0$  such that

$$|g(x) - w(x)| \leq L\varepsilon \quad (3.18)$$

By  $g(x) = y'(x) + c(x)y(x)$ ,  
we get

$$|y'(x) + c(x)y(x) - w(x)| \leq \varepsilon \quad (3.19)$$

Using the same technique as above, we know that

$$\begin{aligned}
 v(x) &= \exp \left\{ \int_a^x (-c(s)) ds \right\} \left[ (y(b) - \varepsilon) \exp \left\{ - \int_a^b (-c(x)) ds \right\} \right. \\
 &\quad \left. - \int_x^b w(s) \exp \left\{ - \int_a^s (-c(t)) dt \right\} ds \right] \\
 &= \exp \left\{ \int_a^x (-c(s)) ds \right\} \left[ (y(b) - \varepsilon) \exp \left\{ \int_a^b (c(s)) ds \right\} - \int_x^b w(s) \exp \left\{ \int_a^s (c(t)) dt \right\} ds \right]
 \end{aligned} \quad (2.11)$$

Satisfying

$$v^r(x) + c(x)v(x) - w(x) = 0, \quad (3.20)$$

and there exists a  $K > 0$  such that

$$|y(x) - v(x)| \leq K\varepsilon. \quad (3.21)$$

The desired conclusion is proved.

**Theorem 3.3.2** Let  $p(x)$ ,  $q(x)$ , and  $r(x)$  be continuous real functions on the interval  $I = (a, b)$  such that  $p(x) \neq 0$  and  $y_0(x)$  is a nonzero bounded particular solution  $p(x)y'' + q(x)y' + r(x)y = 0$ . If  $y : I \rightarrow \mathbb{R}$  is a twice continuously differentiable function, which satisfies the differential inequality

$$|p(x)y'' + q(x)y' + r(x)y| \leq \varepsilon \quad (3.22)$$

for all  $t \in I$  and for some  $\varepsilon > 0$ , then there exists a solution  $v : I \rightarrow \mathbb{R}$  such that

$$|y(x) - v(x)| \leq K\varepsilon \quad (3.23)$$

where  $K > 0$  is a constant, and  $v$  satisfies

$$p(x)v'' + q(x)v' + r(x)v = 0.$$

**Proof.** Setting  $y(x) = y_0(x) \int_a^x z(s) ds$ , we obtain

$$y'(x) = y_0'(x) \int_a^x z(s) ds + y_0(x) z(x)$$

and also

$$y''(x) = y_0''(x) \int_a^x z(s) ds + 2y_0'(x) z(x) + y_0(x) z'(x) \quad (3.24)$$

By a simple calculation, we see that

$$\begin{aligned} & |p(x)y_0(x)z'(x) + [2p(x)y_0'(x) + q(x)y_0(x)] \\ &= |p(x)y''(x) + q(x)y'(x) + r(x)y(x)| \\ &\leq \varepsilon. \end{aligned} \quad (3.25)$$

Without loss of generality, we may assume that  $p(x)y_0(x) > 0$ . Using the similar technique in [14], we know that

$$z_1(x) = \exp \left\{ - \int_a^x \frac{2p(s)y'_0(s) + q(s)}{y_0(s)} ds \right\} \left[ (z(b) - \varepsilon) \exp \left\{ \int_a^b \frac{2p(s)y'_0(s) + q(s)}{y_0(s)} ds \right\} \right]$$

satisfies

$$p(x)y_1(x)z'_1(x) + [2p(x)y'_1(x) + q(x)y_1(x)]z_1(x) = 0$$

and also

$$|z(x) - z_1(x)| \leq L\varepsilon \quad (3.26)$$

for some  $L > 0$

$$z_2(x) = \left( \frac{y(b)}{y_1(b)} - \varepsilon \right) - \int_x^b z_1(s) ds \quad (3.27)$$

satisfies

$$z_2(x) - z_1(x) = 0$$

And also

$$(3.28)$$

$$|z(x) - z_2(x)| \leq Q\varepsilon$$

for some  $Q > 0$ .

Consequently, we have

$$|y(x) - z_2(x)y_0(x)| \leq M\varepsilon \quad (3.29)$$

From the inequalities  $-L\varepsilon \leq z(x) - z_1(x) \leq L\varepsilon$  and the similar technique in [14], we further get that

for some positive constant  $M$ .

Define  $v(x) = z^2(x)y_0(x)$ . It then follows from the above inequality that  $|z(x) - v(x)| \leq M \varepsilon$  holds for every  $x \in I$ . We can easily verify that  $v$  satisfies  $p(x)v''' + q(x)v'' + r(x)v' = 0$ . This completes the proof of our theorem.

We can prove the following corollaries by using an analogous argument. Hence, we omit the proofs.

**Corollary 3.3.3.** *Let  $p(x)$ ,  $q(x)$ , and  $r(x)$  be continuous real functions on the interval  $I = (a, b)$  such that  $p(x) \neq 0$  and  $r^2 + p(x)r + q(x) = 0$ . If  $y : I \rightarrow R$  is a twice continuously differentiable function, which satisfies the differential inequality*

$$|p(x)y'' + q(x)y' + r(x)y| \leq \varepsilon \quad (3.30)$$

for all  $t \in I$  and for some  $\varepsilon > 0$ , then there exists a solution  $v : I \rightarrow R$  such that

$$|y(x) - v(x)| \leq K\varepsilon, \quad (3.31)$$

where  $K > 0$  is a constant, and  $v$  satisfies  $p(x)v'' + q(x)v' + r(x)v = 0$ .

**Corollary 3.3.4** *Let  $p(x)$ ,  $q(x)$ ,  $r(x)$ , and  $s(x)$  be continuous real functions on the interval  $I = (a, b)$  such that  $p(x) \neq 0$  and  $y_0(x)$  is a nonzero bounded particular solution  $p(x)y_0''' + q(x)y_0'' + r(x)y_0' + s(x)y_0 = 0$ . If  $y : I \rightarrow R$  is a twice continuously differentiable function, which satisfies the differential inequality*

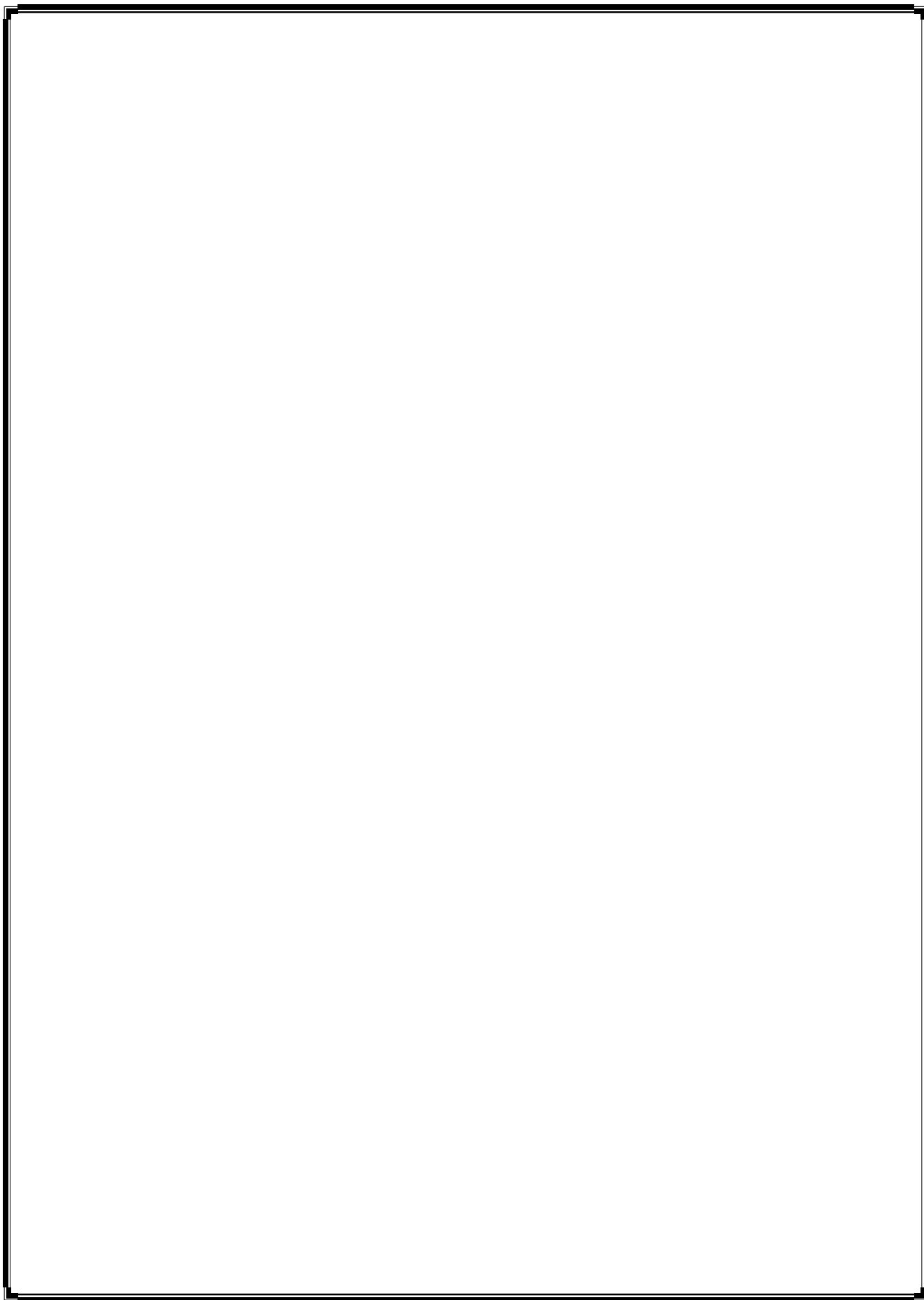
$$|p(x)y''' + q(x)y'' + r(x)y' + s(x)y| \leq \varepsilon \quad (3.32)$$

for all  $t \in I$  and for some  $\varepsilon > 0$ , then there exists a solution  $v : I \rightarrow R$  such that:

$$|y(x) - v(x)| \leq K\varepsilon, \quad (3.33)$$

where  $K > 0$  is a constant, and  $v$  satisfies

$$p(x)v''' + q(x)v'' + r(x)v' + s(x)v = 0.$$



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## المخلص :

في هذه الورقة نقوم بالتحقق عن استقرارية المعادلات  
التفاضليه الخطيه غير المتجانسه من الرتبه  $n$  مع الشروط الاوليه  
من خلال استقرارية هاييرس اولام



جمهورية العراق  
وزارة التعليم العالي والبحث العلمي  
جامعة القادسية  
كلية علوم الحاسبات وتكنولوجيا المعلومات  
قسم الرياضيات

## On some inequities and stability results related to the differential equation

### بحث تخرج

مقدم الى كلية علوم الحاسوب وتكنولوجيا المعلومات جامعة القادسية  
وهو جزء من متطلبات نيل درجة البكالوريوس في علوم الرياضيات .

من قبل الطالبة  
زهراء نبيل كاظم

بإشراف  
أ.م.د. قصي حاتم الكفاري