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من قبل

زهراء صارم فاطمة احمد

إشراف

أ.م.د. وقاص غالب عطشان

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University of AL-Qadisiyah
College of Computer Science and
information Technolog



Application of a Class of Univalent Meromorphic Functions Defined by Linear Operator

A Thesis

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By

Zahraa Saarem

Fatema Ahmed

Supervised by

Assist. Prof. Dr. Waggas Galib Atshan

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

يَا أَيُّهَا النَّاسُ إِنَّا خَلَقْنَاكُمْ مِنْ ذَكَرٍ وَأُنْثَىٰ وَجَعَلْنَاكُمْ شُعُوبًا وَقَبَائِلَ
لِتَعَارَفُوا إِنَّ أَكْرَمَكُمْ عِنْدَ اللَّهِ أَتْقَاكُمْ إِنَّ اللَّهَ عَلِيمٌ خَبِيرٌ

صَدَقَ اللَّهُ الْعَلِيُّ الْعَظِيمُ

(الحجرات - 13)

الاهداء

- امتنان وشكر الى الله سبحانه وتعالى والى النبي محمد (ص) وال بيته (صلوات الله عليهم اجمعين)
- الى منبع الاحترام والتقدير والدي العزيز .
- الى منبع الحب والحنان والدي العزيزة .
- الى الذين مهدوا لنا طريق العلم والمعرفة اساتذتي الاعزاء
- الى منبع الاخاء أخوتي وأخواتي
- الى جميع أصدقائي الاوفياء

الباحث

فاطمة أحمد

نزهراء صابر

شكر وتقدير

- أنه لمن دواعي الفخر والاعتزاز أن أقدم بالشكر الجزيل والتقدير الفائق لأستاذي المشرف على البحث الدكتور وقاص غالب عطشان ولما قدمه لي من توجيهات وارشادات كانت لها الأثر في حبكة البحث ومدلوليته.
- كما أقدم بالشكر والتقدير للأساتذة الأفاضل في قسم الرياضيات ولا استثنى أحداً منهم والشكر موصول الى أعضاء لجنة المناقشة.

والله ولي التوفيق

الباحثان

فاطمة أحمد

نزهة صابر

Abstract

In this paper, we introduce certain class of univalent meromorphic functions defined by linear operator . We obtain coefficient bounds , radii of starlikeness and convexity and convex linear combination .

2000 Mathematics Subject : 30C45.

Keywords: Univalent Meromorphic Function , Linear Operator , Coefficient Bounds , Distortion Bounds , Radius of Starlikeness , Convex Combination .

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1.Introduction:-

Let M denote class of functions of the form :

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, (a_n \geq 0, n \in \mathbb{N} = \{1, 2, \dots\}) \quad (1)$$

which are analytic and univalent in the punctured open unit disk

$U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U - \{0\}$, where U is the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$

A function $f \in M$ is said be meromorphic univalent Starlike functions if

$$-Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, z \in U^*$$

A function $f \in M$ is said be meromorphic univalent convex function if

$$-Re \left\{ 1 + \frac{zf'(z)}{f(z)} \right\} > 0, z \in U^*.$$

A function $f \in M$ is said be meromorphic univalent Starlike function of order α if.

$$-Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U^*, 0 \leq \alpha < 1).$$

The class of all such meromorphic univalent Starlike functions in U^* is denoted by $M(\alpha)$.

Definition (1) [5] : Let $f(z)$ be a function in the class M given by (1) , we define a linear operator L^k by

$$L^0(z) = f(z)$$

$$L^1 f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (n+2) a_n z^n = \frac{(z^2 f(z))'}{z}$$

and in general

$$L^k f(z) = L(L^{k-1} f(z)) = \frac{1}{z} + \sum_{n=1}^{\infty} (n+2) a_n z^n = \frac{(z^2 L^{k-1} f(z))'}{z}, (k \in \mathbb{N}) \quad (2)$$

It is easily verified from (2) that

$$\begin{aligned} (z^2 L^k f(z))' \\ = L^{k+1} f(z) - 2L^k f(z), (f \in M, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \end{aligned}$$

Definition(2): A function $f(z)$ in M is in the class $M(k, \alpha, \beta)$ if it satisfies the condition :-

$$\left| \frac{\frac{z(L_k f(z))' + 1}{L_k f(z)}}{\frac{z(L_k f(z))' + (2\alpha - 1)}{L_k f(z)}} \right| \leq \beta \quad (z \in U, 0 \leq \alpha < 1, 0 < \beta \leq 1).$$

If $k=0$, the class $M(k, \alpha, \beta)$ reduce to the class $M(0, \alpha, \beta) \equiv \Sigma(\alpha, \beta)$ studied by Uralegaddi and Desai[7].

Some authors studied univalent meromorphic functions for other classes , like , Kulkarni and joshi [4] , Ramreddy et . [6] and Atshan and Kulkarni [1,2,3]

2. Coefficient bounds

In the following theorem, we obtain a necessary and sufficient condition for a function $f(z)$ to be in the class $M(k, \alpha, \beta)$.

Theorem 1: Let $f(z) \in M$. Then $f(z) \in M(k, \alpha, \beta)$ if and only if.

$$\sum_{n=1}^{\infty} (n+2)^k (n+1+\beta+n+2\alpha-1) a_n \leq 2\beta(1-\alpha) \quad (2.1)$$

Where, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$

The result (2.1) is sharp.

Proof: Assume (2.1) holds for all admissible value k, α, β . Let the expression

$$M(f) = |z(L_k f(z))' + L_k f(z)| - \beta |z(L_k f(z))' + (2\alpha - 1)L_k(z)| \quad (2.2)$$

Since

$$L_k f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} (n+2)^k a_n z^n. \text{ Then}$$

$$(L_k f(z))' = \frac{1}{z^2} + \sum_{n=1}^{\infty} n(n+2)^k a_n z^{n-1}$$

Therefore, we have for $0 < |z| = r < 1$

$$M(f) = \sum_{n=1}^{\infty} (n+1)(n+2)^k a_n z^{n+1}$$

$$- \beta \left| (2\alpha - 1) + \sum_{n=1}^{\infty} (n+2\alpha - 1)(n+2)^k a_n z^{n+1} \right|$$

$$= \left| \sum_{n=1}^{\infty} (n+1)(n+2)^k a_n z^{n+1} \right|$$

$$\begin{aligned}
& -\beta \left| (2\alpha - 1) + \sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n z^{n+1} \right| \\
& \leq \sum_{n=1}^{\infty} (n + 1)(n + 2)^k a_n z^{n+1} - 2\beta(1 - \alpha) \\
& + \sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n z^{n+1} \\
& = \sum_{n=1}^{\infty} (n + 2)^k (n + 1\beta(n + 2\alpha - 1)) a_n r^{n+1} - \\
& 2\beta(1 - \alpha) \tag{2.3}
\end{aligned}$$

Since the above inequality holds for all $r, 0 < r < 1$, letting $r \rightarrow 1^-$ in (2.3), we get the result (2.1).

Conversely, let $f \in M(K, \alpha, \beta)$. Then

$$\begin{aligned}
& \left| \frac{\frac{z(L_k f(z))'}{L_k f(z)} + 1}{\frac{z(L_k f(z))'}{L_k f(z)} + (2\alpha - 1)} \right| \\
& = \left| \frac{\sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n z^{n+1}}{(2\alpha - 1) + \sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n z^{n+1}} \right| \leq \beta, (z \in U^*)
\end{aligned}$$

Since $R_e\{z\}f$ for all z , it follows that

$$R_e \left\{ \frac{\sum_{n=1}^{\infty} (n + 1)(n + 2)^k a_n z^{n+1}}{(2\alpha - 1) + \sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n z^{n+1}} \right\} \leq \beta, (z \in U^*) \tag{2.4}$$

We choose the values of z on the real axis so that $\frac{z(L_k f(z))'}{L_k f(z)}$ is real. By letting $z \rightarrow 1^-$ through values, we get

$$\sum_{n=1}^{\infty} (n + 1)(n + 2)^k a_n \leq 2\beta(1 - \alpha) - \sum_{n=1}^{\infty} (n + 2\alpha - 1)(n + 2)^k a_n$$

Therefore

$$\sum_{n=1}^{\infty} (n+2)^k (n+1+\beta(n+2\alpha-1)) a_n \leq 2\beta(1-\alpha).$$

which completes the proof of Theorem 1

The result is sharp for

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{(n+2)^k(n+1+\beta(n+2\alpha-1))} z^n \quad (2.5)$$

Corollary (1): Let $f(z) \in M(k, \alpha, \beta)$. Then

$$a_n \leq \frac{2\beta(1-\alpha)}{(n+2)^k(n+1+\beta(n+2\alpha-1))}$$

3. Distortion Bound

In the following theorem , We obtain distortion bounds for

$f(z) \in M(k, \alpha, \beta)$.

Theorem(2): Let $f \in M(k, \alpha, \beta)$. Then

$$\frac{1}{|z|} - \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} |z| \leq |z| \leq \frac{1}{|z|} + \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} |z|. \quad (3.1)$$

The result (3.1) is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} z.$$

Proof : Let $f(z) \in M(k, \alpha, \beta)$. By Theorem 1, We have

$$\sum_{n=1}^{\infty} \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} a_n. \quad \text{Thus} \quad (3.2)$$

$$|f(z)| \leq \frac{1}{|z|} \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n.$$

Now, by using (3.2) , we get

$$|f(z)| \leq \frac{1}{|z|} + \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} |z|.$$

Similarly , we get

$$|f(z)| \leq \frac{1}{|z|} + \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} |z|$$

And this completes the proof of Theorem 2.

The result is sharp for $|f(z)| \leq \frac{1}{|z|} + \frac{2\beta(1-\alpha)}{(3)^k 2(1+\alpha\beta)} |z|$

4. Radii of starlikeness and convexity

Now , We obtain the of univalency , in particular , starlikeness and convexity the $M(k, \alpha, \beta)$.

Theorem 3 : If $f(z) \in M(k, \alpha, \beta)$, then $f(z)$ is univalent

meromorphic starlike of order γ ($0 < \gamma < 1$) in $|z| < r_1 = r_1(k, \alpha, \beta, \gamma)$, where

$$r_1(k, \alpha, \beta, \gamma) = \inf_n \left\{ \frac{((1-\gamma)(n+2)^k(n+1+\beta(n+2\alpha-1)))^{\frac{1}{n+1}}}{(2+n-\gamma) 2\beta(1-\alpha)} \right\}. \quad (4.1)$$

The result (4.1) is sharp.

Proof: Let $f(z) \in M(k, \alpha, \beta)$, Then by Theorem 1

$$\sum_{n=1}^{\infty} \frac{(n+2)^k(n+1+\beta(n+2\alpha-1))}{2\beta(1-\alpha)} a_n \leq 1 \quad (4.2)$$

It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \gamma \quad (4.3)$$

For $|z| < r_1(k, \alpha, \beta, \gamma)$ where $r_1(k, \alpha, \beta, \gamma)$ is given (4.1).

Since

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad \text{Then}$$

$$zf'(z) = z^{-1} + \sum_{n=1}^{\infty} n a_n z^n \quad \text{There fore}$$

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{\sum_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} a_n z^{n+1}} \leq 1 - \gamma$$

Provided

$$\sum_{n=1}^{\infty} \frac{(2 + n - \gamma)}{(1 - \gamma)} a_n |z|^{n+1} \leq 1. \quad (4.4)$$

Now making use of (4.2), we observe that (4.4) holds true if.

$$\begin{aligned} & |z|^{n+1} \\ & \leq \left\{ \frac{(1 - \gamma)(n + 2)^k (n + 1 + \beta(n + 2\alpha - 1))}{2\beta(1 - \alpha)} \right\}^{\frac{1}{n+1}}. \end{aligned} \quad (4.5)$$

Setting $|z| = r_1(k, \alpha, \beta,)$ in (4.5), we get radius of starlikeness, which completes the proof of the Theorem 3.

The result is sharp for the function given by.

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{(n+2)^k(n+1+\beta(n+2\alpha-1))} z^n.$$

Theorem 4 : If $(z) \in M(k, \alpha, \beta,)$ Then $f(z)$ is univalent

Meromorphic convex of order $\gamma(0 \leq \gamma < 1)$ in $|z| < r_2 = r_2(k, \alpha, \beta, \gamma)$ where

$$r_2(k, \alpha, \beta, \gamma) = \inf_n \left\{ \frac{((1-\gamma)(n+2)^k(n+1+\beta(n+2\alpha-1)))^{\frac{1}{n+1}}}{n(2+n-\gamma) 2\beta(1-\alpha)} \right\} \quad (4.6)$$

The result (4.6) is sharp

Proof: Let: If $(z) \in M(k, \alpha, \beta,)$. Then by Theorem 1

$$\sum_{n=1}^{\infty} \frac{(n+2)^k(n+1+\beta(n+2\alpha-1))}{2\beta(1-\alpha)} a_n \leq 1 \quad (4.7)$$

It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \gamma, \quad (4.8)$$

for $|z| < r_2(k, \alpha, \beta, \gamma)$ where $r_2(k, \alpha, \beta, \gamma)$ is given by (4.6).

Since

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n. \text{ Then}$$

$$zf'(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} n a_n z^{n-1} \text{ There fore}$$

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{\sum_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} (n+1) a_n z^{n+1}}{1 + \sum_{n=1}^{\infty} a_n z^{n+1}} \leq 1 - \gamma,$$

Provided that

$$\sum_{n=1}^{\infty} \frac{n(n+2-\gamma)}{(1-\gamma)} a_n |z|^{n+1} \leq 1 \quad (4.9)$$

Now making use of (4.7) , we observe that (4,9) holds true if

$$|z|^{n+1} \leq \frac{(1-\gamma)(n+2)^k(n+1+\beta(n+2\alpha-1))}{n(2+n-\gamma)2\beta(1-\gamma)}$$

Hence

$$\setminus Z \setminus$$

$$\leq \left\{ \frac{((1-\gamma)(n+2)^k(n+1+\beta(n+2\alpha-1)))^{\frac{1}{n+1}}}{n(2+n-\gamma)2\beta(1-\gamma)} \right\} . \quad (4.10)$$

Setting $|z| = r_2(k, \alpha, \beta, \gamma)$ in (4.10) , we get radius of convexity , which completes the proof of the Theorem 4 .

The result is sharp for the function given by

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2\beta(1-\alpha)}{(n+2)^k(n+1+\beta(n+2-\alpha)-1)} z^n.$$

5. Convex linear combination

In continuation of our study , we shall prove that the class $M(k, \alpha, \beta)$ is closed under convex combination .

Theorem 5: Let the function $f_1(z)$ be given by

$$f_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,l} z^n, \quad l = 1, 2, \dots, m, \quad a_{n,l} \geq 0 \quad (5.1)$$

Be in the class $M(k, \alpha, \beta)$ for every $l=1, 2, \dots, m$. Then the function (z) defined by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0 \quad (5.2)$$

Is also in the same class $M(k, \alpha, \beta)$, where

$$b_n = \frac{1}{m} \sum_{l=1}^m a_{n,l} \quad (n = 1, 2, \dots) \quad (5.3)$$

Proof: Since $f_l(z) \in M(k, \alpha, \beta)$, then by Theorem 1, that

$$\sum_{n=1}^{\infty} (n+2)^k (n+1+\beta(n+2-\alpha)-1) a_{n,l} \leq 2\beta(1-\alpha),$$

$k \in \mathbb{N}_0, 0 \leq \alpha < 1$ and $0 < \beta \leq 1$, for every $l = 1, 2, \dots, m$,

Hence

$$\begin{aligned}
& \sum_{n=1}^{\infty} (n+2)^k (n+1+\beta(n+2\alpha-1)) b_n, \\
&= \sum_{n=1}^{\infty} (n+2)^k (n+1+\beta(n+2\alpha-1)) \left[\frac{1}{m} \sum_m^{\infty} a_{n,l} \right] \\
&= \frac{1}{m} \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} (n+2)^k (n+1+\beta(n+2\alpha-1)) a_{n,l} \leq 2\beta(1-\alpha).
\end{aligned}$$

Then by Theorem 2, we get $g(z) \in M(k, \alpha, \beta)$. Then completes the proof of the theorem .

Applications

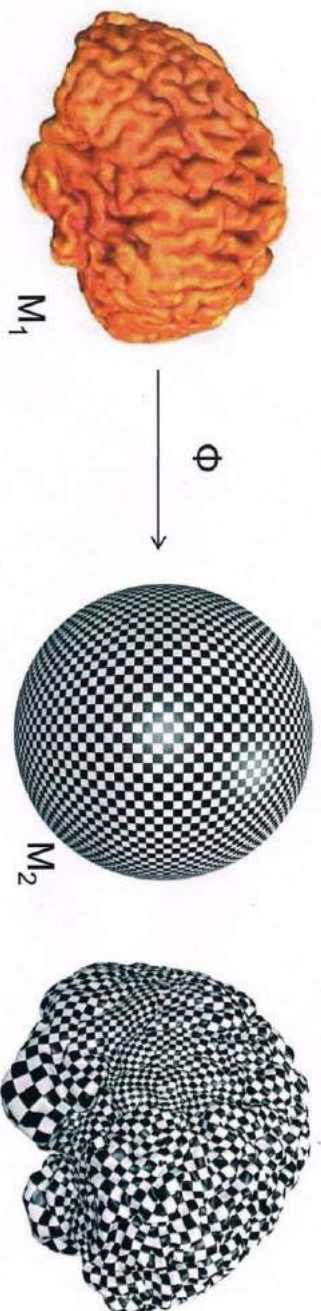
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- Motivation
- **Part I: Conformal Geometry & Applications**
 - **Basic Mathematical Background**
 - Computational Algorithms
 - Applications
- **Part II: Quasi-conformal Geometry & Applications**
- Conclusion



Why Conformal for Brain Mapping?

- Metric preserved up to scaling $\Rightarrow$ Local geometry preserved!
- Angle-preserving $\Rightarrow$ inherits a natural orthogonal grid on the surface.
- Simple (g_{ij}) Matrix $\Rightarrow$ simple differential operator expression on the parameter domain and simple projected equations.



What is Conformal map?

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- Conformal map $f : M \rightarrow N$ = preserves inner product up to a scaling factor (the conformal factor λ).
- Mathematically, $f^*(ds_N^2) = \boxed{\lambda(x_1, x_2)} ds_M^2$ where $ds_M^2 = \sum_{i,j=1}^2 g_{ij} dx^i \wedge dx^j$ ← conformal factor

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المستخلص

الهدف من هذا البحث هو مناقشة صنف اكيد من الدول احادية التكافؤ الميرمورفية والمعرفة بواسطة المؤثر الخطي .
حصانا على حدود المعامل ، حدود التشويه ، انصاف اقطار النجمية ،
التحذب و التركيب الخطي المحدب .