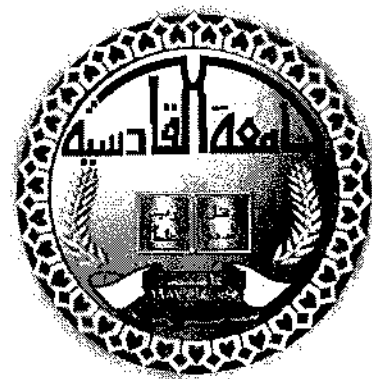


Ministry of Higher Education And Scientific Research
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Department of Mathematics



Subclass of Multivalent Analytic Function Defined by Fractional Derivative Operator

A research

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

(رَبِّ أَوْزِرْ عَنِّي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى
وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي فِي ذُرِّيَّتِي إِنِّي
خُفْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ)

صدق الله العلي العظيم

الإهداء

إلى من ضحوا بمائهم الطاهرة من اجل السلام . . .

أهل البيت ((عليهم السلام))

إلى من أوحى الله بهما برا وأحسانا . . .

أبي وأمي

إلى كل من ضحى من اجل الحق والعلم . . .

العلماء

إلى الذي لم يخل علي بعلمه وإمر شاده . . .

أساتذتي الفاضل

إلى من مرافقني سنوات الدراسة . . .

أصدقائي وزملائي

والى من يقدر المجهود العلمي اهديه ثمره جهودي

شكر وتقدير

قال رسول الله صل الله عليه واله وسلم ((من لا يشكر الناس لا يشكر الله))

اشكر في هذا الصدد الأستاذ المشرف عباس كريم وناس لنصائحه الثمينة

وتوجيهاته القيمة

كما أتوجه بالشكر إلى من أوصاني الله بطاعتها وبرهما . . . أبي وأمي وإلى

كل من يد لي يد العون والمساعدة في إتمام هذا البحث

ولله الحمد من قبل ومن بعد

Abstract

In this paper, we introduce a subclass of Multivalent analytic function with positive coefficients defined by using fractional derivative operator to study some of the interesting properties, Like, Coefficient inequalities, Distortion bounds, Closure theorems, Integral representation and Radii of starlikeness and convexity

2010 Mathematics Subject Classification: 30C45, 30C50

Key words: Multivalent function, Fractional derivative, Coefficient inequalities, Distortion bounds, Integral representation.

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1. Introduction

Let A_p be the class of p -valent analytic functions with positive coefficients of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in U = \{z \in \mathbb{C} : |z| < 1\}. \quad (1.1)$$

For functions f given by (1.1) and

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (1.2)$$

The Hadamard product (or convolution) of f and g denoted by $f * g$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

A function $f \in A_p$ is said to be p -valently starlike of order α if it satisfies the inequality:

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in U, 0 \leq \alpha < p, p \in \mathbb{N}). \quad (1.3)$$

We denote by $A_p^*(\alpha)$ the class of all p -valently starlike functions of order α .

A function $f \in A_p$ is said to be p -valently convex of order α if it satisfies the inequality:

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (z \in U, 0 \leq \alpha < p, p \in \mathbb{N}). \quad (1.4)$$

We denote by $C_p(\alpha)$ the class of all p -valently convex functions of order α . We note that (see for example Duren [2] and Goodman [3]).

$$f(z) \in C_p(\alpha) \iff \underset{p}{z} f'(z) \in A_p^*(\alpha) \quad (0 < \alpha < p, p \in \mathbb{N}).$$

The classes $A_p^*(\alpha)$ and $C_p(\alpha)$ are studied by Gura [5].

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (cf., e.g. [1], [4], [6] and [7]). See also the various references cited therein.

For our present investigations, we recall the following definitions.

Definition 1.1 [7]. The fractional integral operator of order λ ($\lambda > 0$) is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\epsilon)}{(z-\epsilon)^{1-\lambda}} d\epsilon, \quad (1.5)$$

where f is analytic function in simply connected region of z plane containing the origin and multiplicity of $(z-\epsilon)^{1-\lambda}$ is removed by requiring $\log(z-\epsilon)$ to be real when $z-\epsilon > 0$.

Definition 1.2 [7]. The fractional derivative of order λ ($0 \leq \lambda < 1$) is defined for function f by

$$D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\epsilon)}{(z-\epsilon)^\lambda} d\epsilon, \quad (1.6)$$

where f is constrained and the multiplicity of $(z-\epsilon)^\lambda$ is removed as in Definition 1.1.

Definition 1.3 [Z]. Under the hypothesis of Definition 1.2, the fractional derivative of order $n+1$ is defined for a function f by

$$D_z^{n+1} f(z) = \frac{d^n}{dz^n} \int_z^1 f(z) (0 \leq \lambda < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Srivastava and Aouf [6] defined and studied the operator:

$$\mathcal{N}_z^{(\alpha, p)} f(z) = \frac{\Gamma(p+1-\alpha)}{\Gamma(p+1)} z^\alpha D_z^1 f(z) \quad (0 \leq \alpha < 1, p \in \mathbb{N}). \quad (1.7)$$

We can easily see that

$$\mathcal{N}_z^{(\alpha, p)} f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1) \Gamma(p+1-\alpha)}{\Gamma(p+1) \Gamma(k+1-\alpha)} a_k z^k \quad (1.8)$$

Definition 1.4. Let $f \in A_p$ and g is given by (1.2). For $1 < \delta < 1 + \frac{1}{2p}$, $z \in U$, we define the class:

$$A(p, 1, \delta) = \left\{ f \in A_p : \operatorname{Re} \left\{ 1 + \frac{(z(\mathcal{N}_z^{(\alpha, p)}(f * g))''(z))}{(\mathcal{N}_z^{(p)}(f * g)(z))} \right\} < p\delta \right\}. \quad (1.9)$$

2 Coefficient Inequalities

In this section we obtain a necessary and sufficient condition for functions to be in the class $A(p, \lambda, \delta)$.

Theorem 2.1. Let $f \in A_p$. Then $f \in A(p, \lambda, \delta)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\delta)}{p^2(\delta-1)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k b_k \leq 1. \quad (2.1)$$

Proof: Let $f \in A(p, \lambda, \delta)$. Then

$$\operatorname{Re} \left\{ 1 + \frac{\delta \left[p(p-1)z^{p-2} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} k(k-1) a_k b_k z^{k-2} \right]}{p z^{p-1} + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} k a_k b_k z^{k-1}} \right\} < p\delta.$$

Choosing values of z on real axis and letting $z \rightarrow 1^-$ we have

$$\frac{p^2 \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} k^2 a_k b_k}{p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} k a_k b_k} < p\delta,$$

or equivalently

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\delta)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k b_k \leq p^2(\delta-1)$$

* To show the converse, let the inequality (2.1) holds true,

To show the converse, let the inequality (2.1) holds true, So

$$\left| \mathcal{Z}(\mathcal{L}_{\delta}^{(k,p)}(f * g)_{(\mathcal{Z})}) - (p-1)(-\mathcal{L}_{\delta}^{(k,p)}(f * g)_{(\mathcal{Z})}) \right|$$

$$\left| \mathcal{Z}(\mathcal{L}_{\delta}^{(k,p)}(f * g)_{(\mathcal{Z})}) - [2p(1-\delta) - 1 + p](\mathcal{L}_{\delta}^{(k,p)}(f * g)_{(\mathcal{Z})}) \right|$$

$$\sum_{k=p+1}^{\infty} k(k-p) \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k b_k$$

$$2p^2(\delta-1) - \sum_{k=p+1}^{\infty} [k(k-p)(1-2(1-\delta))] \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k b_k$$

Thus, $f \in A(p, \lambda, \delta)$.

Corollary 2.1. If $f \in A(p, \lambda, \delta)$, then

$$a_k \leq \frac{p^2(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)}{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k}, \quad (2.2)$$

the result is sharp for the functions of the form:

$$f_k(z) = z^p + \frac{p^2(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)}{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k} z^k, \quad k = p+1, p+2, \dots \quad (2.3)$$

3. Distortion Bounds

In this section we obtain the distortion bounds for $f \in A(p, \lambda, \delta)$.

Theorem 3.1. If $f \in A(p, \lambda, \delta)$, then for $|z| < 1$, we have

$$|f(z)| \leq r^p + \frac{p^2(\delta-1)\Gamma(p+1)\Gamma(p+2-\lambda)}{(p+1)(p+1-p\delta)\Gamma(p+2)\Gamma(p+1-\lambda)b_{p+1}} r^{p+1} \quad (3.1)$$

and

$$|f(z)| \geq r^p - \frac{p^2(\delta-1)\Gamma(p+1)\Gamma(p+2-\lambda)}{(p+1)(p+1-p\delta)\Gamma(p+2)\Gamma(p+1-\lambda)b_{p+1}} r^{p+1} \quad (3.2)$$

The result is sharp for the function

$$f(z) = z^p + \frac{p^2(\delta-1)\Gamma(p+1)\Gamma(p+2-\lambda)}{(p+1)(p+1-p\delta)\Gamma(p+2)\Gamma(p+1-\lambda)b_{p+1}} z^{p+1} \quad (3.3)$$

Proof: By using (2.1) and (2.2), we obtain

$$b_{p+1}(p+1)(p+1-p\delta) \frac{\Gamma(p+2)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(p+2-\lambda)} \sum_{k=p+1}^{\infty} a_k \\ \leq \sum_{k=p+1}^{\infty} k(k-p\delta) \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k b_k \leq p^2(\delta-1)$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{p^2(\delta-1)\Gamma(p+1)\Gamma(p+2-\lambda)}{(p+1)(p+1-p\delta)\Gamma(p+2)\Gamma(p+1-\lambda)b_{p+1}} \quad (3.4)$$

For the function $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and using (3.4)

and $|z| = r < 1$, we have

$$|f(z)| \leq r^p + \sum_{k=p+1}^{\infty} a_k r^k \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k$$

$$\leq r^p + \frac{p^2(\gamma-1)\Gamma(p+1)\Gamma(p+2-1)}{(p+1)(p+1-p\gamma)\Gamma(p+2)\Gamma(p+1-1)b_{p+1}} r^{p+1}.$$

Also, $|f(z)| \geq r^p - \sum_{k=p+1}^{\infty} a_k r^k$

$$\geq r^p - \frac{p^2(\gamma-1)\Gamma(p+1)\Gamma(p+2-1)}{(p+1)(p+1-p\gamma)\Gamma(p+2)\Gamma(p+1-1)b_{p+1}} r^{p+1}.$$

Hence the proof is complete.

Theorem 3.2. If $f \in A(p, \lambda, \gamma)$, then for $|z| = r < 1$

$$|f'(z)| \leq p r^{p-1} + \frac{p^2(\gamma-1)\Gamma(p+1)\Gamma(p+2-1)}{(p+1-p\gamma)\Gamma(p+2)\Gamma(p+1-1)b_{p+1}} r^p$$

and

$$|f'(z)| \geq p r^{p-1} - \frac{p^2(\gamma-1)\Gamma(p+1)\Gamma(p+2-1)}{(p+1-p\gamma)\Gamma(p+2)\Gamma(p+1-1)b_{p+1}} r^p.$$

The results is sharp for the function given by (3.3).

Proof: The proof is similar to that of Theorem 3.1.

4 Closure Theorems and Integral Representation

In this section, we discuss certain inclusion properties of the class $A(p, \lambda, \delta)$ and we obtain integral representation for $\Omega_{\Sigma}^{(\lambda, p)}(f * g)(z)$.

Theorem 4.1. Let $f(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k$ ($j=1, 2, \dots, q$) be in the class $A(p, \lambda, \delta)$ and $\eta_j \geq 0$ for $j=1, 2, \dots, q$ and $\sum_{j=1}^q \eta_j \leq 1$, then the function

$$F(z) = z^p + \sum_{k=p+1}^{\infty} \left(\sum_{j=1}^q \eta_j a_{k,j} \right) z^k$$

belongs to $A(p, \lambda, \delta)$.

Proof: Since $f_j(z) \in A(p, \lambda, \delta)$, then from Theorem 2.1 for every $j=1, 2, \dots, q$ we have

$$\sum_{k=p+1}^{\infty} k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k a_{k,j} \leq p^2 (\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)$$

Now

$$\begin{aligned} & \sum_{k=p+1}^{\infty} k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k \left(\sum_{j=1}^q \eta_j a_{k,j} \right) \\ &= \sum_{j=1}^q \eta_j \left(\sum_{k=p+1}^{\infty} k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k a_{k,j} \right) \\ &\leq \sum_{j=1}^q \eta_j p^2 (\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda) \\ &\leq p^2 (\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda). \end{aligned}$$

Thus, by Theorem 2.1 $F \in A(p, \lambda, \delta)$.

Corollary 4.1. The class $A(p, \lambda, \delta)$ is closed under convex linear combination.

Theorem 4.2. If $f \in A(p, \lambda, \gamma)$, then

$$\Omega_z^{(\lambda, p)}(f * g)(z) = (p\gamma - 1) \int_0^z e^{\int_0^t \frac{Q(u)}{u} dt} dt.$$

Proof: By letting $\Omega_z^{(\lambda, p)}(f * g)(z) = M(z)$ in (1.9), we have

$$\operatorname{Re} \left\{ 1 + \frac{z M''(z)}{M'(z)} \right\} < p\gamma.$$

Thus

$$\frac{z M''(z)}{M'(z)} < p\gamma - 1$$

or

$$\frac{z M''(z)}{M'(z)} = Q(z)(p\gamma - 1),$$

where $|Q(z)| < 1, z \in U$

So

$$\frac{M''(z)}{M'(z)} = \frac{Q(z)}{z} (p\gamma - 1).$$

After integration we obtain

$$\log(M'(z)) = \int_0^z \frac{Q(u)}{u} (p\gamma - 1) du.$$

Hence

$$M'(z) = \exp \left[\int_0^z \frac{Q(u)}{u} (p\gamma - 1) du \right].$$

After integration we have

$$M(z) = \Omega_z^{(\lambda, p)}(f * g)(z) = \int_0^z \exp \left[\int_0^t \frac{Q(u)}{u} (p\gamma - 1) du \right] dt$$

and this gives the result.

5. Radii of Starlikeness and Convexity

Theorem 5.1. If $f \in A(p, \lambda, \delta)$, then f will be p -valent starlike of order δ ($0 \leq \delta < p$) in the disk $|z| < R_1$, where

$$R_1 = \inf_{k \geq p+1} \left\{ \frac{k(p-\delta)(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k}{p^2 (k-\delta)(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)} \right\}^{\frac{1}{k-p}}$$

The result is sharp for the function f given by (2.3).
 Proof: It is sufficient to show that

$$\left| \frac{z f'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{for } |z| < R_1 \quad (5.1)$$

We have

$$\left| \frac{z f'(z)}{f(z)} - p \right| \leq \frac{\sum_{k=p+1}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k |z|^{k-p}}$$

Thus (5.1) will be satisfied if

$$\frac{\sum_{k=p+1}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k |z|^{k-p}} \leq p - \delta,$$

or if

$$\sum_{k=p+1}^{\infty} \left(\frac{k-\delta}{p-\delta} \right) a_k |z|^{k-p} \leq 1.$$

Since $f \in A(p, \lambda, \delta)$, then we have.

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda)}{p^2 (\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)} a_k b_k \leq 1.$$

Thus (6.2) will hold true if

$$\frac{k-\delta}{p-\delta} |z|^{k-p} \leq \frac{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\delta) b_k}{p^2(\delta-1) \Gamma(p+1) \Gamma(k+1-\delta)},$$

or if

$$|z| \leq \left(\frac{k(p-\delta)(k-p\delta) \Gamma(k+1) \Gamma(p+1-\delta) b_k}{p^2(k-\delta)(\delta-1) \Gamma(p+1) \Gamma(k+1-\delta)} \right)^{\frac{1}{k-p}}, \quad k \geq p+1,$$

which follows the result.

Theorem 5.2. If $f \in A(p, \delta, \alpha)$, then f will be p -valent convex of order δ ($0 \leq \delta \leq p$) in the disk $|z| < R_2$, where

$$R_2 = \inf_{k \geq p+1} \left\{ \frac{(p-\delta)(k-p\delta) \Gamma(k+1) \Gamma(p+1-\delta) b_k}{p(k-\delta)(\delta-1) \Gamma(p+1) \Gamma(k+1-\delta)} \right\}^{\frac{1}{k-p}}.$$

The result is sharp for the function f given by (2.3).

Proof: It is sufficient to show that

$$\left| \frac{z f''(z)}{f'(z)} - (p-1) \right| \leq p-\delta \quad \text{for } |z| < R_2 \quad (5.3)$$

We have

$$\left| \frac{z f''(z)}{f'(z)} - (p-1) \right| \leq \frac{\sum_{k=p+1}^{\infty} k(k-p) a_k |z|^{k-p}}{p - \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}}.$$

Thus (9.3) will be satisfied if

$$\frac{\sum_{k=p+1}^{\infty} k(k-p) a_k |z|^{k-p}}{p \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}} \leq p-\delta,$$

or, if

$$\sum_{k=p+1}^{\infty} \frac{k(k-\delta)}{p(p-\delta)} a_k |z|^{k-p} \leq 1 \quad (5.4)$$

Since $p \in A(p, \lambda, \delta)$, then we have.

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda)}{p^2(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)} a_k b_k \leq 1.$$

Thus (9.4) will hold true if

$$\frac{k(k-\delta)}{p(p-\delta)} |z|^{k-p} \leq \frac{k(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k}{p^2(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)},$$

or if

$$|z| \leq \left(\frac{(p-\delta)(k-p\delta) \Gamma(k+1) \Gamma(p+1-\lambda) b_k}{p(k-\delta)(\delta-1) \Gamma(p+1) \Gamma(k+1-\lambda)} \right)^{\frac{1}{k-p}}, \quad k \geq p+1,$$

which follows the assertion of Theorem 6.2.

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