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A linear Operator Of A class Of Univalent Functions Defined By Generalized Hypergeometric Function

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Abstract

In the present paper , we have studied application of linear operator defined by generalized hyper geometric functions to univalent functions with negative coefficients , here we obtain some geometric properties , like , coefficients estimates , closure properties , distortion and radii of star likeness , convexity and close - to - convexity .

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1- Introduction :-

Let D denote class of functions f(z) of the form :-

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \,, \qquad (1.1)$$

which are analytic and univalent in the open unit disk

U = { z ∈ ℂ : $|z| < 1$ } .

The Hadamard product (or convolution) of two function of D given by (1.1) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_{k b_k} z^k \,, \qquad (1.2)$$

where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad (1.3)$$

Let T denoteasubclass of D consisting of function of the form :-

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \,, (a_k \geq 0) . \qquad (1.4)$$

For { $a_1, a_2, \ldots a_m$ } ∈ ℂ and { $\beta_1, \beta_2, \ldots, \beta_1$ } ∈ ℂ - {0,-1,-2,...}, the generalized hyper geometric function

$_mF_i$ (a_1 , ... , a_m ; β1, ... , βi; z) is defined by

$$mFi(a_1,\dots a_m;\beta_1\dots,\beta_i\,;z) = \sum_{k=1}^{\infty} \frac{(a_1)_k\dots(a_m)_k}{(\beta_1)_k\dots(\beta i)_k}\, a_k z^k \qquad (1.5)$$

($m \leq l + 1$, m , l N_0 = { 0 , 1 , 2 , ... }) ,

where $(x)_k$ is the pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & k = 0 \\ x(x+1)\dots(x+k-1) & k \in N \end{cases} \qquad (1.6)$$

Dziok and Srivastava considered in [3] a linear operator under the

univalent analytic function

$$DS_1^{m,l}\ (a_1, \dots\dots, a_m; \beta_1, \dots.., \beta_1): w_1 \rightarrow w_1$$

defined by the Hadamard product :

$$DS_1^{m,l}(f)(z) = DS_1^{m,l}\ (a_1, \dots\dots, a_m; \beta_1, \dots.., \beta i_{,z})(f)(z)$$

$$= h_1(a_1, \dots\dots, a_m; \beta_1, \dots.., \beta_i; z) * (f)(z) \qquad (1.7)$$

$$h_p(a_1, \dots\dots, a_m; \beta_1, \dots.., \beta i_{,z})\,(f)(z) = z_m F_1\,(a_1,\dots,a_m\,;\,\beta_1 \dots \beta_i, z)$$

if $f \in T$ *is given by* (1.4), *then we can write the Dziok* − *Srivastava operator* (1.7)*as follow* :

$$DS_1^{m,l}(f)(z) = z - \sum_{k=1}^{\infty} \frac{(a_1)_k\dots(a_m)_k}{(\beta_1)_k\dots(\beta_i)_k}\, a_k z^k \qquad (1.8)$$

Special cases of the Dziok - Srivastava linear operator were investicated recently by Liu [5] , Carlson and Shaffer [2] , Ruscheweyh [8,9], Bernardi [1] , Libera [4] , Livingston [6] ,

Srivastava and Owa [10] and others . The motivation stems out from such work of mathematicians .

Definition 1 : Let $0 \leq a < 1, 0 < \beta \leq 1$ and $1/2 < \gamma \leq 1$

$if a = 0, and \frac{1}{2} < \gamma \leq \frac{1}{2a}, if\ a \neq 0$.

We define a class $T(m, l, a, \beta, \gamma)$ as the class of all functions $f(z)$ of the from given in (1.4) , $if\ f(z)$ satisfies the condition :-

$$\left| \frac{\left(DS_1^{m,l}(f)(z)\right)' - 1}{2\gamma\left(\left(DS_1^{m,l}(f)(z)\right)' - a\right) - \left(DS_1^{m,l}(f)(z)\right)' - 1} \right| < \beta .$$

For other classes of univalent functions , we can see the recent work of authors in [5] , [7] .

2- Coefficient Estimates :

Theorem 1 : Let the function f(z) be defined by (1.4) .Then f(z) is in the class T (m, l, a, β, γ) if and only if

$$\sum_{k=2}^{\infty} k\,(1-\beta(1-2\gamma))\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}\;a_{k}\leq 2\beta\gamma\,(1-a) \qquad (2.1)$$

The result (2.1) is sharp .

Proof : Assume that the inequality (2.1) holds and let $|z|=1$.

Then by hypothesis , we have

$$\left|(DS_{1}^{m,l}(f)(z))'-1\right|-\beta\left|2\gamma\left(\left(DS_{1}^{m,l}(f)(z)\right)'-a\right)-\left(\left(DS_{1}^{m,l}(f)(z)\right)'-1\right)\right|$$

$$\left|-\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}z^{k-1}\right|-\beta\left|2\gamma\left(1-\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}z^{k-2}-a\right)+\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}z^{k-1}\right|$$

$$\leq\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}-\beta\left(2\gamma(1-a)-2a\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}\right)-\beta\sum_{k=1}^{\infty}k\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}$$

$$=\sum_{k=1}^{\infty}k\,(1-\beta(1-2\gamma))\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_{1})_{k}\cdots((\beta_{i})_{k}k!}a_{k}-2\beta\gamma(1-a)\leq 0$$

Hence , by maximum modulus theorem f(z) ∈ $T(m, l, a, \beta, \gamma)$.

Conversely , suppose that f(z) ∈ $T(m, l, a, \beta, \gamma)$, *then*

$$\left|\frac{(DS_{1}^{m,l}(f)(z))'-1}{2\gamma\left(\left(DS_{1}^{m,l}(f)(z)\right)'-a\right)-\left(DS_{1}^{m,l}(f)(z)\right)'-1}\right|$$

$$= \left| \frac{DS_1^{m,l}(f)(z)' - 1}{2\gamma\left(\left(DS_1^{m,l}(f)(z)\right)' - a\right) - \left((DS_1^{m,l}(f)(z)\right)' - 1} \right| < \beta \text{ و } z \in U \qquad (2.2)$$

Since $|\mathrm{Re}(z)| < |z|$ for all z , we have .

$$\mathrm{Re}\left\{ \frac{-\sum_{k=1}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots (\beta_i)_k} a_k z^{k-1}}{2y(1-a) + \sum_{k=2}^{\infty} (1-2y) \frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots (\beta_i)_k} a_k z^{k-1}} \right\} < \beta. \qquad (2.3)$$

.

Choose value of z on real axis so that $\left(DS_1^{m,l}(f)(z)\right)'$ is real .

Upon clearing the denominator in (2.3) and letting $z \to 1$

Through real values , we obtain .

$$\sum_{k=1}^{\infty} k \frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_1)_k k!} a_k \le \beta \left(2y(1-a) - 2a \sum_{k=2}^{\infty} (1-2y) \frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!} a_k\right)$$

This gives required result . Finally , for the function ,

$$f(z) = z - \frac{2\beta\gamma(1-a)}{k\,(1-\beta(1-2\gamma))\ \frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}} z^k \qquad (2.4)$$

The result is sharp .

Corollary 1 **:** Let the function f(z) defined by (1.4) be in the class

T (m , i , a , β , γ) . Then

$$a_k \leq \frac{2\beta\gamma(1-a)}{k\,(1-\beta(1-2\gamma))\,\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots (\beta_i)_k k!}}, k \leq 2 .$$

The result is sharp for the function given by (2.4) .

3- Distortion Propertv :

Theorem 2 : Let the function f(z) defined by (1.4) in the class

T (m , l , a , β , γ) . Then we have

$$\mathbf{r} - \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_2 \cdots (a_m)_2}{(\beta_1)_2 \cdots (\beta_i)_2}}\,\mathbf{r}^2 \leq |f(z)| \leq \mathbf{r} - \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_2 \cdots (a_m)_2}{(\beta_1)_2 \cdots (\beta_i)_2}} \quad (3.1)$$

The result (3.1) is sharp .

Proof : Since f(z) ∈ $T(m , l , a , \beta , \gamma)$ *and in view of inequality*

(2.1) of Theorem 1 , we obtain .

$$(1-\beta(1-2\gamma))\,\frac{(a_1)_2 \cdots (a_m)_2}{(\beta_1)_2 \cdots (\beta_i)_2} \sum_{k=1}^{\infty} ak \leq \sum_{k=1}^{\infty} k\,(1-\beta(1-2\gamma))\,\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots (\beta_i)_k k!}\,ak \leq 2\beta y(1-a) \quad (3.2)$$

which implies

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_k \dots (a_m)_2}{(\beta_1)_k \dots ((\beta_i)_2 2}}$$

Therefor , we can show that

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} a_k \geq r - r^2 \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_k \dots (a_m)_2}{(\beta_1)_k \dots ((\beta_i)_2 2}}$$

and

$$|f(z)| \leq r - r^2 \sum_{k=2}^{\infty} a_k \leq r - r^2 \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_k \dots (a_m)_2}{(\beta_1)_k \dots ((\beta_i)_2 2}}$$

This completes the proof . Finally , by taking function ,

$$F(z)= z - \frac{\beta\gamma(1-a)}{(1-\beta(1-2\gamma))\,\frac{(a_1)_2 \dots (a_m)2}{(\beta_1)_2 \dots ((\beta_i)_2 2}} z^2 , \qquad (3.3)$$

we can show the result is sharp .

4- Closure Properties :

Now , we prove the closure properties of a class

$T(m, l, a, \beta, \gamma)$. *Let the function , for* $i = 1,2,3,\dots,m$,

be defined by

$$fi = z - \sum_{k=2}^{\infty} a_{k,i}\, z^k \,,\ a_{k,i} \geq 0, z \in U \ . \qquad (4.1)$$

<u>Theorem 3 :</u> Let the function $f_i\ (z) given\ by\ (4.1) be\ in\ the\ class$

$T\ (m, l, a, \beta, y) for\ every\ i = 1,2,3, \dots, l$.

Then the function h(z) defined by

$$h(z) = \sum_{i=1}^{l} c_i\ f_i\ (z) \qquad (4.2)$$

is also in the class $T\ (m, l, a, \beta, y)$

proof : Observing definition of h(z) in (4.2) , we get .

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{l} c_i a_{k,i} \right) z^k . \qquad (4.3)$$

Further , since f_i (z) are in the class $T\ (m, l, a, \beta, y) for\ every$

$i = 1,2,3, \dots, l\ . we\ get$

$$\sum_{k=2}^{\infty} k(1-\beta(1-2y)) \frac{(a_1)_k \dots (a_m)_k}{(\beta_1)_k \dots ((\beta_i)_k k!} a_{k,i} \ \leq\ 2\beta y(1-a) \qquad (4.4)$$

Henece , in view of (4.4) , we see that

$$\sum_{k=2}^{\infty} k\big(1-\beta(1-2y)\big)\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_1)_{k}\cdots((\beta_i)_{k}k!}\sum_{i=1}^{l} ci\,a_{k,i}$$

$$\sum_{i=1}^{l} ci \quad \Big(\sum_{k=2}^{\infty} k\big(1-\beta(1-2y)\big)\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_1)_{k}\cdots((\beta_i)_{k}k!}\sum_{i=1}^{l} ci\,a_{k,i}\Big) \quad (4.5)$$

$$\leq \sum_{i=1}^{l} ci\,\big(2\beta y(1-a)\big) = 2\beta y(1-a)$$

This immediately implies that h(z) belong to the class

$T\ (m, l, a, \beta, y)$. *This completes the proof of Theorem* 3

<u>Theorem 4 :</u>

$$\text{Let } f1(z) = z \text{ and } fk(z) = z - \frac{2\beta\gamma(1-a)}{k\big(1-\beta(1-2\gamma)\big)\frac{(a_{1})_{k}\cdots(a_{m})_{k}}{(\beta_1)_{k}\cdots((\beta_i)_{k}k!}}\, z^{k}\ ,(k \geq 2) \quad (4.6)$$

$$for\ 0 \leq a < 1, 0 < \beta \leq 1, \frac{1}{2} < \gamma \leq 1\ if\ a = 0\ or\ \frac{1}{2} < \gamma \leq \frac{1}{2}a, if a \neq 0.$$

Then f(z) is in the class $T\ (m, l, a, \beta, y)$ *if and only if it can be expressed in the form.*

$$f(z) = \sum_{k=1}^{\infty} \lambda_{k\, f_k}(z), \qquad (4.7)$$

where

$$\lambda_k \geq 0\,, k \;\geq 1 \; and \; \sum_{k=1}^{\infty} \lambda_k = 1$$

<u>Proof :</u> Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_{k\, f_k}(z) = \text{z} - \sum_{k=1}^{\infty} \frac{2\beta\gamma(1-a)}{\text{k}(\,1-\beta(1-2\gamma))\,\frac{(a_{1)\,k} \cdots (a_{m)\,k}}{(\beta_1)_{\,k} \cdots ((\beta_i)_{\,k}\text{k!}}} \text{z}^k \qquad (4.8)$$

then it follows that

$$\sum_{k=1}^{\infty} \frac{2\beta\gamma(1-a)}{\text{k}(\,1-\beta(1-2\gamma))\,\frac{(a_{1)\,k} \cdots (a_{m)\,k}}{(\beta_1)_{\,k} \cdots ((\beta_1)_{\,k}\text{k!}}} \lambda_k.\, \text{k}(\,1-\beta(1-2\gamma))\,\frac{(a_{1)\,k} \cdots (a_{m)\,k}}{(\beta_1)_{\,k} \cdots ((\beta_i)_{\,k}\text{k!}}$$

$$= 2\beta\gamma(1-a)\sum_{k=2}^{\infty} \lambda_\text{k} \;\; = 2\beta\gamma(1-\text{a})(1-\lambda_1\;) \leq 2\beta\gamma(1-a)\,.$$

Hence , in view of inequality (2.1) of Theorem 1

$, f(z) \in T(m\,, l\,, a\,, \beta\,, \gamma)$.

Conversely , assume that f(z) defined by (1.4) belong to class

$1, f(z) \in T(m, l, a, \beta, \gamma)$, *then*

$$a_k \leq \frac{2\beta\gamma(1-a)}{k(1-\beta(1-2\gamma))\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}, k \geq 2 .$$

setting

$$\lambda_k = \frac{k(1-\beta(1-2\gamma))\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{2\beta\gamma(1-a)} a_k .$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$$

we can see that f(z) can be expressed in the form (4.7) . This complete the proof .

Corollary 2 : The extreme point of class $T(m, l, a, \beta, \gamma)$ are the function $f_k(z)(k \geq 1)$given by (4.6).

5- Radii of Starlikeness , Conexity and Close - to - Conexity :-

Teorem 5 : Let f(z) defined by (1.4) be in the class $T(m, l, a, \beta, \gamma)$

Then f(z) is starlike of order $\rho(0 \leq \rho < 1)$ *in the disc* .

$$|z| < r_1(m, l, a, \beta, \gamma) = \inf_k \left\{ \frac{(1-\rho)k\left(1-\beta(1-2\gamma)\right)\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{(k-\rho)(2\beta\gamma(1-a)} \right\} \quad (5.1)$$

The result is sharp .

Proof : It is suffices to prove

$$\left| \frac{zf^1(z)}{f(z)} - 1 \right| < 1 - \rho , \qquad (5.2)$$

that is , to prove ,

$$\frac{\sum_{k=2}^{\infty} (k-1) a_k |z|^k}{1 - \sum_{k=2}^{\infty} a_k |z|^k} \leq 1 - \rho ,$$

which is equivalent to

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k |z|^{k-1} \leq 1 ,$$

in view of Theorem 1 , it is possible only when

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k |z|^{k-1} \leq \sum_{k=2}^{\infty} \frac{k\left(1-\beta(1-2\gamma)\right)\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{2\beta\gamma(1-a)} a_k \quad .(5.3)$$

After simplification we get required result . This complete proof of Theorem 5 .

Theorem 6 : Let f(z) defined by (1.4) be in the class $T\ (m\ ,l\ ,a\ ,\beta\ ,\gamma)$. Then f(z) is convex of order $\rho(0 \leq \rho < 1)$ *in the disc* .

$$|z| < r_2(m\ ,l\ ,a\ ,\beta\ ,\gamma) = \inf_k \left\{ \frac{(1-\rho)k\left(1-\beta(1-2\gamma)\right)\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{(k-\rho)(2\beta\gamma(1-a)} \right\} . \quad (5.4)$$

Then result is sharp.

Proof : It is suffices to prove

$$\left| \frac{zf''(z)}{f(z)} - 1 \right| < 1-\rho\ , \quad (5.5)$$

that is , to prove ,

$$\frac{\sum_{k=2}^{\infty} (k-1) a_k |z|^k}{1-\sum_{k=2}^{\infty} a_k |z|^k} \leq 1-\rho\ ,$$

which is equivalent to

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k |z|^{k-1} \leq 1 .$$

In view of Theorem 1 , it is possible only when

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k |z|^{k-1} \leq \sum_{k=2}^{\infty} \frac{k\left(1-\beta(1-2\gamma)\right)\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{2\beta\gamma(1-a)} a_k \quad .(5.6)$$

After simplification we get required result . This complete proof of Theorem 6 .

<u>Theorem 7 :</u> Let f(z) defined by (1.4) be in class $T\ (m\ ,l\ ,a\ ,\beta\ ,\gamma)$. Then f(z) is close of order $\rho(0 \leq \rho < 1)$ *in the disc* .

$$|z| < r_2(m\ ,l\ ,a\ ,\beta\ ,\gamma) = \inf_k \left\{ \frac{(1-a)k\left(1-\beta(1-2\gamma)\right)\frac{(a_1)_k \cdots (a_m)_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{k(2\beta\gamma(1-a)} \right\}. \quad (5.7)$$

The result is sharp .

Proof : it is suffices to prove $|f'(z) - 1| < 1-\rho$

That is , to prove ,

$$\sum_{k=2}^{\infty} \mathrm{k}\, a_k |z|^{\,k-1} \le 1$$

In view of Theorem 1 , it is possible only when

$$\sum_{k=2}^{\infty} \frac{(k-\rho)}{1-\rho} a_k |z|^{\,k-1} \le \sum_{k=2}^{\infty} \frac{k\left(1-\beta(1-2\gamma)\right)\frac{(a_{1})_k \cdots (a_{m})_k}{(\beta_1)_k \cdots ((\beta_i)_k k!}}{2\beta\gamma(1-a)} a_k , \qquad (5.8)$$

After simplification we get required result . This complete proof of Theorem 7 .

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