Republic of Iraq
Ministry of Higher Education & Scientific Research
AL-Qadisiyah University
College of Computer Science and Mathematics
Department of Mathematics

Homotopy Perturbation Method for Nonlinear Equation

A Research

Submitted by

Suhair Saad Nama

To the Council of the department of Mathematics / College of Education, University of AL-Qadisiyah as a Partial Fulfilment of the Requirements for the Bachelor Degree in Mathematics

Supervised by

Alaa Kamel Jaber

A. D. 2017 *A.H. 1438*

بسم الله الرحمن الرحيم

وما أوتيتم من العلم إلا قليلا

صدق الله العلي العظيم

سورة الاسراء
آيه (85)

شكر وتقدير

الحمد لله المعلم بالقلم والشكر له على ما جاد وانعم والصلاة والسلام على نبيه وسيد المرسلين محمد(صلى الله عليه واله وسلم)

وبعد... فلا يسعنا ونحن نختم هذا العمل الا ان نخط كلمات صغيرة في حجمها لكنها كبيرة فيما تحمله من معاني الشكر والامتنان الى كل ذي فضل اسهم بجهد او مشورة او دعم في انجاز هذا البحث في البداية اتقدم بالشكر الجزيل الى الدكتور علاء كامل جابر (مشرف البحث) والذي والذي امدنا بالنصح والمشورةوتفضل علينا بالاشراف على البحث ليضهربشكله المتكامل.

ونقدم شكرنا وامتناننا لاساتذتنا الافاضل لجهودهم البناءة في ارواء ظمأنا الى معين العلم والمعرفة.

كما نقدم شكرنا الى كل الزملاء في الدراسة والى الاصدقاء الذين امدونا بدعمهم وعونهم ودعائهم.

وختاما نسأل الله العلي القدير ميسر الامور ومفرج الهموم ان نكون قد وفقنا فيما قدمناه في هذا البحث.

سهير

الاهداء

والى (وسلم واله عليه الله صلى) محمد الكريم رسولنا الى

(السلام عليهم) والزمان العصر صاحب والى الاطهار آله

من منسوجة بخيوط سعادتي حاكت من الى العطاء لايمل الذي الينبوع الى قبلها

العزيزة امي

بشيء يبخل لم الذي والهناء بالراحة لانعم وشقى سعى من الى

العزيز ابي

العلم في عبارات واحلى درر من وكلمات ذهب من حروفا علمونا من الى

حروفا علمهم صاغوا من

الكرام اساتذتي

الطريق نشق ونحن سويا سرنا من الى عروقي في يجري حبهم من الى

وزميلاتي زملائي

سهير

Abstract

In this paper, the homotopy perturbation method (HPM) which doesn't small parameter is applied to solve the linear and nonlinear differential equations. The HPM deforms a difficult problem into a simple problem which can be easily solved. It is implemented with appropriate initial conditions. Comparison of the applied methods with exact solutions reveals that the method is tremendously effective.

Contents

Introduction

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He. Unlike classical techniques, the homotopy perturbation method leads to an analytical approximate and exact solutions of the nonlinear equations easily and elegantly without transforming the equation or linearizing the problem and with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. As a numerical tool, the method provides us with a numerical solution without discretization of the given equation, and therefore, it is not effected by computation round-off errors and one is not faced with the necessity of large computer memory and time. This technique has been employed to solve a large variety of linear and nonlinear problems.

In the present study, homotopy perturbation method has been applied to solve the parabolic equations. The numerical results are compared with the exact solutions. It is shown that the errors are very small. However, recently, Adomian decomposition method has was applied for approximating the solution of the parabolic equations.

Chapter One

Basic Concepts

1.1 Introduction

In this chapter we will discuss the partial differential equations, linear or nonlinear, homogeneous and inhomogeneous. In section 1.2 we introduce the definition of a PDE, in section 1.3 we will discuss the order of a PDE, the linearity property introduce in section 1.4, the homogeneous and inhomogeneous PDEs in section 1.5, the solution of a PDE discuss in 1.6, in 1.7 we introduce the initial and boundary conditions.

1.2 Definition of a PDE

A partial differential equation (PDE) is an equation that contains the dependent variable (the unknown function), and its partial derivatives. It is known that in the ordinary differential equations (ODEs), the dependent variable $u = u(x)$ depends only on one independent variable x. Unlike the ODEs, the dependent variable in the PDEs, such as $u = u(x,t)$ or $u = u(x,y,t)$, must depend on more than one independent variable. If $u = u(x,t)$, then the function u depends on the independent variable x, and on the time variable t. However, if $u = u(x,y,t)$, then the function u depends on the space variables x,y, and on the time variable t. [13]

Examples of the PDEs are given by

$u_t = ku_{xx}$ ………………………………………………………………… (1.1)

$u_t = k(u_{xx} + u_{yy})$ ………………………………………………………… (1.2)

$u_t = k(u_{xx} + u_{yy} + u_{zz})$ ………………………………………….……..... (1.3)

that describe the heat flow in one dimensional space, two dimensional space, and three dimensional space respectively. In (1.1), the dependent variable $u = u(x,t)$ depends on the position x and on the time variable t. However, in (1.2), $u = u(x,y,t)$ depends on three independent variables, the space variables x,y and the time variable t. In (1.3), the dependent variable $u = u(x,y, z,t)$ depends on four independent variables, the space variables x,y, and z, and the time variable t.

Other examples of PDEs are given by

$u_{tt} = c^2u_{xx}$ ………………………………………………………………… (1.4)

$u_{tt} = c^2(u_{xx}+u_{yy})$ ……………………………………………………………… (1.5)

$u_{tt} = c^2(u_{xx}+u_{yy}+u_{zz})$ …………………………………………………….. (1.6)

that describe the wave propagation in one dimensional space, two dimensional space, and three dimensional space respectively. Moreover, the unknown functions in (1.4), (1.5), and (1.6) are defined by $u = u(x,t)$, $u = u(x,y,t)$, and $u = u(x,y, z,t)$ respectively.

1.3 Order of a PDE

The order of a PDE is the order of the highest partial derivative that appears in the equation. For example, the following equations

$u_x-u_y = 0,$

$u_{xx} - u_t = 0$ ………………..…………………………………………(1.12)

$u_y-uu_{xxx} = 0$

are PDEs of first order, second order, and third order respectively.

Example 1 : The order of the following PDEs:

(a) $u_t = u_{xx}+u_{yy}$

The highest partial derivative contained in this equation is u_{xx} or u_{yy}. The PDE is therefore of order two.

(b) $u_x+u_y = 0$

The highest partial derivative contained in this equation is u_x or u_y. The PDE is therefore of order one.

(c) $u^4u_{xx}+u_{xxy} = 2$

The highest partial derivative contained in this equation is u_{xxy}. The PDE is therefore of order three.

(d) $u_{xx} + u_{yyyy} = 1$

The highest partial derivative contained in this equation is u_{yyyy}. The PDE is therefore of order four.

1.4 Linear and Nonlinear PDEs

Partial differential equations are classified as ***linear*** or ***nonlinear***. A partial differential equation is called linear if:

(1) the power of the dependent variable and each partial derivative contained in the equation is one, and

(2) the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear. [13]

Example 2 : The Classify of the following PDEs as *linear* or *nonlinear*:

(a) $xu_{xx}+yu_{yy} = 0$

The power of each partial derivative u_{xx} and u_{yy} is one. In addition, the coefficients of the partial derivatives are the independent variables x and y respectively. Hence, the PDE is linear

(b) $uut + xux = 2$

Although the power of each partial derivative is one, but u_t has the dependent variable u as its coefficient. Therefore, the PDE is nonlinear.

(c) $u_x+\sqrt{u} = x$

The equation is nonlinear because of the term $\sqrt{u}$.

(d) $u_{rr}+\frac{1}{r}u_r + \frac{1}{r2}u_{\theta\theta} = 0,$

The equation is linear because it satisfies the two necessary conditions.

1.4.1 Some Linear Partial Differential Equations

As stated before, linear partial differential equations arise in many areas of scientific applications, such as the diffusion equation and the wave equation. In what follows, we list some of the well-known models that are of important concern:

1. The *heat equation* in one dimensional space is given by

$$u_t = ku_{xx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.13)$$

where k is a constant.

2. The *wave equation* in one dimensional space is given by

$$u_{tt} = c^2u_{xx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.14)$$

where c is a constant.

3. The *Laplace equation* is given by

$$u_{xx}+u_{yy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.15)$$

4. The *Klein-Gordon equation* is given by

$$\nabla^2 u - \frac{1}{c2} u_{tt} = \mu^2 u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.16)$$

where c and μ are constants.

5. The *Linear Schrodinger's equation* is given by

$$iu_t + u_{xx} = 0 \quad , \quad i = \sqrt{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.17)$$

6. The *Telegraph equation* is given by

$$u_{xx} = au_{tt} + bu_t + cu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.18)$$

where a,b and c are constants. It is to be noted that these linear models and others will be studied in details in the forthcoming chapters.

1.4.2 Some Nonlinear Partial Differential Equations

It was mentioned earlier that partial differential equations arise in different areas of mathematical physics and engineering, including fluid dynamics, plasma physics, quantum field theory, nonlinear wave propagation and nonlinear fiber optics. In what follows we list some of the well-known nonlinear models that are of great interest:

1. The *Advection equation* is given by

$$u_t + uu_x = f(x,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.19)$$

2. The *Burgers equation* is given by

$$u_t + uu_x = \alpha u_{xx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.20)$$

3. The *Korteweg de-Vries* (*KdV*) *equation* is given by

$$u_t + auu_x + bu_{xxx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.21)$$

4. The *modified KdV equation* (*mKdV*) is given by

$$u_t - 6u^2 u_x + u_{xxx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.22)$$

5. The *Boussinesq equation* is given by

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.23)$$

6. The *sine-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sin u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.24)$$

7. The *sinh-Gordon equation* is given by

$$u_{tt} - u_{xx} = \alpha \sinh u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.25)$$

8. The *Liouville equation* is given by

$$u_{tt} - u_{xx} = e^{\pm u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.26)$$

9. The *Fisher equation* is

$$u_t = Du_{xx}+u(1-u) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.27)}$$

10. The *Kadomtsev-Petviashvili* (*KP*)*equation* is given by

$$(u_t +auu_x+bu_{xxx})_x+u_{yy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.28)}$$

11. The $K(n,n)$*equation* is given by

$$u_t +a(un)_x +b(un)_{xx} = 0,\ n > 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.29)}$$

1.5 Homogeneous and Inhomogeneous PDEs

Partial differential equations are also classified as **homogeneous** or **inhomogeneous**. A partial differential equation of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, otherwise, it is called an inhomogeneous PDE. This can be illustrated by the following example.

Example 3. The classify of the following partial differential equations as homogeneous or inhomogeneous:

(a) $u_t = 4u_{xx}$

The terms of the equation contain partial derivatives of u only, therefore it is a homogeneous PDE.

(b) $u_t = u_{xx}+x$

The equation is an inhomogeneous PDE, because one term contains the independent variable x.

(c) $u_{xx}+u_{yy} = 0$

The equation is a homogeneous PDE.

(d) $u_x+u_y = u+4$

The equation is an inhomogeneous PDE.

1.6 Solution of a PDE

A solution of a PDE is a function u such that it satisfies the equation under discussion and satisfies the given conditions as well. In other words, for u

to satisfy the equation, the left hand side of the PDE and the right hand side should be the same upon substituting the resulting solution. This concept will be illustrated by examining the following examples. Examples of partial differential equations subject to specific conditions will be examined in the coming chapters.

Example 4. To show that $u(x,t) = \sin x\, e^{-4t}$ is a solution of the following PDE

$$u_t = 4u_{xx} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.30)$$

we have

Left Hand Side (LHS)= $u_t = -4\sin x e^{-4t}$

Right Hand Side (RHS)=$4u_{xx} = -4\sin x e^{-4t}$=LHS

Example 5. To show that $u(x,y)=\sin x\, \sin y+x^2$ is a solution of the following eq.

$$u_{xx} = u_{yy}+2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.31)$$

we have

Left Hand Side (LHS)= $u_{xx} = -\sin x\, \sin y+2$

Right Hand Side (RHS)=$u_{yy}+2 = -\sin x\, \sin y+2$=LHS

Example 6. To show that $u(x,t) = \cos x \cos t$ is a solution of the following PDE

$$u_{tt} = u_{xx} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.32)$$

we have

Left Hand Side (LHS)= $u_{tt} = -\cos x\, \cos t$

Right Hand Side (RHS)=$u_{xx} = -\cos x\, \cos t$=LHS

Example 7. To show that

(a) $u(x,y) = xy$

(b) $u(x,y) = x^2y^2$

(c) $u(x,y) = \sin(xy)$

are solutions of the equation

$$xu_x - yu_y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.33)$$

we have

(a) $u = xy,\ ux = y,\ uy = x,$

$$xux - yuy = xy - yx = 0$$

(b) $u = x2y2,\ ux = 2xy2,\ uy = 2x2y,$

$$xux - yuy = 2x2y2 - 2x2y2 = 0$$

(c) $u = \sin(xy),\ ux = y\cos(xy),\ uy = x\cos(xy),$

$$xux - yuy = xy\cos(xy) - yx\cos(xy) = 0$$

1.7.1 Boundary Conditions

As stated above, the general solution of a PDE is of little use. A particular solution is frequently required that will satisfy prescribed conditions. Given a PDE that controls the mathematical behavior of physical phenomenon in a bounded domain *D*, the dependent variable *u* is usually prescribed at the boundary of the domain *D*. The boundary data is called *boundary conditions*. The boundary conditions are given in three types defined as follows:

1. **Dirichlet Boundary Conditions**: In this case, the function *u* is usually prescribed on the boundary of the bounded domain.

2. **Neumann Boundary Conditions**: In this case, the normal derivative $\frac{du}{dn}$ of *u* along the outward normal to the boundary is prescribed.

3. **Mixed Boundary Conditions**: In this case, a linear combination of the dependent variable *u* and the normal form $\frac{du}{dn}$ is prescribed on the boundary.

It is important to note that it is not always necessary for the domain to be bounded, however one or more parts of the boundary may be at infinity. This type of problems will be discussed in the coming chapters.

1.7.2 Initial Conditions

It was indicated before that the PDEs mostly arise to govern physical phenomenon such as heat distribution, wave propagation phenomena and phenomena of quantum mechanics. Most of the PDEs, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. It will be discussed later that for the heat case, the initial value $u(t = 0)$, that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions $u(t = 0)$ and $u_t(t = 0)$ should also be prescribed.

Chapter Two

Homotopy Perturbation Method

2.1 Introduction

In this chapter we will discuss the ***Homotopy Perturbation Method***, first we will introduce the basic idea of this method in section 2.2, in section 2.3 we will apply this method on some examples .

2.2 The Basic Idea

To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation:

$$A(u) - F(r) = 0, r \in \Omega \qquad (2.1)$$

Considering the boundary conditions of:

$$B\left(u, {\partial u}/{\partial n}\right), r \in \Gamma$$

(2.2)

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. The operator A can be generally divided into two parts of L and N, , where L is the linear part, while N is the nonlinear one. Eq. (2.1) can, therefore, be rewritten as: $L(u) + N(u) - f\ (r) = 0.$
(2.3)

By the homotopy technique, we construct a homotopy as $v(r\ , p\ ): \Omega \times [0,1] \to R$ which satisfies:

$$H(v, p) = (1- p)[L(v) - L(u_0\ )] + p[\ A(v) - f\ (r)] = 0, p \in [0,1], r \in \Omega$$

(2.4)

Or

$$H\ (v, p) = L(v) - L(u_0\ ) + pL(u_0\ ) + p[N\ (v) - f\ (r)] = 0$$

(2.5)

Where $P \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of equation (2.2) which satisfy the boundary conditions. Obviously, considering equation (2.4) and (2.5),we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (2.6)$$

$$H(v,1) = A(v) = f(r) = 0 \qquad (2.7)$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called **deformation**, and $L(v) - L(u_0)$ and $A(v) = f(r)$ are called **homotopy**

According to HPM, we can first use the embedding parameter p as "small parameter", and assume that the solution of equation (2.5) and (2.6) can be written as a power series in p:

$$v = v_0 + pv_1 + pv_2 + \cdots \qquad (2.8)$$

Setting $p = 1$ results in the approximate solution of equation (2.1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \qquad (2.9)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which lessens the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

The series (2.9) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He:

1. The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, that mean $p \to 1$.
2. The norm of $L^{-1}\, \partial N / \partial v$ must be smaller than one so that the series converges.

2.3 Application of HPM

In this section, we demonstrate the main algorithm of homotopy perturbation method on linear and nonlinear equations with initial condition, namely we consider:

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + \emptyset(u) + g(x,t), (x,t) \in [a,b] \times (0, T) \tag{2.10}$$

With the initial condition $u(x,0) = f(x)$ (2.11)

where φ is a function of u. We are looking for the solution satisfying Eqations (2.1) ,(2.2).

2.3.1 Example 1:

This problem was used by Hopkins and Wait [12] to provide an example of a problem with a nonlinear source term:

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + e^{-u} + e^{-2u} ,(x ,t) \in [a,b] \times (0,T) \tag{2.12}$$

with the initial condition $u(x,0) = \ln(x+2)$.

In this example we have $e^{-u} + e^{-2u}$

$$g(x,t) = 0, f(x) = \ln(x+2)$$

we construct the following homotopy:

$$\frac{du}{dt} - \frac{du_0}{dt} = p\left(\frac{d^2u}{dx^2} + e^{-u} + e^{-2u} - \frac{du_0}{dt}\right)$$

(2.13)

Assume the solution of equation (2.13) to be in the form:

$$u = u_0 + pu_1 + pu_2 + pu_3 + \cdots$$

(2.14)

Substitution (2.14) into(2.13) and equation the coefficients of like powers p we get the following set of differential equations:

$$p^0: \frac{du_0}{dt} - \frac{du_0}{dt} = 0$$

$$p^1: \frac{du_1}{dt} = \frac{d^2u_0}{dx^2} + e^{-u_0} + e^{-2u_0} - \frac{du_0}{dt}$$

$$p^2: \frac{du_2}{dt} = \frac{d^2u_1}{dx^2} + u_1(-e^{-u_0} - e^{-2u_0}) \tag{2.15}$$

$$p^3: \frac{du_3}{dt} = \frac{d^2u_2}{dx^2} + (-u_2 + \frac{1}{2}v_1^2)e^{-u_0} + (-2u_2 + 2u_1^2 - \frac{1}{48}u_2u_1^2)e^{-2u_0}$$

Solving the above equations, we obtain

$$u_o = \ln(x+2),$$

$$u_1 = \frac{t}{x+2},$$

$$u_2 = \frac{-t^2}{2(x+2)^2}, \tag{2.16}$$

$$u_3 = \frac{t^3}{3(x+2)^3},$$

$$u_n = \frac{(-t)^{n+1}t^n}{n(x+2)^n},$$

Therefore from the results we can obtain

$$u(x,t) = \ln(x+2) + \frac{t}{x+2} - \frac{t^2}{2(x+2)^2} + \frac{t^3}{3(x+2)^3} + \cdots + \frac{(-t)^{n+1}t^n}{n(x+2)^n} + \cdots$$

$$= \ln(x+2) + \ln\left(\frac{t}{x+2}, +1\right) = \ln(x+t+2), \tag{2.17}$$

<u>2.3.2 Example 2:</u>

The problem was used by Lawson and Et. Al. as the form

$$\frac{du}{dx} = \frac{d^2u}{dx^2} + (\pi^2 - 1 - p)u + p(pe^{-t} + e^{-pt}), (x,t) \in [a,b] \times (0,T) \tag{2.18}$$

With the initial condition

$$u(x,0) = 2\sin(\pi x) \tag{2.19}$$

In this example we have

$$\emptyset(u) = (\pi^2 - 1 - p)u,\ \ g(x,t) = pe^{-t} + e^{-pt}, f(x) = 2\sin(\pi x).$$

We construct the following homotopy:

$$\frac{du}{dt} - \frac{du_0}{dt} = p\left(\frac{d^2u}{dx^2} + (\pi^2 - 1 - p)u + pe^{-u} + e^{-pt}) - \frac{du_0}{dt}\right) \tag{2.20}$$

Substituting (2.5) into (2.20) and equating the coefficients of like powers p, we get following set of differential equations:

$$p^0: \frac{du_0}{dt} - \frac{du_0}{dt} = 0$$

$$p^1: \frac{du_1}{dt} = \frac{d^2u_0}{dx^2} + ((\pi^2 - 1 - p)u_0(pe^{-u} + e^{-pt}) - \frac{du_0}{dt} \tag{2.21}$$

$$p^2: \frac{du_2}{dt} = \frac{d^2u_1}{dx^2}\ + (\pi^2 - 1 - p)u_1$$

$$p^3: \frac{du_3}{dt} = \frac{d^2u_2}{dx^2} + (\pi^2 - 1 - p)u_2$$

Solving the above equations, we obtain

$$u_o = [2 - pe^{-pt} + \frac{1}{p^2}e^{-pt} + \left(p + \frac{1}{p^2}\right)]\sin(\pi x) \tag{2.22}$$

$$u_1 = -(1 - p)[2T + pe^{-t} + \frac{1}{p^2}e^{-pt} + \left(p + \frac{1}{p}\right)t - \left(p + \frac{1}{p^2}\right)]\sin(\pi x)$$

(2.23)

$$u_2 = (1 - p)^2[t^2 + pe^{-t} + \frac{1}{p^3}e^{-pt} + \left(p + \frac{1}{p}\right)\frac{t^2}{2!} - \left(p + \frac{1}{p^2}\right)t + (p + \frac{1}{p^3})]\sin(\pi x) \tag{2.24}$$

And so on. Therefore from the equations, we have

$$u(x,t) = [2 -\ pe^{-pt} + \frac{1}{p^2}e^{-pt} + \left(p + \frac{1}{p}\right)t - \left(p + \frac{1}{p^2}\right)]\sin(\pi x)$$

$$-(1 + p)\left[2T + pe^{-t} + \frac{1}{p^2}e^{-pt} + \left(p + \frac{1}{p}\right)t - \left(p + \frac{1}{p^2}\right)\right]\sin(\pi x) +$$

$(1-p)^2[t^2 + pe^{-t} + \frac{1}{p^3}e^{-pt} + \left(p+\frac{1}{p}\right)\frac{t^2}{2!} - \left(p+\frac{1}{p^2}\right)t + (p +$

$\frac{1}{p^3})] \sin(\pi x) + \cdots = (e^{-t} + e^{-pt}) \sin(\pi x)$

2.2.3 Example 3: Firstly, we consider the linear Schrödinger Equation:

$$u_t + iu_{xx} = 0,\ u(x,0) = 1 + \cosh 2x \tag{2.25}$$

where $u(x,t)$ is a complex function and $i^{\,2} = -1$.
According to ($H(V,p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, p \in [0,1], r \in \Omega$), a homotopy $(x,t,p) : \Omega \times [0,1] \to C$ can be constructed as follows:

$$(1-p)(V_t - u_{0,t}) + p(V_t + iV_{xx}) = 0, p \in [0,1], (x,t) \in \Omega, \tag{2.26}$$

where $u_0(x,t) = V_0(x,0) = u(x,0)$ and $u_{0,t} = \partial u_0/\partial t$.
We now try to get a solution of (2.26) in the form

$$V(x,t) = V_0(x,t) + pV_1(x,t) + p2V_2(x,t) + \cdots. \tag{2.27}$$

Substituting (2.27) into (2.28), and equating the terms with the identical powers of p, yields

$$\begin{aligned} p^0 &: V_{0,t} = 0, \\ p^1 &: V_{1,t} + iV_{0,xx} = 0, \\ p^2 &: V_{2,t} + iV_{1,xx} = 0, \\ &. \\ &. \\ &. \\ p^n &: V_{n,t} + iV_{n-1,xx} = 0, \quad n = 3,4,5,\cdots, \end{aligned} \tag{2.28}$$

with the following initial conditions:

$$V_i(x,0) = \begin{cases} 1 + cosh(2x),\ i = 0, & i = 0 \\ 0 & i = 1,2,3,... \end{cases} \tag{2.29}$$

The solution of the system (2.29), with the initial conditions (2.30), can be easily obtained as follows:

$$\begin{aligned} V_0(x,t) &= 1 + cosh(2x), \\ V_1(x,t) &= -4itcosh(2x), \\ V_2(x,t) &= -8t^2cosh(2x), \\ V_3(x,t) &= \frac{32}{3}it^3cosh(2x), \\ V_4(x,t) &= \frac{32}{3}t^4cosh(2x), \end{aligned} \tag{2.31}$$

$V_5(x,t) = -\frac{128}{15} it^5 cosh(2x).$

In this manner the other components can be easily obtained. Substituting (2.31) into ($u = \lim_{p\to 1} V = V_0 + V_1 + V_3 + \cdots$) yields

$u(x,t) = \big(1 + cosh(2x)\big)1 - 4it - 8t^2 + \frac{32}{3} it^3 + \frac{32}{3} t^4 - \frac{128}{15} it^5 - \cdots$

(2.32)

Consequently, the exact solution of (2.25)

$$u(x,t) = 1 + cosh(2x)e^{-4it}, \qquad (2.33)$$

is readily obtained upon using the Taylor series expansion of e^{-4it} .

2.3. Conclusion

In the present study the homotopy perturbation method was applied on some periodic equations. The solution has been compared with the exact solution. The results show that while the traditional perturbation method depends on small parameter assumption, and the obtained results, in most cases, end up with a non physical result, the numerical method leads to inaccurate results when the equation is intensively dependent on time, while He's homotopy perturbation method (HPM) overcomes completely the above shortcomings, revealing that the HPM is very convenient and effective.

References

[1]. D.D. Ganji, 2006. The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Physics Letters A, 353(4-5): 337-341.

[2]. J.H. He, 1998. An approximate solution technique depending on an artificial parameter: A special example, Communications in Nonlinear Science and Numerical Simulation, 3(2): 92-97.

[3]. J.H. He, 1999. Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering; 178: 257-262

[4]. J.H. He, 2000, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-Linear Mechanics, 35(1): 37-43.

[5]. J.H. He, 2003. A simple perturbation approach to Blasius equation, Applied Mathematics and Computation, 140(2-3): 217-222.

[6]. J.H. He, 2003. Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation, 135:73-79

[7]. J.H. He, 2004, The homotopy perturbation method for nonlinear oscillators with discontinuities, Applied Mathematics and Computation, 151(1): 287-292.

[8]. J.H. He, 2005. Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons & Fractals, 26(3): 695-700.

[9]. J.H. He, 2006. Homotopy perturbation method for solving boundary value problems, Physics Letters A, 30(1-2): 87-88.

[10]. M. Javidi, A. Golbabai, 2007. Adomian Decomposition Method for Approximating the Solution of the Parabolic Equations, Applied Mathematical Sciences, 1(5): 219 – 225.

[11]. S. Q. Wang, J.H. He, 2008. Nonlinear oscillator with discontinuity by parameter-expansion method, Chaos, Solitons & Fractals, 35(4): 688-691.

[12]. T.R. Hopkins and R. Wait, 1978. 1978A comparison of galerkin collocation and the method of lines for PDE’s, Int. J. Numer. Meth. Engin., 12(7): 1081-1107.

[13]. Wazwaz, Abdul-Majid, "Partial Differential Equations and Solitary Waves Theory ", Saint Xavier University, 2009.