



وزارة التعليم العالي والبحث العلمي
جامعة القادسية
كلية علوم الحاسوب وتكنولوجيا
المعلومات

ON d -IDEALS IN d -ALGEBRAS

بحث

مُقدم الى كلية علوم الحاسوب وتكنولوجيا المعلومات/قسم الرياضيات
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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

((وَاقِلْ رَأْيِي زَادَ نِيَّ حَلِيًّا))

حَسْبُكَ اللَّهُ الْعَلِيُّ الْعَظِيمُ

كلمة شكر

لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة ... في بناء جيل الغد لتبعث الأمة من جديد وقبل أن نمضي تقدم أسمى آيات الشكر والامتنان والتقدير . والمحبة إلى الذين حملوا أقدس رسالة في الحياة . إلى الذين مهدوا لنا طريق العلم والمعرفة إلى جميع أساتذتنا الأفاضل كن عالما .. فإن لم تستطع فكن متعلما ، فإن لم تستطع فأحب " العلماء ، فإن لم تستطع فلا تبغضهم وأخص بالتقدير والشكر

الاستاذ ((احمد طالب)

الذي نقول له بشراك قول رسول الله ﷺ
"إن الحوت في البحر ، والطير في السماء ، ليصلون على معلم
الناس الخير "

الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب
إلى من كلت أنامله ليقدّم لنا لحظة سعادة
إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم
إلى القلب الكبير
(والدي العزيز)

إلى من أرضعتني الحب والحنان
إلى رمز الحب وبلسم الشفاء
إلى القلب الناصع بالبياض
(والدتي الحبيبة)

إلى القلوب الطاهرة الرقيقة والنفوس البريئة إلى رياحين
حياتي
(إخوتي)

الآن تفتح الأشرعة وترفع المرساة لتنتطلق السفينة في
عرض بحر

واسع مظلم هو بحر الحياة وفي هذه الظلمة لا يضيء إلا قنديل
الذكريات ذكريات الأخوة البعيدة إلى الذين أحببتهم وأحبوني
(أصدقائي)

ABSTRACT. We introduce the notions of $\mathcal{A}$ -subalgebra, $\mathcal{A}$ -ideal, $\mathcal{A}^\#$ -ideal and $\mathcal{A}^*$ -ideal in $\mathcal{A}$ -algebras, and investigate relations among them. Furthermore, we are able to define the idea of a quotient $\mathcal{A}$ -algebra and to prove a fundamental theorem of $\mathcal{A}$ -morphisms for $\mathcal{A}$ -algebras as a consequence.

1. Introduction

Y. Imai and K. Iseki [II] and K. Iseki [Is] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1], [2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H.S. Kim [NK] introduced the notion of $\mathcal{A}$ -algebras which is another generalization of BCK-algebras, and investigated relations between $\mathcal{A}$ -algebras and BCK-algebras. In this paper we discuss the ideal theory in $\mathcal{A}$ -algebras. We introduce the notions of $\mathcal{A}$ -subalgebra, $\mathcal{A}$ -ideal, $\mathcal{A}^\#$ -ideal and $\mathcal{A}^*$ -ideal, and investigate relations among them. Furthermore, we are able to define the idea of a quotient $\mathcal{A}$ -algebra and to prove a fundamental theorem of $\mathcal{A}$ -morphisms for $\mathcal{A}$ -algebras as a consequence.

2. Preliminaries

DEFINITION 1: A *d-algebra* is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

- (I) $x * x = 0$,
- (II) $0 * x = 0$,
- (III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X .

DEFINITION : A *BCK-algebra* is a *d-algebra* $(X, *, 0)$ satisfying the following additional axioms:

- (IV) $((x * y) * (x * z)) * (z * y) = 0$,
- (V) $(x * (x * y)) * y = 0$

for all x, y, z in X .

In a BCK-algebra $(X, *, 0)$ the following hold:

- (1) $(x * y) * x = 0$,
- (2) $((x * z) * (y * z)) * (x * y) = 0$

for arbitrary $x, y, z \in X$.

A non-empty subset I of a BCK-algebra X is called a *BCK-ideal* of X if

- (i) $0 \in I$
- (ii) $x \in I$ and $y * x \in I$ imply $y \in I$,

for all $x, y \in X$.

PROPOSITION 2.1 .let X be a *d-algebra*. If $x \neq y$ and $x * y = 0$ then $y * x \neq 0$.

Proof :- By (III), it is straightforward. •

3. d-ideals

DEFINITION 3.1 . Let $(X, *, 0)$ be a *d-algebra* and $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies:

- (D_0) $0 \in I$,
- (D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

- (D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$

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EXAMPLE 3.2. Let $X = \{0, a, b, c, d\}$ be a d -algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

Then $I = \{0, a\}$ is a d -ideal of X .

EXAMPLE 3.3. Let $X = \{0, a, b, c\}$ be a d -algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Then $J = \{0, a, c\}$ satisfies (D_2) , but not (D_1) since $b * c = 0 \in J$ and $c \in J$, but $b \notin J$, i.e., J is a d -subalgebra, but not a BCK-ideal of X .

In a d -algebra, a BCK-ideal need not be a d -subalgebra, and also a d -subalgebra need not be a BCK-ideal as shown in the following example.

EXAMPLE 3.4. Let $X = \{0, a, b, c\}$ be a d -algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	0
c	c	c	c	0

Then $I = \{0, a, b\}$ is a BCK-ideal which is not a d-subalgebra of X , while $J = \{0, c\}$ is a d-subalgebra which is not a BCK-ideal of X

Clearly, $\{0\}$ is a d-subalgebra of every d-algebra X and every d-ideal of X is a d-subalgebra, but the converse need not be true.

EXAMPLE 3.5. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $I = \{0, a\}$ is a d-subalgebra of X , but not a d-ideal of X since $a * c = b \notin I$.

LEMMA 3.6. *If I is a d-ideal of a d-algebra X , then $0 \in I$.*

Proof. Since $I \neq \emptyset$, there exists x in I and hence $0 = x * x \in I$ by (D_2) .

Note that every d-ideal of a d-algebra is a BCK-ideal, but the converse need not be true. In Example 3.5, $I = \{0, a\}$ is a BCK-ideal of X but not a d-ideal of X

PROPOSITION 3.7. *Let I be a d-ideal of a d-algebra X . If $x \in I$ and $y * x = 0$, then $y \in I$.*

Proof. Assume that $x \in I$ and $y * x = 0$. By Lemma 3.6 and (D_1) , we have $y \in I$. This completes the proof. •

DEFINITION 3.8. Let X be a d-algebra. A d-ideal I of X is called a $d^\#$ -ideal of X if, for arbitrary $x, y, z \in X$,

$$(D_3) \quad x * z \in I \text{ whenever } x * y \in I \text{ and } y * z \in I.$$

EXAMPLE 3.9. Let X be a d-algebra as in Example 3.5. Then $K = \{0, a, b\}$ is a $d^\#$ -ideal of X .

Obviously, every $d^\#$ -ideal is a d-ideal, but the converse need not be true.

EXAMPLE 3.10. Let X be a d -algebra as in Example 3.2. Then $L = \{0, a\}$ is a d -ideal which is not a $d^\#$ -ideal of X , since $b * d = 0 \in L$, $d * c = a \in L$, but $b * c = c \notin L$.

Note that we can see that $d^\#$ -ideal $\subsetneq$ d -ideal $\subsetneq$ d -subalgebra and $d^\#$ -ideal $\subsetneq$ d -ideal $\subsetneq$ BCK-ideal in d -algebras.

In a d -algebra X the identity $(x * y) * x = 0$ does not hold in general. For instance, in Example 3.5, we know that $(a * c) * a = b * a = b \neq 0$.

DEFINITION 3.11 . A d -algebra X is called a d^* -algebra if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^* -algebra, but the converse need not be true.

EXAMPLE 3.12. Let $X = \{0, 1, 2, \dots\}$ and let the binary operation $*$ be defined as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{otherwise} \end{cases}$$

Then $(X, *, 0)$ is a d -algebra which is not a BCK-algebra (see [NK; Example 2.8]). We can easily see that $(X, *, 0)$ is a d^* -algebra.

THEOREM 3.13 . In a d^* -algebra, every BCK-ideal is a d -ideal.

Proof . Let I be a BCK-ideal of a d^* -algebra X and let $x \in I$, $y \in X$. Since $(x * y) * x = 0$ for all $x, y \in X$, it follows from Proposition 3.7 that $x * y \in I$. Hence I is a d -ideal of X . $\bullet$

The following corollary is obvious.

COROLLARY 3.14. In a d^* -algebra, every BCK-ideal is a d -subalgebra.

DEFINITION 3.15. If a $d^\#$ -ideal I of a d -algebra X satisfies

(D_4) $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$ for all $x, y, z \in X$, then we say that I is a d^* -ideal of X .

In Example 3.3, the set $I = \{0, a\}$ is a d^* -ideal of X . Obviously, every d^* -ideal in a d -algebra is a $d^\#$ -ideal, but the converse does not hold in general.

EXAMPLE 3.16. Let $X = \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

Then $(X, *, 0)$ is a d-algebra, but not a BCK-algebra. We can see that $I = \{0, a\}$ is a $d^\#$ -ideal, but not d^* -ideal, since $0 * a = 0 \in I$ and $a * 0 = a \in I$, but $(c * 0) * (c * a) = c * b = b \notin I$.

LEMMA 3.17. (Iseki et al. [IT1]) *Let I be a BCK-ideal of a BCK-algebra X . If $x \in I$ and $y * x = 0$ then $y \in I$.*

THEOREM 3.18. *If $(X, *, 0)$ is a BCK-algebra, then every BCK-ideal of X is a d^* -ideal of X .*

Proof : Let I be a BCK-ideal of X and let $x \in I$ and $y \in X$. Since $(x * y) * x = 0$ by (1), it follows from Lemma 3.17 that $x * y \in I$, proving (D_2) .

Assume that $x * y \in I$ and $y * z \in I$ for all $x, y, z \in I$. Then $((x * z) * (y * z)) * (x * y) = 0$ by (2), and hence $(x * z) * (y * z) \in I$. Since $y * z \in I$ and since I is a BCK-ideal of X , it follows that $x * z \in I$. This proves (D_3) .

Let $x * y, y * x \in I$ for all $x, y \in X$. Then, by (IV) and (2), we have

$$((z * x) * (z * y)) * (y * x) = 0 \quad \text{and} \quad ((x * z) * (y * z)) * (x * y) = 0,$$

respectively. It follows from Lemma 3.17 that $(z * x) * (z * y) \in I$ and $(x * z) * (y * z) \in I$ proving (D_4) . This completes the proof. •

Remark 3.19.

- (i) In a d^* -algebra, the concept of d-ideal, d-subalgebra and BCK-ideal coincide.
- (ii) In a BCK-algebra, the concept of d-ideal, $d^\#$ -ideal, d^* -ideal and BCK-ideal coincide.

4. Quotient d-algebras

Let $(X, *, 0_x)$ and $(Y, *, 0_y)$ be d-algebras. A mapping $f : X \rightarrow Y$ is called a *d-morphism* ([NK]) if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(0_x) = 0_y$. A d-algebra $(X, *, 0_x)$ is said to be *d-transitive* ([NK]) if $x * z = 0$ and $z * y = 0$ imply $x * y = 0$. Every BCK-algebra is a d-transitive d-algebra, but the converse does not hold in general. See Example 3.2.

Let I be a d^* -ideal of a d-algebra $(X, *, 0_x)$. For any x, y in X , we define $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. We claim that $\sim$ is an equivalence relation on X . Since $0 \in I$, we have $x * x = 0 \in I$, i.e., $x \sim x$, for any $x \in X$. If $x \sim y$ and $y \sim z$, then $x * y, y * x \in I$ and $y * z, z * y \in I$. By (D_3) $x * z, z * x \in I$ and hence $x \sim z$. This proves that $\sim$ is transitive. The symmetry of $\sim$ is trivial. By (D_4) we can easily see that $\sim$ is a congruence relation on X . Using the notion of d-transitivity we obtain:

PROPOSITION 4.1 . Let $f : X \rightarrow Y$ be a d -morphism from a d -algebra X into a d -transitive d -algebra Y . Then $\text{Ker } f$ is a d^* -ideal of X .

Proof . The properties D_1 and D_2 are simple. If $x * y, y * z \in \text{Ker } f$, then $f(x) * f(y) = 0_y = f(y) * f(z)$. Since Y is d -transitive, we obtain $f(x) * f(z) = 0$ and hence $x * z \in \text{Ker } f$, which proves (D_3) . Let $x * y, y * x \in \text{Ker } f$. Then $f(x) * f(y) = 0_y = f(y) * f(x)$. By (III) we obtain $f(x) = f(y)$. It follows that $f(x * z) * f(y * z) = f(x * z) * f(y * z) = (f(x) * f(z)) * f(y) * f(z) = 0_y$ and hence $(x * z) * (y * z) \in \text{Ker } f$. Similarly, $(z * x) * (z * y) \in \text{Ker } f$, which proves (D_4) . •

EXAMPLE 4.2. Let X be a d -algebra as in Example 3.3, and let Y be a d -transitive d -algebra as in Example 3.2. Define a map $f : X \rightarrow Y$ by $f(0) = f(a) = 0$, $f(b) = f(c) = a$. Then f is a d -morphism. Obviously, $\text{Ker } f = \{0, a\}$ is a d^* -ideal of X .

We denote the congruence class containing x by $[x]_I$, i.e., $[x]_I = \{y \in X \mid x \sim y\}$. We see that $x \sim y$ if and only if $[x]_I = [y]_I$. Denote the set of all equivalence classes of X by X/I , i.e., $X/I = \{[x]_I \mid x \in X\}$.

LEMMA 4.3. Let I be a d^* -ideal of a d -algebra $(X, *, 0)$. Then $I = [0]_I$.

Proof . If $x \in I$ then $x * 0 \in I * x \subseteq I$ and hence $x \in [0]_I = \{x \in X \mid \text{i.e. } I \subseteq [0]_I$

since

$$\begin{aligned} [0]_I &= \{x \in X \mid x \sim 0, \} \\ &= \{x \in X \mid x * 0, 0 * x \in I\} \\ &= \{x \in X \mid x * 0 \in I\} & (0 \in I) \\ &\subseteq I & ((D1)) \end{aligned}$$

it follows that $I = [0]_I$.

THEOREM 4.4. let $(X, *, 0)$ be a d -algebra and I be a d^* -ideal of X . If we define $[x]_I * [y]_I = [x * y]_I$ ($x, y \in X$), then $(X/I, *, 0)$ is a d -algebra, called the quotient d -algebra.

Proof . Since $\sim$ is a congruence relation on X , $x * y \sim x' * y'$ for any $x \sim x', y \sim y'$. This means that $[x]_I * [y]_I = [x * y]_I$ is well-defined. Let $[x]_I, [y]_I \in X/I$ with $[x]_I * [y]_I = [0]_I = [y]_I * [x]_I$. Then $[x * y]_I = [0]_I = [y * x]_I$ and $x * y, y * x \in I$. Thus $x \sim y$ and $[x]_I = [y]_I$. The rest is trivial, and so we omit the proof. •

PROPOSITION 4.5 . Let I be a d^* -ideal of the d -algebra X . Then the mapping $\pi : X \rightarrow X/I$ defined by $\pi(x) = [x]_I$ is a d -morphism of X onto the quotient d -algebra X/I and the kernel of π is precisely the set I .

Proof . Since $[x * y]_I = [x]_I * [y]_I$, π is a d-morphism. By Lemma 4.3 we know that

$$\begin{aligned} \text{Ker } \pi &= \{x \in X \mid \pi(x) = [0]_I\} \\ &= \{x \in X \mid [x]_I = [0]_I\} \\ &= \{x \in X \mid x \sim 0\} \\ &= [0]_I \\ &= I \end{aligned}$$

THEOREM 4.6 . If $f : X \rightarrow Y$ is a d-morphism from a d-algebra X onto a d-transitive d-algebra Y , then $X/\text{Ker } f \cong Y$.

Proof . Assume $\mu : X/\text{Ker } f \rightarrow Y$ such that $\mu([x]_{\text{Ker } f}) = f(x)$. If

$[x]_{\text{Ker } f} = [y]_{\text{Ker } f}$ then $x * y, y * x \in \text{Ker } f$, and so $f(x) * f(y) = 0 = f(y) * f(x)$.

By

(III) we have $f(x) = f(y)$, i.e., $\mu([x]_{\text{Ker } f}) = \mu([y]_{\text{Ker } f})$. This means that μ is well-defined. For any $y \in Y$, there is an $x \in X$ such that $y = f(x)$ since f is onto. Hence $\mu([x]_{\text{Ker } f}) = f(x) = y$, which means that μ is onto. If $\mu([x]_{\text{Ker } f}) \neq \mu([y]_{\text{Ker } f})$ then either $x * y \notin \text{Ker } f$ or $y * x \notin \text{Ker } f$. Without loss of generality, we may assume $x * y \notin \text{Ker } f$. It follows that $f(x) * f(y) = f(x * y) \neq 0$ and hence $f(x) \neq f(y)$. This means that μ is one-one. Since $\mu([x]_{\text{Ker } f}) * \mu([y]_{\text{Ker } f}) = \mu([x * y]_{\text{Ker } f}) = f(x * y) = f(x) * f(y) = \mu([x]_{\text{Ker } f}) * \mu([y]_{\text{Ker } f})$, μ is a d-morphism. Thus we have $X/\text{Ker } f \cong Y$, completing the proof. •

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