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Application of Subclass of Meromorphic Multivalent Functions Defined by linear operator

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

لَوْ يَفْعَلُ اللَّهُ الْكَذِبَ مَا آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا
الْعِلْمَ دَرَجَاتٍ جُودًا بِمَا تَعْمَلُونَ خَيْرٌ ۖ

صدق الله العلي العظيم

سورة المجادلة - ١١

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الإهداء

أهدي هذا الجهد البسيط إلى من سببهم
توصلت إلى هذه المرحلة الدراسية ولولا دمائهم الطاهرة لما
استطعت إن أكمل مسيرتي الدراسية الا وهم شهداء العراق
ومن تبقى في جبهات القتال من الحشد الشعبي المقدس
والجيش العراقي وكل فصائل المقاومة
اهدي لكم هذا العمل المتواضع

شكر وتقدير

قال رسول الله صل الله عليه واله وسلم

((من لا يشكر الناس لا يشكر الله))

اشكر في هذا الصدد الأستاذ المشرف الدكتور

وقاص خاليج عطشان

لنصائحه الثمينة وتوجيهاته القيمة

كما أتوجه بالشكر إلى من أوصاني الله بطايعتهما وبرهما ... أبي و

أمي وإلى كل من يد لي يد العون والمساعدة في إتمام هذا

البحث

ولله الحمد من قبل ومن بعد

1- Introduction :

Let Σ_p denote the class of function $F(Z)$

of the form:-

$$f(z) = \frac{1}{z^p} = \sum_{n=p}^{\infty} a_n z^n, \quad p \in \mathbb{N} \quad (1,1)$$

which are analytic and multivalent in punctured

$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$ Unit disk

We denote by $\Sigma^*(\beta, P)$ and $\Sigma_k(\beta, P)$

respectively the classes of meromorphic multivalent starlike function of order β and meromorphic multivalent convex function of order β .

Definition 1 :- A Function $F(z)$ in the class Σ_P is in the class Σ_p^+ if it is given by

$$f(z) = \frac{1}{z^p} = \sum_{n=p}^{\infty} a_n z^n, \quad (a_n \geq 0, p \in \mathbb{N}) \quad (1.2)$$

We define the Hadmard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n b_n z^n = (g * f)z. \quad (1.3)$$

For positive real values of $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$

$(\beta_j \neq 0, -1, \dots, j=1, 2, \dots, s),$

We now define the generalized by pergemetric function

$${}_qF_s (\alpha_1, \dots, \alpha_q)(\beta_1, \dots, \beta_s; z)$$

by

$${}_qF_s (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n z^n}{(\beta_1)_n \dots (\beta_s)_n n!} \quad (1.4)$$

$$(q \leq s + 1; q, s \in \mathbb{N} \cup \{0\}; z \in \mathbb{C}),$$

(1.5)

Where (0)

$$v = \begin{cases} 0 & v = 0 \\ \theta(\theta + 1)(\theta + 2) \dots (\theta + r - 1) & v \in \mathbb{N} \end{cases}$$

Corresponding to the function ${}_pF_q (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by :

$${}_pF_q (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p} {}_qF_s (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \quad (1.6)$$

We consider a linear operator

$$H_p (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) : \sum p \rightarrow \sum p.$$

Which is defined by means of the following Hadamard product (or convolution) :

$$H_p (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z) = {}_pF_q (\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z) \quad (1.7)$$

We observe that for a function $f(z)$ of the form (1.2); we have

$$H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) f(z) = \frac{1}{z^p} - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{a_n}{n!} z^n. \quad (1.8)$$

If , for convergence , we write

$$H_p, q, s(\alpha_1) = H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s), \quad (1.9)$$

Then one can easily verify from the definition (1.7) that

$$Z(H_p, q, s(\alpha_1) f'(z)) = \alpha_1 H_p, q, s(\alpha_1 + 1) f(Z) - (\alpha_1 + p) H_p, q, s(\alpha_1) f(Z) \quad (1.10)$$

The linear operator $H_p, q, s(\alpha_1)$ was investigated recently by Liu and Srivastava [10].

Some interesting sub classes of analytic function associated with the generalized hypergeometric function , were considered recently by (for example) Dziok and Srivastava ([6] and [7]).

Gangadheran et al. [8] and Liu [9] .

Definition : A function $f(z)$ in the class Σ_p is in the class $\Sigma_p(q, s, \alpha_1, A, B, \lambda)$ if it satisfies the condition .

$$\left| \frac{\frac{z(H_p, q, s(\alpha_1)f(z))'}{H_p, q, s(\alpha_1)f(z)} + p}{\beta \frac{z(H_p, q, s(\alpha_1)f(z))'}{H_p, q, s(\alpha_1)f(z)} + [pB + (A-B)(p-\lambda)]} \right| < 1, (z \in U) \quad (1.11)$$

for fixed parameters A, B, λ ($-1 \leq B < A \leq 1$; $0 \leq \lambda < p$; $p \in \mathbb{N}$) and $I_p^\sigma f(Z)$ is given by (1.4)

Let us write $\Sigma_p^+((q,s,\alpha_1,A,B,\lambda)) = \Sigma_p^+ \cap \Sigma_p((q,s,\alpha,A,B,\lambda))$

Here, we undertake the systematic study of a new subclass

$$\Sigma_p^+(q,s,\alpha_1,A,B,\lambda).$$

Such type of study has been already undertaken by several authors, like Aouf [1,2] Choetal [5],

Mograetal [11] and Atshan and Kulkarni [3,4]

2- coefficient bounds

In the following theorem , we obtain a sufficient condition for a function $f(z)$ to be in the class $\Sigma_p(q, s, \alpha_1, A, B, \lambda)$

Theorem :- Let $f(z) \in \Sigma_p$. If

$$\sum_{n=p}^{\infty} \left((x+p)(1+B) + (A-B)(p-\lambda) \right) \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} |a_n| \leq (A-B)(p-\lambda) \quad (2-1)$$

Then $f(z)$ is in the class $\Sigma_p(q, s, \alpha_1, A, B, \lambda)$

proof:- Assume (2-1) holds for all admissible values A, B, λ

Let the expression .

$$J(f) = \left| Z(H, p, q, s, \alpha_1) \overline{f(z)} + p(H, p, q, s, \alpha_1) f(z) \right| - \left| B_z(H, p, q, s, \alpha_1) \overline{f(z)} + [pB + (A-B)(p-\lambda)] H, p, q, s, \alpha_1 f(z) \right| \quad (2.2)$$

Since

$$I_p^\sigma f(Z) = \frac{1}{Z^p} + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n Z^n \text{ then}$$

$$(I_p^\sigma \overline{F(z)}) = -p \frac{1}{Z^{p+1}} + \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n Z^{n-1}$$

Therefore , we have for $0 < |Z| = r < 1$

$$\begin{aligned}
J(f) &= \left| -p \frac{1}{z^p} + \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^n + p \frac{1}{z^p} \right. \\
&\quad \left. \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^n \right| \\
&= \left| B(-p \frac{1}{z^p} + \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^n) + \right. \\
&\quad \left. [pB + (A-B)(p-\lambda)] \left(\frac{1}{z^p} + \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^n \right) \right| \\
&= \left| \sum_{n=p}^{\infty} (n+p) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^{n+p} \right| \\
&\quad \left| (A-B)(p-\lambda) + \sum_{n=p}^{\infty} p^n \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^{n+p} + \sum_{n=p}^{\infty} p^n (pB + \right. \\
&\quad \left. (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^{n+p} \right| \\
&= \left| \sum_{n=p}^{\infty} (n+p) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^{n+p} \right| - \left| (A-B)(p-\lambda) + \right. \\
&\quad \left. \sum_{n=p}^{\infty} ((p+n)B + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n z^{n+p} \right| \\
&\leq \sum_{n=p}^{\infty} (n+p) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) |a_n| r^{n+p} - (A-B)(p-\lambda) \\
&\quad + \sum_{n=p}^{\infty} ((p+n)B + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) |a_n| r^{n+p} \\
&\quad \sum_{n=p}^{\infty} ((n+p)((1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) |a_n| r^{n+p} \\
&\quad - (A-B)(p-\lambda) \quad (2.3)
\end{aligned}$$

Since the

above in equality holds for all r , $0 < r < 1$, letting $r \rightarrow 1$, in (2 .

3) we get the result (2.1) This completes the proof .

Theorem 2:- Let $f(z) \in \Sigma_p^+$. Then $f(z) \in$

$\Sigma_p^+ (q, s, \alpha_1, A, B, \lambda)$ if and only if

$$\sum_{n=p}^{\infty} \frac{(n+p)!}{(1+B) + (A-B)(p-\lambda)} \left(\frac{(\alpha_1)_n \dots (\alpha_s)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n \leq (A-B)(p-\lambda) \quad (2.4)$$

The result (2.4) is sharp

Proof :- from Theorem 1 ; it is sufficient to prove

The "only if" part . Let $f \in \Sigma_p^+ (q, s, \alpha_1, A, B, \lambda)$, the

$$\left| \frac{\frac{z(Hp, q, s(\alpha_1)f(z))'}{Hp, q, s(\alpha_1)f(z)} + p}{B \frac{z(Hp, q, s(\alpha_1)f(z))'}{Hp, q, s(\alpha_1)f(z)} + [pB + (A-B)(p-\lambda)]} \right|$$

$$= \frac{\sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_s)_n}{(\beta_1)_n \dots (\beta_s)_n n!} |a_n| |z|^{n+p}}{(A-B)(p-\lambda) + \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_s)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n |z|^{n+p}} < 1, (z \in U^*)$$

Since $\operatorname{Re} \{z\} \leq |z|$ for all z , it follows that

$$\sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n z^{n+p}$$

$$\operatorname{Re}\left\{ \frac{z(Hp, q, s(\alpha_1)f(z))'}{Hp, q, s(\alpha_1)f(z)} \right\} < 1$$

$$(A-B)(p - \lambda) - \sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n z^{n+p}, (z \in U^*) . \quad (2.5)$$

We choose the values of z on the real axis so that

$$\frac{z(Hp, q, s(\alpha_1)f(z))'}{Hp, q, s(\alpha_1)f(z)} \text{ is real, by letting } z \rightarrow 1^-$$

Through real values, we get.

$$\sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n \leq (A-B)(p - \lambda) -$$

$$\sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n$$

Therefore

$$\sum_{n=p}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} a_n \leq (p+n)(1+B) + (A-B)(p - \lambda)$$

$$(A-B)(p - \lambda),$$

Which completes the proof of Theorem 2.

The result is sharp for

$$f_n(z) = \frac{1}{z^p} + \frac{(A-B)(p - \lambda)}{(n+p)(1+B) + (A-B)(p - \lambda)} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!}, p \in \mathbb{N}$$

corollary 1:- Let $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$, Then

$$a_n \leq \frac{(A - B)(p - \lambda)}{(n + p)(1 + B) + (A - B)(p - \lambda)} \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)$$

3- Distortion Theorem :

In The following Theorem , we obtain distortion bounds for $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$

Theorem 3:- Let $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$.Then

$$\begin{aligned} & \frac{1}{|z|^p} - \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)} |z|^p \leq |f(z)| \\ & \leq \frac{1}{|z|^p} + \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)} |z|^p \quad (3.1) \end{aligned}$$

The result (3.1) is sharp for the function

$$F(z) = \frac{1}{z^p} + \frac{(A-B)(p-\lambda) z^p}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)}$$

Proof:- Let $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$. By Theorem2, we have

$$\sum_{n=p}^{\infty} a_n \leq \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)}, \quad p \in \mathbb{N} \quad (3.2)$$

Thus

$$|f(z)| \leq \frac{1}{|z|^p} + \sum_{n=p}^{\infty} a_n |z|^n \leq \frac{1}{|z|^p} + |z|^p \sum_{n=p}^{\infty} a_n.$$

Now ,by using (3,2) we get

$$|f(z)| \leq \frac{1}{|z|^{\overline{p}+}} |z|^p \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_p}{(\beta_1)_n \dots (\beta_s)_p p!} \right)}$$

Similarly , we get

$$|f(z)| \geq \frac{1}{|z|^{\overline{p}-}} |z|^p \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)}$$

And this completes The proof of Theorem 3 .

The result is sharp for : $f(z) =$

$$\frac{1}{|z|^{\overline{p}+}} \frac{(A-B)(p-\lambda)}{(2p(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_p \dots (\alpha_q)_p}{(\beta_1)_p \dots (\beta_s)_p p!} \right)} z^p$$

4. Radii of starlikeness and convexity:

In the following theorems, we obtain the region of univalence , in particular , starlikeness and convexity for the class

$$\Sigma_p^+(q, s, \alpha_1, A, B, \lambda).$$

Theorem 4 :-

If $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$, Then $f(z)$ is meromorphic multivalent starlike of order

Θ ($0 \leq \Theta < p$) in the disk $|z| < r_1(q, s, \alpha_1, A, B, \lambda, \Theta)$ where:-

$$r_1(q, s, \alpha_1, A, B, \lambda, \Theta)$$

$$= \inf_n \left\{ \frac{(p - \Theta) + (n + p)(1 + B) + (A - B)(p - \lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)^{\frac{1}{n+p}}}{(2p + n - \Theta)(A - B)(p - \lambda)} \right\} \quad (4.1)$$

The result (4.1) is sharp

Proof :- Let $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$, Then by Theorem 2:-

$$\sum_{n=p}^{\infty} \frac{(n + p)(1 + B) + (a - B)(p - \lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)}{(A - B)(p - \lambda)} \quad \text{an} \leq 1 \quad (4.2)$$

It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \Theta \quad (4.3)$$

For $|z| < r_1(q, s, \alpha_1, A, B, \lambda, \Theta)$ where $r_1(q, s, \alpha_1, A, B, \lambda, \Theta)$

Is given by (4.1)

Since

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n, \text{ Then } Zf'(z) = -pZ^{-p} + \sum_{n=p}^{\infty} na_n z^n.$$

Therefore :

$$\left| \frac{Z f'(z)}{f(z)} + p \right| = \left| \frac{\sum_{n=p}^{\infty} (n+p) a_n z^{n+p}}{1 + \sum_{n=p}^{\infty} a_n z^{n+p}} \right| \leq \frac{\sum_{n=p}^{\infty} (n+p) a_n |z|^{n+p}}{1 - \sum_{n=p}^{\infty} a_n |z|^{n+p}} \leq p - \theta.$$

Provided that :

$$\sum_{n=p}^{\infty} \frac{(2p+n-\theta)}{(p-\theta)} a_n |z|^{n+p} \leq 1 \quad (4.4)$$

Now making use of (4.2), we observe that (4.4) hold true if:

$$|z|^{n+p} \leq \frac{(p-\theta)(n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)}{(2p+n-\theta)(A-B)(p-\lambda)} \quad p \in \mathbb{N}$$

Hence

$$|z| \leq \left\{ \frac{(p-\theta)(n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)}{(2p+n-\theta)(A-B)(p-\lambda)} \right\}^{\frac{1}{n+p}} \quad (4.5)$$

Setting $|Z| = r(q, s, \alpha_1, A, B, \lambda)$ in (4.5) we get the radius of starlikeness, which completes the proof of the Theorem 4.

The result is sharp for the function given by:

$$F_n(z) = \frac{1}{z^p} + \frac{(A-B)(p-\lambda)}{(n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)} z^n.$$

Theorem 5 :

If $f(z) \in \Sigma_p^+(q, s, \alpha_1, A, B, \lambda)$, Then $f(z)$ is moromorphicmultiva

Lent convex of order Θ ($0 \leq \Theta < p$) in the disk $|Z| < r_2$

$(q, s, \alpha_1, A, B, \lambda, \Theta)$ we have

$$r_2(q, s, \alpha_1, A, B, \lambda, \Theta) = \inf_{in} \left\{ \frac{p(p - \Theta(n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right))}{n(2p + n - \Theta)(A-B)(p-\lambda)} \right\}^{\frac{1}{n+p}} \quad (4.6)$$

The result (4.6) is sharp .

Proof :- let $f(z) \in \Sigma_p^+(q, s, \alpha, A, B, \lambda)$ Then by Theorem 2

$$\sum_{n=p}^{\infty} \frac{(n+p)(1+B) + (A+B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)}{(A-B)(p-\lambda)} a_n \leq 1$$

It is sufficient to show that :-

$$\left| \frac{zf'(z)}{f(z)} + 1 + p \right| \leq p - \Theta \quad (4.7)$$

For $|Z| < r_2(q, s, \alpha_1, A, B, \lambda, \Theta)$ where $r_2(q, s, \alpha_1, A, B, \lambda, \Theta)$

Is given by (4.6)

Since $f(z) = \frac{1}{|z|^p} + \sum_{n=p}^{\infty} a_n z^n$. Then

$zf'(z) = -p|z|^{-p-1} + \sum_{n=p}^{\infty} n a_n z^{n-1}$. There for :

$$\left| \frac{zf'(z)}{f(z)} + 1+p \right| = \left| \frac{\sum_{n=p}^{\infty} n(n+p)a_n z^{n+p}}{-p + \sum_{n=p}^{\infty} n a_n z^{n+p}} \right|$$

$$\leq \frac{\sum_{n=p}^{\infty} n(n+p)a_n |z|^{n+p}}{p - \sum_{n=p}^{\infty} n a_n |z|^{n+p}} \leq p - \Theta ,$$

Provided that :

$$\sum_{n=p}^{\infty} \frac{n(2p+n+\Theta)}{p(p-\Theta)} a_n |z|^{n+p} \leq 1 \quad (4.8)$$

Holds true if :

$$|z|^{n+p} \leq \frac{(p(p-\theta)(n+p)(1+B) + (A-B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)}{n(2p+n-\theta)(A-B)(p-\lambda)}$$

, $p \in \mathbb{N}$

Hence .

$$|z| \leq \left\{ \frac{(p(p-\theta)(n+p)(1+B) + (A+B)(p-\lambda)) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)^{\frac{1}{n+p}}}{n(2p+n-\theta)(A-B)(p-\lambda)} \right\}^{\frac{1}{n+p}}$$

4.9)

Setting $|z| = r_2(q, s, \alpha_1, A, B, \lambda, \Theta)$ in (4.9), we get the radians of convexity, which completes the proof of Theorem 5.

The result is sharp for the function given by :-

$$F_n(z) = \frac{1}{z^p} + \frac{(A-B)(p-\lambda)}{((n+p)(1+B)(A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)} z^n.$$

5- convex combination:

In condinnation of our study we shell prove that the class $\sum_p^+(q, s, \alpha_1, A, B, \lambda)$ is closed under convex combination .

Theorem 6:- let The functions $F_k(z)$ be given by :

$$F_k(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n k z^n, k=1, \dots, m, a_n, k \geq 0 \quad (5.1)$$

Be in the class $\sum_p^+(q, s, \alpha_1, A, B, \lambda)$ for every $k=1, 2, \dots, m$.

Then the function , $g(z)$ defined by :

$$g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n, b_n \geq 0 \quad (5.2)$$

Is also in the same class $\sum_p^+(q, s, \alpha_1, A, B, \lambda)$ where

$$b_n = \frac{1}{m} \sum_{k=1}^m a_n, k(n=p, p+1, \dots) \quad (5.3)$$

proof :- since $F_k(z) \in \sum_p^+(q, s, \alpha_1, A, B, \lambda)$ by Theorem 2 ,That

$$\begin{aligned} \sum_{n=p}^{\infty} (n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) a_n \\ \leq (A-B)(p-\lambda), p \in \mathbb{N} \end{aligned}$$

Hence

$$\sum_{n=p}^{\infty} (n+p)(1+B) + (A-B)(p-\lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) b_n$$

$$= \sum_{n=p}^{\infty} (\mathbf{n} + \mathbf{p})(\mathbf{1} + \mathbf{B}) + (\mathbf{A} - \mathbf{B})(\mathbf{p} - \lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right)$$

$$\left[\frac{1}{m} \sum_{k=1}^m a_n, k \right]$$

$$= \frac{1}{m} \sum_{k=1}^m \sum_{n=p}^{\infty} (\mathbf{n} + \mathbf{p})(\mathbf{1} + \mathbf{B}) - (\mathbf{A} - \mathbf{B})(\mathbf{p} - \lambda) \left(\frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n n!} \right) \mathbf{a}_n, \mathbf{k}$$

$$\leq (\mathbf{A} - \mathbf{B})(\mathbf{p} - \lambda)$$

Then by Theorem 2 , we get $g(z) \in \sum_p^+(q, s, \alpha_1, A, B, \lambda)$

This completes The proof .

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قسم الرياضيات

تطبيق صنف من الدوال متعددة التكافؤ الميرومورفية المعرفة
بواسطة المؤثر الخطي

بحث مقدم الى قسم الرياضيات/كلية علوم الحاسوب وتكنولوجيا
المعلومات جامعة القادسية كأستكمال جزئي لمتطلبات نيل درجة
بكالوريوس علوم في الرياضيات

من قبل الطالبة:

دعاء محمد جبر

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