



وزارة التعليم العالي والبحث العلمي

جامعة القادسية

كلية علوم الحاسوب وتكنولوجيا

Some properties for class of Meromorphic Univalent Function Involving Linear operator

بحث

مقدم الى كلية علوم الحاسوب وتكنولوجيا المعلومات/قسم الرياضيات

جامعة القادسية كمتطلب أساسي لنيل درجة البكالوريوس

في علوم الرياضيات

من قِبَل الطالب

سجاد عبدالله شبلاوي

تحت إشراف

المدرس المساعد : عباس كريم وناس

2017

1438

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ (سُورَةُ الْحَمْدِ)

((وَاقِلْ رَأْيِي زَادَ نِيَّ حَلِيمًا))

حَسْبُكَ اللَّهُ (سُورَةُ الْحَمْدِ)

كلمة شكر

لابد لنا ونحن نخطو خطواتنا الأخيرة في الحياة الجامعية من
وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا
الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة

في بناء جيل الغد لتبعث الأمة من جديد...
وقبل أن نمضي تقدم أسمى آيات الشكر والامتنان والتقدير
والمحبة إلى الذين حملوا أقدس رسالة في الحياة.
إلى الذين مهدوا لنا طريق العلم والمعرفة.
إلى جميع أساتذتنا الأفاضل

"كن عالما .. فإن لم تستطع فكن متعلما ، فإن لم تستطع فأحب
العلماء ، فإن لم تستطع فلا تبغضهم"
وأخص بالتقدير والشكر:

الاستاذ (عباس كريم وناس)

الذي نقول له بشراك قول رسول الله صلى الله عليه وسلم:
إن الحوت في البحر ، والطير في السماء ، ليصلون على معلم"
الناس الخير "

الإهداء

إلى من جرع الكأس فارغاً ليسقيني قطرة حب
إلى من كلّت أنامله ليقدّم لنا لحظة سعادة
إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم
إلى القلب الكبير

(والدي العزيز)

إلى من أرضعتني الحب والحنان
إلى رمز الحب وبلسم الشفاء
إلى القلب الناصع بالبياض

(والدتي الحبيبة)

إلى القلوب الطاهرة الرقيقة والنفوس البريئة إلى رياحين
حياتي

(إخوتي)

الآن تفتح الأشرعة وترفع المرساة لتنتطلق السفينة في

عرض بحر

واسع مظلم هو بحر الحياة وفي هذه الظلمة لا يضيء إلا قنديل
الذكريات ذكريات الأخوة البعيدة إلى الذين أحببتهم وأحبوني

(أصدقائي)

Abstract

By making use of certain extended derivative operator of Ruscheweyh type , We introduce and investigate class of Meromorphic univalent analytic function in the punctured unit disk. The results obtained here for this class include coefficients estimates, Growth and distortion theorems, Closure theorems, Neighborhoods and integral operators.

2010 AMS Subject Classification:30C45,30C50,30C55

Key words and phrases:

Meromorphic functions, Ruscheweyh derivative , Hadamard product, neighborhood , Integral operators.

1-Introduction:

Let Σ denote the class of normalized Meromorphic functions f the form:

$$f_{(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$$

Define on the punctured unit disk $U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$

A function $f \in \Sigma$ is meromorphic starlink of order $\alpha(0 \leq \alpha < 1)$ if

$$-R_e \left(\frac{f'_{(z)}}{f_{(z)}} \right) > \alpha .$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$.

A function $f \in \Sigma$ is meromorphic convex of order $\alpha(0 \leq \alpha < 1)$ if

$$-R_e \left(1 + \frac{f''_{(z)}}{f'_{(z)}} \right) > \alpha .$$

The class of all such functions is denoted by $\Sigma_k(\alpha)$.

Note that $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ are subclass of Σ consisting meromorphic univalent functions which are respectively starlike

And convex of order $\alpha(0 \leq \alpha < 1)$. We refer to Liu and Srivastava [2], Mogra[3], Raina and Srivastava [4] and Xu and Yang[7] for related work on the subject of meromorphic functions.

For functions f given by (1.1 and

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \quad (1.4)$$

We define the Hadamard product or convolution of f and g by

$$(f * g)_{(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k. \quad (1.5)$$

The extended linear derivative operator of Ruscheweyh type for the functions belongs to the class Σ

$$D_*^\lambda: \Sigma \rightarrow \Sigma$$

Is defined by the following convolution :

$$D_*^\lambda f_{(z)} = \frac{1}{z(1-z)^{\lambda+1}} * f_{(z)} \quad , \quad (\lambda > -1, f \in \Sigma) \quad (1.6)$$

In terms of binomial coefficients, (1.6) can be written as

$$D_*^\lambda f_{(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} \binom{\lambda+k}{k} a_k z^k. \quad (1.7)$$

In particular when $\lambda=n$ ($n \in N_0 = N \cup \{0\}$), it is easily observed from (1.6) and (1.7) that

$$D_*^n f_{(z)} = \frac{(z^{n+1} f_{(z)})^{(n)}}{z n!}. \quad (1.8)$$

The definition (1.6) of linear operator D_*^λ is motivated essentially by familiar Ruscheweyh operator D^λ which has been used widely on the space of analytic and univalent functions (see for details [5]).

Now , by using the operator D_*^λ ($\lambda > -1$) given by (1.6), we introduce a class of meromorphically univalent analytic functions defined as follows:

Definition 1.1: A function $f \in \Sigma$ is said to be in the class $J(\lambda, \alpha, \beta)$ if and only if :

$$Re \left(\frac{z(D_*^\lambda f(z))'}{(\beta-1)D_*^\lambda f(z) + \beta z(D_*^\lambda f(z))'} \right) > \alpha \quad (1.9)$$

Where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and ($\lambda > -1$)

2-Coefficients Estimates

Our first theorem gives a necessary and Sufficient condition for a function f to be in the class $J(\lambda, \alpha, \beta)$.

Theorem 2.1 let $f \in \Sigma$ be given by (1.1). then $f \in J(\lambda, \alpha, \beta)$ if and only if

$$\sum_{k=1}^{\infty} [k + \alpha - \alpha\beta(1 + k)] \binom{\lambda+k}{k} a_k \leq 1 - \alpha \quad (2.1)$$

Poof: it $f \in J(\lambda, \alpha, \beta)$, then

$$\begin{aligned} & Re \left(\frac{z(D_*^\lambda f(z))'}{(\beta-1)D_*^\lambda f(z) + \beta z(D_*^\lambda f(z))'} \right) \\ &= Re \left(\frac{-1 + \sum_{k=1}^{\infty} k \binom{\lambda+k}{k} z^{k+1}}{-1 + \sum_{k=1}^{\infty} (\beta-1 + \beta k) \binom{\lambda+k}{k} a_k z^{k+1}} \right) > \alpha \end{aligned}$$

By letting $z \rightarrow 1$, we have

$$\left\{ \frac{-1 + \sum_{k=1}^{\infty} k \binom{\lambda+k}{k} a_k}{-1 + \sum_{k=1}^{\infty} (\beta-1 + \beta k) \binom{\lambda+k}{k} a_k} \right\} > \alpha$$

This shows that (2.1) hold.

Conversely, assume that (2.1) hold.

Since $R_e(w) > \alpha$ if and only if $|w-1| < |w+1-2\alpha|$, it is Sufficient to show that

$$\left| \frac{z(D_*^\lambda f_{(z)})' - [(\beta - 1)D_*^\lambda f_{(z)} + \beta z(D_*^\lambda f_{(z)})']}{z(D_*^\lambda f_{(z)})' + (1 - 2\alpha)[(\beta - 1)D_*^\lambda f_{(z)} + \beta z(D_*^\lambda f_{(z)})']} \right| < 1$$

Using (2.1), we see that

$$\begin{aligned} & \left| \frac{z(D_*^\lambda f_{(z)})' - [(\beta - 1)D_*^\lambda f_{(z)} + \beta z(D_*^\lambda f_{(z)})']}{z(D_*^\lambda f_{(z)})' + (1 - 2\alpha)[(\beta - 1)D_*^\lambda f_{(z)} + \beta z(D_*^\lambda f_{(z)})']} \right| \\ &= \left| \frac{\sum_{k=1}^{\infty} (1-\beta)(k+1) \binom{\lambda+k}{k} a_k z^{k+1}}{-(1-2\alpha) + \sum_{k=1}^{\infty} [k(1+(1-2\alpha)\beta) + (1-2\alpha)(\beta-1)] \binom{\lambda+k}{k} a_k z^{k+1}} \right| \\ &\leq \left| \frac{\sum_{k=1}^{\infty} (1-\beta)(k+1) \binom{\lambda+k}{k} a_k}{(1-2\alpha) - \sum_{k=1}^{\infty} [k(1+(1-2\alpha)\beta) + (1-2\alpha)(\beta-1)] \binom{\lambda+k}{k} a_k} \right| \\ &\leq 1 \end{aligned}$$

Tuse we have $f \in J(\lambda, \alpha, \beta)$.

For the sake of brevity throughout this paper we let

$$d_k(\beta, \alpha) = k + \alpha - \alpha\beta(1 + k)$$

Corollary 2.1 If $f \in J(\lambda, \alpha, \beta)$, then

$$a_k \leq \frac{1-\alpha}{d_k(\beta, \alpha) \binom{\lambda+k}{k}}, \quad k=1, 2, 3, \dots$$

The result is sharp for the function f_k given by

$$f_k(z) = \frac{1}{z} + \frac{1-\alpha}{d_k(\beta, \alpha) \binom{\lambda+k}{k}} z^k \quad k=1, 2, 3, \dots \quad (2.2)$$

Proof : If $f \in J(\lambda, \alpha, \beta)$, then we have for each k

$$d_k(\beta, \alpha) a_k \leq \sum_{k=1}^{\infty} d_k(\beta, \alpha) \binom{\lambda+k}{k} a_k \leq 1 - \alpha$$

Therefore we have

$$a_k \leq \frac{1 - \alpha}{d_k(\beta, \alpha) \binom{\lambda+k}{k}}.$$

3.Growth and Distortion Theorems

Theorem 3.1 if $f \in J(\lambda, \alpha, \beta)$, then for $|z|=r<1$

$$\frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} r \leq |f(z)| \leq \frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} r \quad (3.1)$$

The result is sharp for the function given by

$$f(z) = \frac{1}{z} + \frac{1 - \alpha}{(1 + \alpha - 2\alpha\beta)(\lambda + 1)} z$$

Proof: let $f \in J(\lambda, \alpha, \beta)$. then we have

$$\sum_{k=1}^{\infty} a_k \leq \frac{1 - \alpha}{(1 + \alpha - 2\alpha\beta)(\lambda + 1)}$$

Hence

$$\begin{aligned} |f(z)| &\leq \frac{1}{r} + \sum_{k=1}^{\infty} a_k r^k \\ &\leq \frac{1}{r} + r \sum_{k=1}^{\infty} a_k \\ &\leq \frac{1}{r} + \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} r \end{aligned} \quad (3.3)$$

And

$$\begin{aligned}
 |f_{(z)}| &\geq \frac{1}{r} - \sum_{k=1}^{\infty} a_k r^k \\
 &\geq \frac{1}{r} - r \sum_{k=1}^{\infty} a_k \\
 &\geq \frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} r \quad (3.4)
 \end{aligned}$$

From (3.3) and (3.4), we obtain (3.1) and the proof is complete

Theorem 3.2 if $f \in J(\lambda, \alpha, \beta)$, then for $|z|=r<1$

$$\frac{1}{r^2} - \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} \leq |f'_{(z)}| \leq \frac{1}{r^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)} \quad (3.5)$$

The result is sharp for the function given by (3.2).

Proof: let $f \in J(\lambda, \alpha, \beta)$. then we have

$$\sum_{k=1}^{\infty} k a_k \leq \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)}$$

Hence

$$\begin{aligned}
 |f'_{(z)}| &\leq \frac{1}{r^2} + \sum_{k=1}^{\infty} k a_k r^{k-1} \\
 &\leq \frac{1}{r^2} + \sum_{k=1}^{\infty} k a_k \\
 &\leq \frac{1}{r^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(\lambda+1)}
 \end{aligned}$$

And

$$\begin{aligned}
|f'_{(z)}| &\geq \frac{1}{r^2} - \sum_{k=1}^{\infty} k a_k r^{k-1} \\
&\geq \frac{1}{r^2} - \sum_{k=1}^{\infty} k a_k \\
&\geq \frac{1}{r^2} - \frac{1 - \alpha}{(1 + \alpha - 2\alpha\beta)(\lambda + 1)}
\end{aligned}$$

From (3.6) and (3.7), we obtain (3.5) and the proof is complete .

4- closure theorems

Let the function $f_j(z)$ be given by

$$F_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k, \quad j=1,2,\dots,m \quad (4.1)$$

we shall prove the following closure theorems for the class $J(\lambda, \alpha, \beta)$

theorem 4.1 :- let the function f_j defined by (4.1) be in the class $J(\lambda, \alpha, \beta)$

for every $j=1,2,\dots,m$. then the function

$$h_1(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^m a_{k,j} \right) z^k$$

Belongs to the class $J(\lambda, \alpha, \beta)$.

Proof: since $F_j \in J(\lambda, \alpha, \beta)$. It follows from theorem 2.1 , that

$$\sum_{k=1}^{\infty} d_k(\alpha, \beta) \binom{\lambda+k}{k} a_{k,j} \leq 1-\alpha \quad \forall j = 1, 2, \dots, m \quad (4.2)$$

$$\text{Hence } \sum_{k=1}^{\infty} d_k(\alpha, \beta) \binom{\lambda+k}{k} \left(\frac{1}{m} \sum_{j=1}^m a_{k,j} \right) = \frac{1}{m}$$

$$\sum_{j=1}^m \left(\sum_{k=1}^{\infty} d_k(\alpha, \beta) \binom{\lambda+k}{k} a_{k,j} \right) \leq \frac{1}{m} \sum_{j=1}^m (1-\alpha) = 1-\alpha .$$

Thus $h_j \in J(\lambda, \alpha, \beta)$

Theorem4.2:- the class $J(\lambda, \alpha, \beta)$ is closed under convex linear combination .

Proof:- let the function F_j given by (4.1) be in the class $J(\lambda, \alpha, \beta)$.

Then it is enough to show that the function

$$h_2(z) = t F_1(z) + (1-t)F_2(z) \quad (0 \leq t \leq 1)$$

Is also in the class $J(\lambda, \alpha, \beta)$

We have

$$h_2(z) = \frac{1}{z} \sum_{k=1}^{\infty} [t a_{k,1} + (1-t)a_{k,2}] z^k$$

Hence

$$\sum_{k=1}^{\infty} [t a_{k,1} + (1-t)a_{k,2}] d_k(\alpha, \beta) \binom{\lambda+k}{k}$$

$$\begin{aligned}
&= t \sum_{k=1}^{\infty} d_k(\alpha, \beta) \binom{\lambda+k}{k} a_{k,1} + (1-t) \sum_{k=1}^{\infty} d_k(\alpha, \beta) \binom{\lambda+k}{k} a_{k,2} \\
&\leq t(1-\alpha) + (1-t)(1-\alpha) = 1-\alpha
\end{aligned}$$

Thus , by theorem 2.1 , $h_2 \in J(\lambda, \alpha, \beta)$

5- Neighborhoods for the class $J(\lambda, \alpha, \beta)$

Following the earlier works on Neighborhoods of analytic function by Goodman[1] and Rucheweyh [6] , we begin by introducing here the δ – Neighborhood of a function $f \in \mathcal{E}$ of the form (1.1) means of the definition below

$$N_{\delta}(e) = \{g \in \mathcal{E} : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \text{ and } \sum_{k=1}^{\infty} k|a_k - b_k| \leq \delta\} \quad (5.1)$$

Particular for the identity function $e(z) = \frac{1}{z}$. We have

$$N_{\delta}(e) = \{g \in \mathcal{E} : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k|b_k| \leq \delta\} \quad (5.2)$$

Theorem 5.1 :- if

$$\delta = \frac{1-\alpha}{(1+(1-2\beta)\alpha)(\lambda+1)} , \quad (5.3)$$

Then $J(\lambda, \alpha, \beta) \subset N_{\delta}(e)$

Proof :- For $f \in J(\lambda, \alpha, \beta)$, theorem 2.1 , immediately yields

$$d_1(\beta, \alpha)(\lambda+1) \sum_{k=1}^{\infty} a_k \leq 1-\alpha ,$$

So that

$$\sum_{k=1}^{\infty} a_k \leq \frac{1-\alpha}{d_1(\beta, \alpha)(\lambda+1)} \quad (5.4)$$

On the other hand , from (2.1) and (5.4) that

$$\begin{aligned}
(\lambda + 1) \sum_{k=2}^{\infty} k a_k &\leq (1 - \alpha) - (1 - 2\beta)\alpha(\lambda + 1) \sum_{k=1}^{\infty} a_k \\
&\leq (1 - \alpha) - (1 - 2\beta)\alpha(\lambda + 1) \frac{1-\alpha}{d_1(\beta, \alpha)(\lambda+1)} \\
&\leq \frac{1-\alpha}{1+(1-2\beta)\alpha}
\end{aligned}$$

That is

$$\sum_{k=2}^{\infty} k a_k \leq \frac{1-\alpha}{(1+(1-2\beta)\alpha)(\lambda+1)} = \delta ,$$

Which in view of the definition (5.2) proves theorem (5.1)

Definition 5.1:- A function $f \in \Sigma$ is said to be in the class $J(\lambda, \alpha, \beta)$ if there exists a function $g \in J(\lambda, \alpha, \beta)$ such

That

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma \quad z \in U^* \quad 0 \leq \gamma < 1$$

Theorem 5.2:- if $g \in J(\lambda, \alpha, \beta)$ and

$$\gamma = 1 - \frac{\delta d_1(\beta, \alpha)(\lambda+1)}{d_1(\beta, \alpha)(\lambda+1) - (1-\alpha)} , \quad (5.5)$$

then $N_\delta(g) \subset J(\lambda, \alpha, \beta, \gamma)$.

proof:- let $f \in N_\delta(g)$. then we find from (5.1) that

$$\sum_{k=1}^{\infty} k |a_k - b_k| \leq \delta ,$$

Which implies the coefficient in equality

$$\sum_{k=1}^{\infty} |a_k - b_k| \leq \delta \quad , \quad (k \in N)$$

Since $g \in J(\lambda, \alpha, \beta)$, then we have

$$\sum_{k=1}^{\infty} b_k \leq \frac{1-\alpha}{d_1(\beta, \alpha)(\lambda+1)},$$

So that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{k=1}^{\infty} |a_k - b_k|}{1 - \sum_{k=1}^{\infty} b_k} = \frac{\delta \partial_1(\beta, \alpha)(\lambda+1)}{\partial_1(\beta, \alpha)(\lambda+1) - (1-\alpha)} \\ &= 1 - \gamma, \end{aligned}$$

Where γ is given by (5.5).

Hence by definition 5.1, $f \in J(\lambda, \alpha, \beta, \gamma)$ and the proof is complete.

6-In tegral operators

In this section, we consider in tegrals forms of functions in the class $J(\lambda, \alpha, \beta)$.

Theorem 6.1. Let the function f given by (1.1) be in the class $J(\lambda, \alpha, \beta)$

Then the in tegral operator

$$G(z) = c \int_0^1 u^c f(uz) du \quad 0 < u < 1, 0 < c < \infty$$

Is in the class $J(\lambda, \alpha, \beta)$, where

$$\delta \leq \frac{(c+k+1)(\beta-\alpha)d_k - ck(1-\alpha)}{ck(1-\alpha)[1-\beta(1-\alpha)] + d_k(\beta-\alpha)(c+K+1)}$$

The result is sharp the function f given by

$$F(z) = \frac{1}{z} + \frac{1-\alpha}{(1+\alpha-2\alpha\beta)(1-\lambda)} z.$$

Proof:- let $f \in J(\lambda, \alpha, \beta)$ then

$$G(z) = c \int_0^1 u^c f(uz) du = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c}{c+k+1} a_k z^k.$$

It is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{cd_k(\beta, \delta) \binom{\lambda+k}{k}}{(c+k+1)(1-\delta)} a_k \leq 1 \quad (6.1)$$

Since $f \in J(\lambda, \alpha, \beta)$ we have

$$\sum_{k=1}^{\infty} \frac{d_k(\beta, \alpha) \binom{\lambda+k}{k}}{(1-\alpha)} a_k \leq 1$$

Note that (6.1) is satisfied if

$$\frac{c d_k(\beta, \delta) \binom{\lambda+k}{k}}{(c+k+1)(1-\delta)} \leq \frac{d_k(\beta, \alpha) \binom{\lambda+k}{k}}{(1-\alpha)}$$

Solving for δ , we have

$$\delta \leq \frac{(c+k+1)d_k(\beta, \alpha) - ck(1-\alpha)}{c(1-\alpha)[1-\beta(1+n)]d_k(\beta, \alpha)(c+k+1)} = \Phi(k).$$

A simple computation will show that $\Phi(k)$ is increasing and $\Phi(k) \geq \Phi(1)$.

Using this, the results follows.

Theorem 6.2 :- let F given by (1.1) be in the class $J(\lambda, \alpha, \beta)$,

$$F(z) = \frac{1}{z} [(c+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c+k+1}{c} a_k z^k, c > 0.$$

Then $F \in J(\lambda, \alpha, \beta)$ for $|z| \leq r(\alpha, \beta, \eta)$ where

$$r(\alpha, \beta, \eta) = \inf_k \left\{ \frac{c(1-\eta)d_k(\beta, \alpha)}{(1-\alpha)(c+k+1)d_k(\beta, \alpha)} \right\}^{\frac{1}{k+1}}, \quad k=1,2,\dots,$$

The result is sharp for the function given by (2.2)

Proof :- suppose that

$$w = \frac{zf'(z)}{(\beta-1)f(z) + \beta zf'(z)}$$

Then it is sufficient to show that

$$\left| \frac{\omega-1}{\omega+1-2\eta} \right| < 1$$

AA simple computation shows that this is satisfied if

$$\sum_{k=1}^{\infty} \frac{(c+k+1)d_k(\beta, \eta) \binom{\lambda+k}{k}}{c(1-\eta)} a_k |z|^{k+1} \leq 1 \quad (6.2)$$

Since $f \in J(\lambda, \alpha, \beta)$, then by Theorem 2.1, we have

$$\sum_{k=1}^{\infty} d_k(\beta, \alpha) \binom{\lambda+k}{k} a_k \leq 1 - \alpha$$

The equation (6.2) is satisfied if

$$\frac{(c+k+1)d_k(\beta, \eta) \binom{\lambda+k}{k}}{c(1-\eta)} a_k |z|^{k+1} \leq \frac{d_k(\beta, \alpha) \binom{\lambda+k}{k}}{1-\alpha} a_k$$

Solving for $|z|$, we get the desired results.

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