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A subclass of Meromorphic univalent Function Defined by Ruscheweyh Derivative

A research

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

"رَبِّ أَوْزِرْ عَنِّي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَى
وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَصْلِحْ لِي فِي ذُرِّيَّتِي إِنِّي
خُفْتُ إِلَيْكَ وَإِنِّي مِنَ الْمُسْلِمِينَ"

سورة الاحقاف ايه (١٥)

الاهداء

يطيب لي ان اهدي ثمرة جهدي المتواضع الى من بعثهُ رحمة
للعاملين مرسولنا محمد (ص) إلى سيدي ومولاي قسيم
الجنة والنار واول الناس في قلبي ((علي بن ابي طالب عليه
السلام))

الى القلوب الطاهرة الرقيقة ((مرياحي حياتي)) عائلتي
الكريمة .

الى المصاييح المضيئة في طريقي نحو النجاح التفوق اساتذتي
الافاضل .

شكر وتقدير

كل الشكر والثناء لله سبحانه لأن التوفيق من عنده وفي هذه اللحظات يتوقف القلم ليفكر قبل أن يخط الحروف ليجمعها . . .

تبعثر الأحرف وعبثاً إن يحاول تجميعها في سطور كثيرة تمر في الخيال ولا يبقى لنا في نهاية المطاف الا قليلاً من الذكريات وصور تجمعنا برفاق كانوا الى جانبنا فواجب علينا شكرهم ووداعهم ونحن نخطو خطواتنا في غمار الحياة ونخص بالشكر الجزيل والعرفان لكل من اشعل شمعة في دروس علمنا وإلى من وقف على المنابر واعطى من حصيلة فكره لينير دربنا إلى الأساتذة واتوجه بالشكر الجزيل الى الدكتور

وقاص غالب عطشان

الذي تفضل بالأشراف فجزاه الله عنا كل خير

وله منا كل التقدير والاحترام .

Abstract

In the present paper, we obtain distortion theorem and the Hadamard product of function belonging the class $Th^+(\rho, \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ of meromorphic functions with positive and missing coefficients defined by Ruscheweyh derivative Also some properties of Neighborhoods of function in the class $Th(\rho, \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ are discussed .

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1- In trodution

Let Σ denote the class of function of the form :

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \dots\dots\dots(1.1)$$

which are analytic and univalent in the punctured unit disc

$U^* = \{z: z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$ and which have simple pole at the origin with residue on there . Let Σ_p denote the class of function defined by (1.1) with

$a_j = 0$ ($j = 1, 2, \dots, p-1$; $p \in \mathbb{N} = \{1, 2, \dots\}$) that is , by

$$f(z) = \frac{1}{z} + \sum_{k=p}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \dots\dots\dots(1.2)$$

which are analytic and univalent in U^* . setting

$$f(z) = (1-\lambda)f(z) + \lambda z f'(z) \quad (f \in \Sigma_p; 0 \leq \lambda < \frac{1}{2}). \dots\dots\dots(1.3)$$

So that obviously

$$F(z) = \frac{1-2\lambda}{z} + \sum_{k=p}^{\infty} [1 + \lambda(k+1)] a_k z^k \quad (p \in \mathbb{N}, 0 \leq \lambda < \frac{1}{2}).$$

Since $f(z) \in \Sigma_p$ is given by (1.2) .

For function $f(z) \in \Sigma_p$, we say that $f(z)$ is a member of the class $Th(\rho, \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ if the function $f(z)$ defined by (1.3) satisfies the inequality:

$$\left| \frac{z^2 (D^\sigma F(z))' + (1 - 2\lambda)}{[(B - A)Y - B]Z^2(D^\sigma F(z))' + 1 - 2\lambda} [(B - A)Y\alpha - B] \right| < \beta(z \in U^*) \quad (1.4)$$

Where the parameters $\alpha, \beta, \gamma, A, B, \lambda$ and m are constrained as follows : $0 \leq \alpha < 1; 0 \leq \beta < 1; 1 \leq A < B \leq 1; 0 < B \leq 1;$

$$\frac{B}{B-A} < \gamma \leq \left\{ \frac{B}{(B-A)\alpha} \begin{matrix} (\alpha \neq 0) \\ (\alpha = 0) \end{matrix} \right\}, \quad \sigma \geq 0$$

And $D^\sigma f(z)$ is the Ruscheweyh derivative of the function F introduced in [13] as follows :

$$D^\sigma f(z) = \frac{1-2\lambda}{z} + \sum_{k=p}^{\infty} [1 + \lambda(k-1)] D_k(\sigma) \alpha_k z^k \dots\dots\dots(1.5)$$

$$\text{Where } D_k(\sigma) = \frac{(\sigma+1)(\sigma+2)\dots(\sigma+k+1)}{(k+1)!}, \quad \sigma > -1, z \in U^*$$

We note that

$$\text{Th}(1; \alpha, \beta, \gamma, A, B, \lambda, \sigma) = \text{Th}(\sigma, \beta, \gamma, A, B, \lambda, \sigma).$$

Furthermore, we say that a function

$f(z) \in \text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ whenever $f(z)$ is of the form

[cf. Equation (1.2)]:

$$f(z) = \frac{1}{z} + \sum_{k=p}^{\infty} \alpha_k z^k, \quad (k \geq p; p \in \mathbb{N}) \dots\dots\dots(1.6)$$

The class $\text{Th}(p, \alpha, \beta, \gamma, A, B, \lambda, 0)$ was introduced and studied by Aout and Frasin[4].

The class $\text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ was introduced and studied by joshi et al. [9] we note that

$$(i) \quad \text{Th}^+(p; \alpha, \beta, \gamma, -1, 1, 0, 0) = \sum_p(\alpha, \beta, \gamma)$$

$$(0 \leq \alpha < 1; 0 < \beta \leq 1; \frac{1}{2} \leq \gamma \leq 1)$$

(cho et al . [7]);

$$(ii) \quad \text{Th}^+(1; \alpha, \beta, \gamma, -1, 1, 0, 0) = \sum_1(\alpha, \beta, \gamma) \quad (0 \leq \alpha < 1; 0 <$$

$$\beta \leq 1; \frac{1}{2} \leq \gamma \leq 1)$$

(cho et al . [6]);

$$(iii) \quad \text{Th}^+(1; 0, 1, 1, -A, -B, 0, 0) = \sum_p(A, B) \quad (-1 \leq B < A \leq$$

$$1; -1 \leq B \leq 0)$$

(cho [5]);

2- coefficient bounds :

Theorem 1: Let $f(z) \in \sum_p^+$ be given by (1.6) then

$f(z) \in \text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ is and only if

$$\sum_{k=p}^{\infty} k \{1+B [B-A] \gamma - B\} [1+ \lambda (k-1)] |D_k(\sigma)| \alpha_k \leq$$

$$(B-A)\beta\gamma (1-\alpha) (1-2\lambda) \dots (2.1)$$

Proof : Assume (2.1) holds for all admissible values

$\alpha, \beta, \gamma, \sigma, A, B$. let the expression .

$$T(f) = \left| z^2 (D^\sigma f(z))' + (1 - 2\lambda) [-B] [(B - A) \gamma - \right. \\ \left. (B) z^2 (D^\sigma f(z))' + (1 - 2\lambda) [(B - A) \gamma - B] \right| \dots\dots\dots(2.2)$$

$$\text{Since } D^\sigma f(z) = \frac{1-2\lambda}{z} + \sum_{k=p}^{\infty} [1+\lambda(k-1)] D_k(\sigma) \alpha_k z^k$$

$$\text{Then } D^\sigma f(z) = -\frac{1-2\lambda}{z^2} + \sum_{k=p}^{\infty} [1+\lambda(k-1)] D_k(\sigma) \alpha_k z^{k-1}$$

Therefore, we have for $0 < |Z| = r < 1$

$$T(f) = \left| \sum_{k=p}^{\infty} k [1+\lambda(k-1)] D_k(\sigma) \alpha_k z^{k+1} \right| - \\ \beta \left| (1-2\lambda)(\alpha-1) \gamma (B-A) + \sum_{k=p}^{\infty} k [1+\lambda(k-1)] [(B-A) \gamma - B] \right. \\ \left. D_k(\sigma) \alpha_k z^{k+1} \right| \\ = \left| \sum_{k=p}^{\infty} k [1+\lambda(k-1)] D_k(\sigma) \alpha_k z^{k+1} \right| - \beta |(B-A) \gamma (1-\alpha) (1-2\lambda) - \\ \sum_{k=p}^{\infty} k [1+\lambda(k-1)] [(B-A) \gamma - B] D_k(\sigma) \alpha_k z^{k+1}| \\ \leq \left| \sum_{k=p}^{\infty} k [1+\lambda(k-1)] D_k(\sigma) \alpha_k r^{n+1} \right| - (B-A) \beta \gamma (1-\alpha) (1-2\lambda) + \\ \sum_{k=p}^{\infty} \beta k [1+\lambda(k-1)] [(B-A) \gamma - B] D_k(\sigma) \alpha_k r^{n+1} \\ = \sum_{k=p}^{\infty} k \{1+\beta[(B-A) \gamma - B]\} [1+\lambda(k-1)] D_k(\sigma) \alpha_k r^{n+1} - \\ (B-A) \beta \gamma (1-\alpha) (1-2\lambda) \dots\dots\dots(2.3)$$

Since the above inequality holds for all r , $0 < r < 1$, letting $r \rightarrow 1$ in (2.3) we get the result (2.2) The completes the proof .

Conversely , let $f \in \text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$

$$\left| \frac{Z^2 (D^\sigma f(z))' + (1 - 2\lambda)}{[(B-A)\gamma - B]Z^2 (D^\sigma f(z))' + (1 - 2\lambda)[(B-A)\gamma\alpha - B]} \right|$$

$$= \left| \frac{\sum_{k=p}^{\infty} p^k [1 + \lambda(k-1)] D_k(\sigma) |\alpha_k| z^{k+1}}{[(B-A)(1-\alpha)(1-2\lambda)] - \sum_{k=p}^{\infty} p^k [1 + \lambda(k-1)] [(B-A)\gamma - B] |\alpha_k| D_k(\sigma) z^{k+1}} \right|$$

$< \beta, z \in U^*$.since $\text{Re}(z) \leq |z|$ for all z , it follows that

$$\text{Re} \left\{ \frac{\sum_{k=p}^{\infty} p^k [1 + \lambda(k-1)] D_k(\sigma) |\alpha_k| z^{k+1}}{[(B-A)(1-\alpha)(1-2\lambda)] - \sum_{k=p}^{\infty} p^k [1 + \lambda(k-1)] [(B-A)\gamma - B] D_k(\sigma) |\alpha_k| z^{k+1}} \right\} < \beta, z \in$$

$$U^* \dots\dots\dots(2.4)$$

We choose the values of z on the real axis so that $Z^2 (D^\sigma f(z))'$

is real . By letting $z \rightarrow -1$ through real values , we get

$$\sum_{k=p}^{\infty} p^k [1 + \lambda(k-1)] D_k(\sigma) |\alpha_k| \leq (B-A)\beta\gamma (1-\alpha) (1-2\lambda)$$

$$- \sum_{k=p}^{\infty} p^k \beta [1 + \lambda(k-1)] [(B-A)\gamma - B] D_k(\sigma) |\alpha_k|.$$

Therefor

$$\sum_{k=p}^{\infty} p^k \{1 + \beta [(B-A)\gamma - B]\} [1 + \lambda(k-1)] D_k(\sigma) |\alpha_k| \leq (B-A)\beta\gamma (1-\alpha)(1-2\lambda),$$

Which completes the proof of theorem 1.

The result is sharp for

$$f_k(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma (1-\alpha)(1-2\lambda)}{k\{1 + \beta[(B-A)\gamma - B]\} [1 + \lambda(k-1)] D_k(\sigma)} z^k, k \geq p \dots\dots\dots(2.5)$$

Corollary (1) : Let $f \in \text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$. Then

$$|\alpha_k| \leq \frac{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{k\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(k-1)D_k(\sigma)]}, \text{ where } k \geq p, p \in$$

$$N; 0 \leq \alpha < 1; 0 < \beta \leq 1; -1 \leq A < B \leq 1; 0 < B \leq 1; \sigma \geq$$

$$0 \text{ and } \frac{B}{(B-A)} < \gamma \begin{cases} \frac{B}{(B-A)^\alpha} & (\alpha \neq 0) \\ 1 & (\alpha = 0) \end{cases}$$

3- Distortion theorem :

Theorem 2: If a function (z) define by (1.6) is in the class

$\text{Th}^+(p; \alpha, \beta, \gamma, A, B, \lambda, \sigma)$. Then

$$\left\{ n! - \frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{(p-n)!\{1+\beta[B-A]\gamma-B\}[1+\lambda(p-1)]D_p(\sigma)} r^{p+1} \right\}$$

$$\leq r^{-(n+1)} |f^{(n)}(z)|$$

$$\leq \left\{ n! + \frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{(p-n)!\{1+\beta[B-A]\gamma-B\}[1+\lambda(p-1)]D_p(\sigma)} r^{p+1} \right\}$$

$$\leq r^{-(n+1)},$$

$$(0 < |z| = r < 1; p \in N; 0 \leq n < p) \dots \dots \dots (3.1)$$

The result is sharp for the function f(z) given

$$f(x) = \frac{1}{z} + \frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(p-1)]D_p(\sigma)} z^p \quad (p \in N)$$

(3.2)

Proof : in view of theorem 1, we have

$$\frac{p\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(p-1)]}{p!D_p(\sigma)} \sum_{k=p}^{\infty} k! |\alpha_k| \leq \sum_{k=p}^{\infty} k! \{1 + \beta[(B-A)\gamma-A]\}[1+\lambda(k-1)]D_k(\sigma) |\alpha_k| \leq (B-A)\beta\gamma(1-\alpha)(1-2\lambda)$$

Which yields

$$\sum_{k=0}^{\infty} p^k |\alpha_k| \frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{\{1+\beta[B-A]\gamma-A\}[1+\lambda(p-1)]Dp(\sigma)} , (p \in \mathbb{N}) \dots (3.3)$$

Now , by differentiating both sides of (1.6) n times with respect to z , we have

$$f^{(n)}(z) = (-1)^n n! z^{-(n+1)} + \sum_{k=p}^{\infty} \frac{k!}{(k-n)!} |\alpha_k| z^{k-n} , (p \in \mathbb{N} ; 0 \leq n < p) \dots \dots \dots (3.4)$$

And Theorem 2 follows easily from (2.3) and (2.4). finally , it is easy to see that the bounds in (3.1) are attained for the function f (z) given by (3.2) at the points

$$Z=r , \pm ir (0 < r < 1).$$

4- Neighborhood and partial sums :

Follow the earlier words (based upon the familiar concept of Neighborhoods of analytic function)by goodman [8] and Ruscheweyh [14] , and (more recently). By Altıntaş et al ([1],[2] and [3] . Liu [10] and Liu and Srivastava ([11]and [12] , we begin by introducing here the δ – neighborhood of a function

f (z) $\in \Sigma$ of the form (1.1) by means of the definition given below :

$$N\delta(f) - \{g \in \Sigma : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \text{ and } \sum_{k=1}^{\infty} \frac{k\{1+\beta(B-A)\gamma-B\}[1+\lambda(k-1)]D_k(\sigma)}{(B-A)\beta\gamma(1-\alpha)} |\alpha_k - b_k| \leq$$

$$\delta(-1 \leq A < B \leq 1, 0 \leq \lambda < \frac{1}{2}, \sigma > -1, \delta > 0)\} \dots \dots \dots (4.1)$$

Making use of the definition (4.1) , we now prove Theorem 3 below :

Theorem 3 : Let the function $f(z)$ defined by (1.1) be in the class $Th(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$. If $f(z)$ satisfies the following

Condition : $\frac{f(z) + \varepsilon z^{-1}}{1 + \varepsilon} \in Th(\alpha, \beta, \gamma, A, B, \sigma)$ ($\varepsilon \in \mathbb{C}, |\varepsilon| < \delta, \delta > 0$)(4.2)

then $N_\delta(f) \subset Th(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$ (4.3)

proof : It is easily seen from (1.4) that $g(z) \in$

$Th(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$ if and only for any complex number η . with $|\eta| = 1$,

$$\frac{z^2(D^\sigma F(z))' + 1 - 2\lambda}{[(B-A)\gamma - B](D^\sigma F(z))' + (1-2\lambda)[(B-A)\gamma\alpha - B]} \neq \eta\beta \quad (z \in U)$$

Which is equivalent to

$$\frac{(g*h)(z)}{z^{-1}} \neq 0 \quad (z \in U) \dots\dots\dots(4.4)$$

Where , for convenience ,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} c_k z^k$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{k(1+\beta\eta(B-A)\gamma-B)[1+\lambda(k-1)D_k(\sigma)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} z^k \dots\dots\dots(4.5)$$

Form (4.5), we have

$$|C_k| \leq \frac{k\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(k-1)]D_k(\sigma)}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}$$

$(K \in N, 0 \leq \lambda < \frac{1}{2})$. now, if $(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in \Sigma$

Satisfies the condition (4.2) then (4.4) yields

$$\left| \frac{(g*h)(z)}{z^{-1}} \right| \geq \delta \quad (z \in U; \delta > 0) . \text{ By letting}$$

$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \in N_{\delta}(f)$, so that

$$\left| \frac{[g_{(z)} * h_{(z)}] * h_{(z)}}{z^{-1}} \right| = \left| \sum_{k=1}^{\infty} (b_k - \alpha_k) c_k z^{k+1} \right|$$

$$\leq |z| \sum_{k=1}^{\infty} \frac{k(1+\beta[(B-A)Y-A]\{(1+\lambda(k-1))D_k(\sigma)\}}{(B-A)\beta Y(1-\alpha)(1-2\lambda)} b_k$$

$$- \alpha_k | < \delta \quad (z \in U; \delta > 0)$$

Thus, we have (4.4) and hence also for any $\eta \in \mathbb{C}$ such that $|\eta| = 1$, which implies

That $g(z) \in \text{Th}(\alpha, \beta, Y, A, B, \lambda, \sigma)$. this evidently proves the assertion (4.3) of theorem 3.

Next, we prove the following result.

Theorem 4: Let $f(z) \in \Sigma$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_m(z)$ as follow :

$$S_1(z) = \frac{1}{z} \text{ and } S_n(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (n \in N)$$

suppose also that

$$\sum_{k=1}^{\infty} d_k z^k \leq 1 \quad d_k = \left(d_k = \frac{k\{1+\beta[(B-A)Y-A]\{1+\lambda(k-1)\}D_k(\sigma)}{(B-A)\beta Y(1-\alpha)(1-2\lambda)} \right) . \quad (4.6)$$

Then, we have

$$(i) \quad f(z) \in \text{Th}(\alpha, \beta, Y, A, B, \lambda, \sigma)$$

$$(ii) \operatorname{Re} \left\{ \frac{f(z)}{S_n(z)} \right\} > 1 - \frac{1}{d_n} \quad (z \in U; n \in \mathbb{N}) \dots\dots\dots(4.7)$$

$$\text{And (III) } \operatorname{Re} \left\{ \frac{S_n(z)}{f(z)} \right\} > \frac{d_n}{1+d_n} \quad (z \in U; n \in \mathbb{N}) \dots\dots\dots(4.8)$$

The estimates (4.7) and (4.8) are sharp for each $n \in \mathbb{N}$

Proof : It is not difficult to see that

$Z^{-1} \in \operatorname{Th}(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$, thus . Theorem 3 and the hypothesis (4.6) of theorem 4 .we have

$$N_1(Z^{-1}) \subset \operatorname{Th}(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$$

Which shows that $f(z) \in \operatorname{Th}(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$ as asserted by theorem 4.

(ii) for the coefficients d_k given by (4.6) , it is not difficult to verify that

$$d_{k+1} > d_k > 1 \quad (n \in \mathbb{N})$$

Therefore , we have

$$\sum_{k=1}^{\infty} |\alpha_k| + d_n \sum_{k=n}^{\infty} |\alpha_k| \leq \sum_{k=1}^{\infty} d_k |\alpha_k| \leq 1 \dots\dots\dots(4.9)$$

Where we have used the hypothesis (4.6) again .

By setting

$$h_1(z) = d_n \left\{ \frac{f(z)}{S_n(z)} - \left(1 - \frac{1}{d_n} \right) \right\} = 1 + \frac{\sum_{k=n}^{\infty} \alpha_k z^{k+1}}{1 + \sum_{k=n}^{\infty} \alpha_k z^{k+1}}$$

and applying (4.9) , we that

$$\left| \frac{h_1(z)-1}{h_1(z)+1} \right| \leq \frac{d_n \sum_{k=n}^{\infty} |\alpha_k|}{2 - 2 \sum_{k=1}^{n-1} |\alpha_k| - d_n \sum_{k=n}^{\infty} |\alpha_k|} \leq (z \in U)$$

Which readily yields the assertion (4.7) of Theorem 4 . If we take

$$f(z) = \frac{1}{z} - \frac{z^n}{d_n} \dots\dots\dots (4.10)$$

$$\text{Then } \frac{f(z)}{S_n(z)} = 1 - \frac{z^{n+1}}{d_n} \rightarrow 1 - \frac{1}{d_n} , \text{ as } z \rightarrow -1$$

Which shows that the bounds in (4.7) is the best possible for each $n \in N$.

(iii) just as in part (ii) above , if we put

$$\begin{aligned} h_2(z) &= (1 + d_n) \left(\frac{S_n(z)}{f(z)} - \frac{d_n}{1+d_n} \right) \\ &= 1 - \frac{(1+d_n) \sum_{k=n}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{\infty} a_k z^{k+1}} \end{aligned}$$

And make use of (4.9) , we can deduce that

$$\begin{aligned} &\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \\ &\leq \frac{(1+d_n) \sum_{k=n}^{\infty} |a_k|}{2 - 2(1+d_n) \sum_{k=1}^{n-1} |a_k| - (1-d_n) \sum_{k=n}^{\infty} |a_k|} \leq 1 (z \in U) \end{aligned}$$

Which leads us immediately to the assertion in (4.8) Theorem 4 . The bounds is sharp for each $n \in N$, with the function $f(z)$ given by (4.10) . the proof of theorem 4 is thus completed .

5 - convolution properties :

For the function

$$f_j(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |\alpha_{k,j}| z^k \quad (j = 1, 2; p \in \mathbb{N}) \dots \dots \dots (5.1)$$

We denote by $(f_1 * f_2)(z)$ the hadamard product (or convolution of the function $f_1(z)$ and $f_2(z)$ that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |\alpha_{k,1}| |\alpha_{k,2}| z^k$$

Theorem 5 : Let the function $f_j(z)$ ($j = 1, 2$) defined by (5.1) be in the class $Th^+(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$. Then $(f_1 * f_2)(z) \in Th^+(\alpha, \beta, \gamma, A, B, \lambda, \sigma)$, where

$$\delta = 1 - \frac{(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_p(\sigma)}$$

The result is sharp for the function

$$f_j(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_p(\sigma)} z^p, \quad (j = 1, 2; p \in \mathbb{N}) \dots \dots \dots (5.2)$$

Proof : Employing the technique used earlier by Schild and Silverman [15], we need to find the largest δ such that

$$\sum_{k=p}^{\infty} \frac{k\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(k-1)]D_k(\sigma)}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} |\alpha_{k,1}| |\alpha_{k,2}| < 1 \text{ for}$$

$f_j(z) \in Th^+(p, \alpha, \beta, \gamma, A, B, \lambda, \sigma)$ ($j = 1, 2$) . since

$f_j(z) Th^+(p, \alpha, \beta, \gamma, A, B, \lambda, \sigma) (j = 1, 2)$ we see

That

$$\sum_{k=p}^{\infty} \frac{k\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(k-1)]D_k(\sigma)}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} |\alpha_{k,j}| \leq 1 (j = 1, 2)$$

Therefore by the Cauchy – Schwarz inequality , we obtain

$$\sum_{k=p}^{\infty} \frac{k\{1+\beta[(B-A)\gamma-A]\}[1+\lambda(k-1)]D_k(\sigma)}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} \sqrt{|\alpha_{k,1}||\alpha_{k,2}|} \leq 1$$

....(5.3)

This implies that we need only to show that

$$\frac{|\alpha_{k,1}||\alpha_{k,2}|}{(1-\delta)} \leq \frac{\sqrt{|\alpha_{k,1}||\alpha_{k,2}|}}{(1-a)} (k \geq p)$$

Or , equivalently that $\sqrt{|\alpha_{k,1}||\alpha_{k,2}|} \leq \frac{(1-\delta)}{(1-a)} (k \leq p)$

hence , by the equivalently (5.3) , it is to prove that

$$\frac{(B-A)\beta\gamma(1-a)^2(1-2\lambda)}{k\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(k-1)]} \leq \frac{(1-\delta)}{(1-a)} (k \geq p) \dots\dots\dots(5.4)$$

It follows from (5.4) that

$$\delta \leq 1 - \frac{(B-A)\beta\gamma(1-a)^2(1-2\lambda)}{k\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(k-1)]D_k(\sigma)} (k \geq p)$$

Now , defining the function $Q(k)$ by .

$$Q(k) = 1 - \frac{(B-A)\beta\gamma(1-a)^2(1-2\lambda)}{k\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(k-1)]D_k(\sigma)} (k \geq p) \text{ we}$$

See that $Q(k)$ is an increasing function of k . Therefore , we conclude that

$$\delta \leq Q(p) = 1 - \frac{(B-A)\beta\gamma(1-a)(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_p(\sigma)}$$

Which evidently completes the proof of Theorem 5 . using arguments similar to those in the proof of theorem 5 , we obtain the following result .

Theorem 6 : Let the function $f_1(z)$ defined by (5.1) be the class $Th^+(p, \varepsilon, \beta, \gamma, A, B, \lambda, \sigma)$ suppose also that the function $f_2(z)$ defined by (5.1) be in the class $Th^+(p; \varepsilon, \beta, \gamma, A, B, \lambda, \sigma)$. Then

$$\varepsilon = 1 - \frac{(B-A)\beta\gamma(1-a)^2(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_p(\sigma)}$$

The result is sharp for the function $f_j(z)$ ($j = 1, 2$) given by

$$f_1(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-a)(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_p(\sigma)} z^p, (p \in \mathbb{N})$$

$$\text{And } f_2(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-a)(1-2\lambda)}{p\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]D_\sigma(\sigma)} z^p, (p \in \mathbb{N})$$

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المستخلص

في البحث حصلنا على مبرهنة التشويه وضرب هادمرد للدوال الموجودة في الصنف

$$\text{Th}+(p, \alpha, \beta, \gamma, A, B, \lambda, \sigma)$$

من الدوال الميرومورفية ذات المعاملات الموجبة والمعرفة بواسطة مشتقة رشاولية ، ايضا
بعض الخواص للجوارات في الصنف $(\text{Th}+(p, \alpha, \beta, \gamma, A, B, \lambda, \sigma))$
نوقشت .



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كلية علوم الحاسوب وتكنولوجيا المعلومات
قسم الرياضيات

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الرياضيات

من قبل الطالبة

اصالة علي فنجان

بإشراف

د. وقاص غالب عطشان

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