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Results On a Subclass of Meromorphic Univalent Functions

A Thesis

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

لِيَرْفَعَ اللَّهُ الَّذِينَ آمَنُوا مِنْكُمْ وَالَّذِينَ أُوتُوا
الْعِلْمَ دَرَجَاتٍ وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

صدق الله العلي العظيم

سورة المجادلة-11

الاهداء

- امتنان وشكر الى الله سبحانه وتعالى والى النبي محمد (ص) والى بيته (صلوات الله عليهم اجمعين)
- الى منيع الاحترام والتقدير والدي العزيز.
- الى منيع الحب والحنان والدي العزيزة.
- الى الذين مهدوا لنا طريق العلم والمعرفة اساتذتي الاعزاء
- الى منيع الاخاء اخوتي وأخواتي
- الى جميع أصدقائي الاوفياء

الباحثان

عباس نعمة جواد مرفل سعيد مرشيد

شكر وتقدير

- أنه لمن دواعي الفخر والاعتزاز والتقدير أن اتقدم بحميم الشكر الجزيل والتقدير الفائق لأستاذي المشرف على البحث الدكتور وقاص غالب عطشان لما قدمه لي من توجيهات وإرشادات كانت لها الأثر في تكامل البحث.

- كما اتقدم بالشكر والتقدير للأساتذة الأفاضل في قسم الرياضيات ولا استثنى أحداً منهم والشكر موصول إلى أعضاء لجنة المناقشة.
- والله ولي التوفيق

الباحث

عباس نعمة جواد مرفل سعيد مرشيد

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Results on a Subclass of Meromorphic Univalent Functions

Abstract: In The present paper , we study and discuss a subclass of meromorphic univalent Functions $SH^*(\alpha, \gamma, \lambda)$ defined by Ruscheweyh derivative. We obtain some geometric Properties ,like, Coefficient inequality ,convex set , arithmetic mean , distortion bounds , radii of starlikeness and convexity and Hadamard product property ,with some applications on Conformal mappings

AMS subject Classification: Primary 30C45, Secondary 30C50.

Keyword: Meromorphic Univalent Function , Ruscheweyh Derivative , Convex Set, Distortion Bounds, Hadamard product.

1-Introduction :

Let SH be a class of functions f of the Form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, \quad (1)$$

Which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$

Let SH^* be a subclass of SH consisting of functions of the Form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, (a_k \geq 0, k \in \mathbb{N} = \{1, 2, \dots\}) \quad (2)$$

Which are analytic and meromorphic univalent in U .

For $f \in SH^*$ given by (2) and $g \in SH^*$ given by.

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k, (b_k \geq 0, k \in \mathbb{N} = \{1, 2, \dots\}), \quad (3)$$

The Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^{-1} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Definition(1): A function $f \in SH^*$ is said to be in the class $SH^*(\alpha, \gamma, \lambda)$

If and only if

$$\left| \frac{z(z^2(D^\lambda f(z))')' - z^2(D^\lambda f(z))'}{\gamma - z^2(D^\lambda f(z))'} \right| < \alpha, \text{ where } (0 < \alpha \leq 1, \gamma > 0 \text{ and } \lambda > -1), \quad (4)$$

and D^λ is the Ruscheweyh derivative [7,8] defined by

$$D^\lambda f(z) = f(z) * \frac{1}{z(1-z)^{\lambda+1}} = z^{-1} + \sum_{k=1}^{\infty} a_k D_k(\lambda) z^k,$$

Where $D_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k+1)}{(k+1)!}$, $\lambda > -1, z \in U^*$.

Some authors studied meromorphic univalent functions for another classes, like,

Aouf [1,2], Atshan [3], Astshan and Abdul-Kadhim [4], Atshan and kulkarni [5] and Ghanim and Darus [6].

2-Coefficient Inequality:

The first the theorem gives a necessary and Sufficient condition for a function f to be in the class $SH^*(\alpha, \gamma, \lambda)$

Theorem 1: Let $f \in SH^*$. then $f \in SH^*(\alpha, \gamma, \lambda)$ if and only if

$$\sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) a_k \leq \alpha(1+\gamma), (0 < \alpha < 1, \gamma > 0 \text{ and } \lambda > -1) \quad (5)$$

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1+\gamma)}{k(k+\alpha) D_k(\lambda)} z^k, (k \geq 1). \quad (6)$$

Proof: suppose that the inequality (5) holds true and $|z| = 1$. then, we have

$$\left| z(z^2(D^\lambda f(z))')' - z^2(D^\lambda f(z))' \right| - \alpha \left| \gamma - z^2(D^\lambda f(z))' \right|$$

$$\begin{aligned}
&= \left| \sum_{k=1}^{\infty} k^2 D_k(\lambda) a_k z^{k+1} \right| - \alpha \left| (1 + \gamma) - \sum_{k=1}^{\infty} k D_k(\lambda) a_k z^{k+1} \right| \\
&\leq \sum_{k=1}^{\infty} k^2 D_k(\lambda) a_k |z|^{k+1} - \alpha (1 + \gamma) + \sum_{k=1}^{\infty} \alpha k D_k(\lambda) a_k |z|^{k+1} \\
&= \sum_{k=1}^{\infty} k(k + \alpha) D_k(\lambda) a_k - \alpha (1 + \gamma) \leq 0, \text{ by hypothesis.}
\end{aligned}$$

Thus by maximum modulus principle, $f(z) \in SH^*(\alpha, \gamma, \lambda)$.

To show The Converse, Suppose that $f(z) \in SH^*(\alpha, \gamma, \lambda)$. Then From (4), we have

$$\begin{aligned}
&\left| \frac{z \left(z^2 \left(D^\lambda f(z) \right)' \right)' - z^2 \left(D^\lambda f(z) \right)'}{\gamma - z^2 \left(D^\lambda f(z) \right)'} \right| \\
&= \left| \frac{\sum_{k=1}^{\infty} k^2 D_k(\lambda) a_k z^{k+1}}{(1 + \gamma) - \sum_{k=1}^{\infty} k D_k(\lambda) a_k z^{k+1}} \right| < \alpha.
\end{aligned}$$

Since $\operatorname{Re}(z) \leq |z|$ for all z ($z \in U$), we have

$$\operatorname{Re} \left(\frac{\sum_{k=1}^{\infty} k^2 D_k(\lambda) a_k z^{k+1}}{(1 + \gamma) - \sum_{k=1}^{\infty} k D_k(\lambda) a_k z^{k+1}} \right) \leq \alpha. \quad (7)$$

We Choose the value of z on the real axis so that

$z^2 \left(D^\lambda f(z) \right)'$ is real. Upon clearing the denominator of (7) and

letting $z \rightarrow 1^-$, through real values so we can write (7) as

$$\sum_{k=1}^{\infty} k(k + \alpha) D_k(\lambda) a_k \leq \alpha (1 + \gamma).$$

Sharpness of the result follows by setting

$$f(z) = z^{-1} + \frac{\alpha (1 + \gamma)}{k(k + \alpha) D_k(\lambda)} z^k, (k \geq 1).$$

Corollary 1: Let $f \in SH^*(\alpha, \gamma, \lambda)$. Then

$$a_k \leq \frac{\alpha (1 + \gamma)}{k(k+\alpha)D_k(\lambda)}, (k \geq 1).$$

3- convex set:

In the following theorem, we show that the class $SH^*(\alpha, \gamma, \lambda)$ is a convex set.

Theorem2: The class $SH^*(\alpha, \gamma, \lambda)$ is a convex set.

Proof: let f_1 and f_2 be the arbitrary elements of $SH^*(\alpha, \gamma, \lambda)$ then for every t ($0 \leq t \leq 1$)

, we show that $(1 - t)f_1 + tf_2 \in SH^*(\alpha, \gamma, \lambda)$. thus, we have.

$$(1 - t)f_1 + tf_2 = z^{-1} + \sum_{k=1}^{\infty} [(1 - t)a_k + tb_k] z^k$$

Hence,

$$\begin{aligned} & \sum_{k=1}^{\infty} k(k+\alpha)D_k(\lambda)[(1 - t)a_k + tb_k] \\ &= (1 - t) \sum_{k=1}^{\infty} k(k+\alpha)D_k(\lambda) + t \sum_{k=1}^{\infty} k(k+\alpha)D_k(\lambda)b_k \\ &\leq (1 - t) \alpha (1 + \gamma) + t \alpha (1 + \gamma) = \alpha (1 + \gamma) \end{aligned}$$

This completes the proof.

4-Arithmetic mean:

In the following theorem, we obtain the arithmetic mean of the function in the class $SH^*(\alpha, \gamma, \lambda)$.

Theorem3: Let the functions f_n defined by

$$f_n(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k,n} z^k, (a_{k,n} \geq 0, k \in \mathbb{N}, n = 1, 2, \dots, \ell)$$

be in the class $SH^*(\alpha, \gamma, \lambda)$ for every $n = 1, 2, \dots, \ell$, then the function h defined by

$$h(z) = z^{-1} + \sum_{k=1}^{\infty} e_k z^k, (e_k \geq 0, k \in \mathbb{N})$$

also belong to the class $SH^*(\alpha, \gamma, \lambda)$, where $e_k = \frac{1}{\ell} \sum_{n=1}^{\infty} a_{k,n}$.

Proof: since $f_n \in SH^*(\alpha, \gamma, \lambda)$, it Follows from Theorem 1 that

$$\sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) a_{k,n} \leq (1 + \gamma),$$

For every $n = 1, 2, 3, \dots, l$ Hence

$$\begin{aligned} \sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) e_k &= \sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) \left(\frac{1}{\ell} \sum_{n=1}^{\ell} a_{k,n} \right) \\ &= \\ \frac{1}{\ell} \left(\sum_{n=1}^{\ell} \sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) a_{k,n} \right) &\leq \frac{1}{\ell} \sum_{n=1}^{\ell} (1 + \gamma) = \alpha(1 + \gamma) \end{aligned}$$

Then $h \in SH^*(\alpha, \gamma, \lambda)$.

5- Distortion Bounds:

In the Following theorem ,we obtain the distortion bounds for the function f in the class $SH^*(\alpha, \gamma, \lambda)$

Theorem4: If $f \in SH^*(\alpha, \gamma, \lambda)$, then

$$\begin{aligned} \frac{1}{r} - \frac{\alpha(1 + \gamma)}{(1+\alpha)D_1(\lambda)} r &\leq |f(z)| \\ &\leq \frac{1}{r} + \frac{\alpha(1 + \gamma)}{(1+\alpha)D_1(\lambda)} r, (|z| = r < 1) \quad (8) \end{aligned}$$

Where $D_1(\lambda) = \frac{1}{2}(\lambda + 1)(\lambda + 2)$.

The result is sharp for the function

$$f(z) = z^{-1} + \frac{\alpha(1 + \gamma)}{(1+\alpha)D_1(\lambda)} z. \quad (9)$$

Proof: let $f \in SH^*(\alpha, \gamma, \lambda)$. then by Theorem 1, we get

$$(1+\alpha)D_1(\lambda) \sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} k(k+\alpha) D_k(\lambda) a_k \leq \alpha(1 + \gamma).$$

or

$$\sum_{k=1}^{\infty} a_k \leq \frac{\alpha (1 + \gamma)}{(1 + \alpha) D_1(\lambda)}. \quad (10)$$

Hence

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + \sum_{k=1}^{\infty} a_k |z|^k \leq \frac{1}{|z|} + |z| \sum_{k=1}^{\infty} a_k \\ &= \frac{1}{r} + r \sum_{k=1}^{\infty} a_k \leq \frac{1}{r} + \frac{\alpha (1 + \gamma)}{(1 + \alpha) D_1(\lambda)} r. \end{aligned} \quad (11)$$

Similarly,

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - \sum_{k=1}^{\infty} a_k |z|^k \geq \frac{1}{|z|} - |z| \sum_{k=1}^{\infty} a_k \\ &= \frac{1}{r} - r \sum_{k=1}^{\infty} a_k \geq \frac{1}{r} - \frac{\alpha (1 + \gamma)}{(1 + \alpha) D_1(\lambda)} r \end{aligned} \quad (12)$$

From (11) and (12), we get (8) and the proof is complete.

6-Radii of starlikeness and convexity

Now, we obtain the radii of starlikeness and convexity.

Theorem 5: If $f \in SH^*(\alpha, \gamma, \lambda)$, then f is univalent meromorphic starlike of order φ ($0 \leq \varphi < 1$) in the disk $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{k(1 - \varphi)(k + \alpha) D_n(\lambda)}{(k - \varphi + 2) \alpha (1 + \gamma)} \right\}^{\frac{1}{k+1}}, \quad (6)$$

The result is sharp for the function f given by (6).

Proof : It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \varphi \text{ for } |z| < r_1. \quad (13)$$

But

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{zf'(z) + f(z)}{f(z)} \right| \leq \frac{\sum_{k=1}^{\infty} (k + 1) a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k+1}}$$

Thus, (13) will be satisfied if

$$\frac{\sum_{k=1}^{\infty} (k + 1) a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k+1}} \leq 1 - \varphi,$$

or if

$$\sum_{k=1}^{\infty} \frac{(k - \varphi + 2)}{1 - \varphi} a_k |z|^{k+1} \leq 1. \quad (14)$$

Since $f \in SH^*(\alpha, \gamma, \lambda)$, we have

$$\sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha(1+\gamma)} a_k \leq 1.$$

Hence, (14) will be true if

$$\frac{(k - \varphi + 2)}{1 - \varphi} |z|^{k+1} \leq \frac{k(k+\alpha)D_k(\lambda)}{\alpha(1+\gamma)},$$

Or equivalently

$$|z| \leq \left\{ \frac{k(1-\varphi)(k+\alpha)D_k(\lambda)}{(k-\varphi+2)\alpha(1+\gamma)} \right\}^{\frac{1}{k+1}}, \quad (k \geq 1),$$

Which follows the result.

Theorem 6: If $f \in SH^*(\alpha, \gamma, \lambda)$, then f is univalent meromorphic convex of order

φ ($0 \leq \varphi < 1$) in the disk $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(1-\varphi)(k+\alpha)D_n(\lambda)}{\alpha(k-\varphi+2)(1+\gamma)} \right\}^{\frac{1}{k+1}}$$

The result is sharp for the function f given by (6).

Proof : it is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \varphi \text{ for } |z| < r_2. \quad (15)$$

But

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \frac{zf''(z) + 2f'(z)}{f'(z)} \right| \leq \frac{\sum_{k=1}^{\infty} k(k+1)a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} ka_k |z|^{k+1}}$$

Thus, (15) will be satisfied if

$$\frac{\sum_{k=1}^{\infty} k(k+1)a_k|z|^{k+1}}{1 - \sum_{k=1}^{\infty} ka_k|z|^{k+1}} \leq 1 - \varphi,$$

Or if

$$\sum_{k=1}^{\infty} \frac{k(k - \varphi + 2)}{1 - \varphi} a_k |z|^{k+1} \leq 1. \quad (16)$$

since $f \in SH^*(\alpha, \gamma, \lambda)$, we have

$$\sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha(1+\gamma)} a_k \leq 1.$$

Hence, (16) will be true if

$$\frac{k(k - \varphi + 2)}{1 - \varphi} |z|^{k+1} \leq \frac{k(k+\alpha)D_k(\lambda)}{\alpha(1+\gamma)}$$

Or equivalently

$$|z| \leq \left\{ \frac{(1 - \varphi)(k+\alpha)D_k(\lambda)}{\alpha(k - \varphi + 2)(1 + \gamma)} \right\}^{\frac{1}{k+1}}, (k \geq 1),$$

which follows the result

7- Hadamard product (convolution):

In the following theorem, we obtain the convolution property of the class $SH^*(\alpha, \gamma, \lambda)$

Theorem 7: let the functions f and g of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k, g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k$$

belong to the class $SH^*(\alpha, \gamma, \lambda)$. then the Hadamard product (or convolution)

Of two functions f and g belong to the class $SH^*(\alpha, \gamma_1, \lambda)$, where

$$\gamma_1 = \frac{\alpha (1 + \gamma)^2}{k(k+\alpha)D_k(\lambda)} - 1$$

Proof : since f and $g \in SH^*(\alpha, \gamma, \lambda)$. then from theorem (1), we have

$$\sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma)} a_k \leq 1, \sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma)} b_k \leq 1.$$

We need to find largest number γ_1 Such that

$$\sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma_1)} a_k b_k \leq 1.$$

By using cauchy – schwarz inequality , we have

$$\sum_{k=1}^{\infty} \frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma)} \sqrt{a_k b_k} \leq 1. \quad (17)$$

Thus it is sufficient to show that

$$\frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma_1)} a_k b_k \leq \frac{k(k+\alpha)D_k(\lambda)}{\alpha (1 + \gamma)} \sqrt{a_k b_k}$$

This is equivalent to

$$\sqrt{a_k b_k} \leq \frac{1 + \gamma_1}{1 + \gamma} \quad (18)$$

From (17) ,we get

$$\sqrt{a_k b_k} \leq \frac{\alpha (1 + \gamma)}{k(k+\alpha)D_k(\lambda)}$$

Thus ,it is sufficient to show that

$$\frac{\alpha (1 + \gamma)}{k(k+\alpha)D_k(\lambda)} \leq \frac{1 + \gamma_1}{1 + \gamma}$$

Which implies

$$\gamma_1 \geq \frac{\alpha (1 + \gamma)^2}{k(k+\alpha)D_k(\lambda)} - 1.$$

المستخلص

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