



وزارة التعليم العالي والبحث العلمي
جامعة القادسية
كلية علوم الحاسوب وتكنولوجيا المعلومات
قسم الرياضيات

COMPLETE BCK - IDEAL

بحث

مُقدم الى كلية علوم الحاسوب وتكنولوجيا المعلومات/قسم الرياضيات
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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

((وَقُلْ رَأَيْتُمْ نَارَ اللَّهِ تَنْزِلُ))

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

شكر وتقدير

لابد لنا ونحن نخطوا خطواتنا الأخيرة في الحياة الجامعية من وقفة نعود إلى أعوام قضيناها في رحاب الجامعة مع أساتذتنا الكرام الذين قدموا لنا الكثير باذلين بذلك جهودا كبيرة في بناء جيل الغد

وقبل أن نمضي أتقدم بأسمى آيات الشكر والامتنان والتقدير والمحبة إلى الذين حملوا أقدم رسالة في الحياة إلى الذين مهدوا لنا طريق العلم والمعرفة إلى جميع أساتذتنا الأفاضل أهديكم هذا المشروع

الإهداء

إلهي لا يطيب الليل إلا بشرك ولا يطيب النهار إلى بطاعتك .. ولا تطيب
اللحظات إلا بذكرك .. ولا تطيب الآخرة إلا بعفوك .. ولا تطيب الجنة إلا برويتك
"الله جل جلاله"

إلى من بلغ الرسالة وأدى الأمانة .. ونصح الأمة .. إلى نبي الرحمة ونور العالمين
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إلى من كثر الله بالهيبة والوقار .. إلى من علمني العطاء بدون انتظار .. إلى من أحمل
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.. وإلى الأبد

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إلى ملاكي في الحياة .. إلى معنى الحب وإلى معنى الحنان و التفاني .. إلى بسملة الحياة
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إلي كل من أضاء بعلمه عقل غيره
أو هدى بالجواب الصحيح حيرة سائله
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Complete BCK-Ideal

Abstract

In this Research, we study a new type of ideals called complete BCK-ideal, and also we defined terms closed complete BCK-ideal, closed with respect to BCK-ideal, implicative with respect to BCK-ideal, and also some of theorems, we make clear its relationships with ideal in BCK-algebra.

Keywords: BCK-algebra, complete BCK-ideal, ideal in BCK-algebra, implicative ideal in BCK-algebra, commutative ideal in BCK-algebra.

Introduction

In 1966, Y. Imai and K. Iseki introduced the a new notion called a BCK-algebra [7], thereafter K. Iseki is the first defined an ideal in BCK-algebra in 1975, and he then returned Iseki with S. Tanaka for study introduction for ideal theorem of BCK-algebra in 1976 [3], and also we defined a commutative ideal in 1978 [4], another then came Meng introduced the concept of implicative ideal in BCK-algebra in 1994 [5].

1. Preliminaries

Definition 1-1:- An algebra $(X, 0, *)$ of type $(2, 0)$ is said to be a BCK-algebra provided the following axioms are satisfied :

1. $(x * y) * (x * z) \leq z * y$
2. $x * (x * y) \leq y$
3. $x \leq x$
4. $0 \leq x$
5. $x \leq y, y \leq x \Rightarrow x = y$

For all $x, y, z \in X$, Where $x \leq y$ is defined by $x * y = 0$. In any BCK-algebra, the following hold :[2]

1. $(x * y) * z = (x * z) * y$,
2. $x * y \leq x$,
3. $x * 0 = x$,
4. $(x * z) * (y * z) \leq x * y$,
5. $x * (x * (x * y)) = x * y$,
6. $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$, For all $x, y, z \in X$.

Definition 1-2:- A BCK-algebra X is said to be commutative if satisfies

$$x * (x * y) = y * (y * x), \text{ for all } x, y \in X$$

And for all $x, y \in X$, $y * (y * x)$ is denoted by $x \wedge y$.

Definition 1-3:- A BCK-algebra X is said to be implicative if satisfies the identity

$$x * (y * x) = x, \text{ for all } x, y \in X$$

Definition 1-4:- A sub set J of a BCK-algebra X is said to be an ideal if

- 1- $0 \in I$
- 2- $x * y \in I \text{ \& } y \in I \Rightarrow x \in I, \forall x \in X, y \in I$

Proposition 1-5:- Let I be an ideal of BCK-algebra X if

$$x, y \in X \text{ and } y \in I \text{ such that } x \leq y \text{ then } x \in I.$$

Proposition 1-6:- Let I be an ideal of BCK-algebra. If $x \in I$, then $x * y \in I, \forall y \in X$.

Proof:

Let $y \in X \text{ \& } x \in I$, since $0 \in I \text{ \& } (x * y) * x = 0$

Then $x * y \in I, \forall y \in X$.

Definition 1-7:- A non-empty subset I of a BCK-algebra X is said to be an implicative ideal if :-

- 1- $0 \in I$
- 2- $(x * (y * x)) * z \in I \text{ and } z \in I \text{ implies } x \in I$.

Proposition 1-8:- Every implicative ideal in BCK-algebra X is ideal.

Proof :

Let $x * y \in I, y \in I$

Since $x * (x * x) = x$, then

$$(x * (x * x)) * y \in I, y \in I$$

Since I is implicative ideal, then $x \in I$

Thus I is an ideal.

Definition 1-9:- A non-empty subset I of a BCK-algebra X is said to be commutative ideal if :-

- 1- $0 \in I$
- 2- $(x * y) * z \in I, z \in I \text{ implies } (x * (y * (y * x))) \in I$.

Proposition 1-10:- Every commutative ideal in BCK-algebra X is ideal. Proof :

Let $x * y \in I, y \in I$

Since $x * 0 = x$, then

$$(x * 0) * y \in I, y \in I$$

Since I is a commutative ideal, then

$$(x * 0) * (0 * x) \in I, \text{ thus } x \in I \text{ Then } I \text{ is an ideal.}$$

Definition 1-11:- A non-empty subset S of a BCK-algebra X is said to be sub-algebra of X if for all $x, y \in S$, then $x * y \in S$.

Proposition 1-12:- In BCK-algebra every ideal is sub-algebra.

Corollary 1-13:- In BCK-algebra every implicative ideal is a sub-algebra.

Corollary 1-13:- In BCK-algebra every implicative ideal is a sub-algebra.

Corollary 1-14:- In BCK-algebra every commutative ideal is a sub-algebra .

Definition 1-15:- A sub-algebra Sof BCK-algebra X is said to be strongly sub-algebra if

$x * y \leq z$ where $y, z \in S$, then $x \in S$.

Proposition 1-16:- Let S be a sub-algebra of BCK-algebra X. Then Sis strongly sub-algebra if and only ifS is ideal .

Proof : $\Rightarrow$ let S be strongly sub-algebra Since $x * x = 0$, for all $x \in S$, then $0 \in S$.

Let $x * y \in S, y \in S$

Since $(x * (x * y)) * y = 0 \in S$ Then $x * (x * y) \leq y$, then $x \in S$ Thus S is ideal Conversely

Let S be an ideal and let $y, z \in S, x \in X$ such that $(x * y) * z = 0$, since $z \in S$ and S is ideal then $x * y \in S$.

Thus $x \in S$, then S is strongly sub-algebra

2. c-BCK-ideal

In this item will know a new kind of ideal we called complete BCK-ideal

Definition 2-1:- A subset I of a BCK-algebra X is said to be complete BCK-ideal (c-BCK ideal)

1- $0 \in I$

2- $x * y \in I, \forall y \in I$ such that $y \neq 0$ implies $x \in I$.

Example 2-2:-Let $X = \{0, a, b\}$ and a binary operation $*$ is defined by

*	0	1	2
0	0	0	0
1	1	0	1
2	2	0	0

It is clear that $(X, *, 0)$ is a BCK-algebra and $I = \{0, 2\}$ is c-BCK-ideal but $J = \{0, 1\}$ is not c-BCK-ideal since $2 * 1 = 0 \in J$ & $1 \neq 0$ but $2 \notin J$

Proposition 2-3:- In BCK-algebra X every ideal is a c-BCK-ideal .

Proof:Let I be an ideal and

Let $x * y \in I, \forall y \in I$ such that $y \neq 0$

If $I = \{0\}$, then I is c-BCK-ideal.

If $I \neq \{0\}$, then $\exists y \in I$, such that $y \neq 0$ & $x * y \in I$,

Since I is ideal, then $x \in I$. Thus I is c-BCK-ideal .

Remark 2-4:-The converse ofproposition above needs not be true in general as in the following example

Example 2-5:- Let $X = \{0, 1, 2, 3\}$ and a binary operation $*$ is defined

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

Then $I = \{0, 1, 3\}$ is c-BCK-ideal, but it's not ideal, since

$2 * 3 = 0 \in I$ & $3 \in I$ but $2 \notin I$

Corollary 2-6:- In BCK-algebra every implicative ideal is c-BCK-ideal

Corollary 2-7:- In BCK-algebra every commutative ideal is c-BCK-ideal

Definition 2-8:- Let f be a mapping from a BCK-algebra X into a BCK-algebra Y . Then f is called Homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. [5] Epimorphism if f is homomorphism and onto. Monomorphism if f is homomorphism and one to one. Isomorphism if f is epimorphism and monomorphism.

Proposition 2-9:- Let f be isomorphism from a BCK-algebra X into a BCK-algebra Y . If I is c-BCK-ideal in X , then $f(I)$ is c-BCK-ideal in Y .

Proof :

1- since $f(0) = 0$, then $f(0) \in f(I)$

2- Let $a * b \in f(I)$, $\forall b \in f(I)$, $b \neq 0$ such that $a = f(x)$, $b = f(y)$ (since f is onto) Then $f(x) * f(y) \in f(I)$, $\forall f(y) \in f(I)$, $f(y) \neq 0$. Since f homomorphism, then $f(x * y) \in f(I)$, $\forall f(y) \in f(I)$, $f(y) \neq 0$

Since f one to one, then $x * y \in I$, $\forall y \in I$, $y \neq 0$

Since I is c-BCK-ideal, then $x \in I$

Thus $f(x) \in f(I)$, then $f(x) \in f(I)$, thus $f(I)$ is c-BCK-ideal.

Proposition 2-10:- Let f be epimorphism from a BCK-algebra X into a BCK-algebra Y . If J is c-BCK-ideal in Y , then $f^{-1}(J)$ is c-BCK-ideal in X .

Proof :

1. Since $f(0) = 0 \in J$, then $0 \in f^{-1}(J)$

2. Let $x * y \in f^{-1}(J)$, $\forall y \in f^{-1}(J)$, $y \neq 0$

Since $(x * y) \in J$, $\forall y \in f^{-1}(J)$, $y \neq 0$, then

$f(x) * f(y) \in J$, $\forall f(y) \in J$, $f(y) \neq 0$, (since f is onto) Thus $f(x) \in J$, then $x \in f^{-1}(J)$. Then $f^{-1}(J)$ is c-BCK-ideal.

Definition 2-11:- An c-BCK-ideal I in BCK-algebra X is said to be closed c-BCK-ideal if it also is sub-algebra.

Example 2-12:- In example (2-5), then $I = \{0, 1\}$ is closed c-BCK-ideal.

Remark 2-13:- In general not every c-BCK-ideal is sub-algebra. As it is shown in the following example.

Example 2-14:- In example (2-5), then $I = \{0, 1, 3\}$ is c-BCK-ideal but not is sub-algebra, since $3, 1 \in I$ but $3 * 1 = 2 \notin I$.

Proposition 2-15:- Let f be isomorphism from a BCK-algebra X into a BCK-algebra Y . If I is closed c-BCK-ideal in X , then $f(I)$ is closed c-BCK-ideal in Y .

Proof :

Let $a, b \in f(I)$ such that $a = f(x)$, $b = f(y)$, when $x, y \in I$

Since $a * b = f(x) * f(y) = f(x * y)$

Since I is closed c-BCK-ideal, then $x * y \in I$, thus $f(x * y) \in f(I)$

Then $f(I)$ is closed c-BCK-ideal.

Proposition 2-16:- Let f be epimorphism from a BCK-algebra X into a BCK-algebra Y . If J is closed c-BCK-ideal in Y , then $f^{-1}(J)$ is closed c-BCK-ideal in X .

Proof :

Let $x, y \in f^{-1}(J)$, since $f(x), f(y) \in J$ & J is closed c-BCK-ideal Then

$f(x) * f(y) \in J$, thus $f(x * y) \in J$

Then $x * y \in f^{-1}(J)$, thus $f^{-1}(J)$ is closed c-BCK-ideal .

Definition 2-17:- Let I and J be two subset of X such that $I \subseteq J$. Then I is said to be closed with respect to J if $x * y \in J, \forall y \in I, y \neq 0$, then $x * y \in I$.

Proposition 2-19:- The union of family of closed with respect to J is closed with respect to J .

Proof :

Let $\{ I_i : i \in \Delta \}$ be a family of closed with respect to J and

Let $x * y \in J, \forall y \in \bigcup_{i \in \Delta} I_i, y \neq 0$, since $\forall i \in \Delta, I_i$ is closed with respect to J then $\exists j \in \Delta$ such that $x * y \in J, \forall y \in I_j, y \neq 0$, then $x * y \in I_j$

Thus $x * y \in \bigcup_{i \in \Delta} I_i$, then $\bigcup_{i \in \Delta} I_i$ is closed with respect to J .

Example 2-21:- In example (2-5), let $J = \{ 0, 1, 3 \}$ and $I_1 = \{ 0, 1, 2 \}$ & $I_2 = \{ 0, 2, 3 \}$ are closed with respect to J , but $I_1 \cap I_2 = \{ 0, 2 \}$ is not closed with respect to J , since $3 * 2 = 1 \in J, 2 \in I_1 \cap I_2$, but $3 * 2 = 1 \notin I_1 \cap I_2$.

Proposition 2-22:- Let I be c-BCK-ideal and $\emptyset \neq I \subseteq J$, if I is closed with respect to J , then J is c-BCK-ideal .

Proof:

Let $x * y \in J, \forall y \in J, y \neq 0$

Since $I \subseteq J$, then $x * y \in J, \forall y \in I, y \neq 0$.

Since I is closed with respect to J , then $x * y \in I, \forall y \in I, y \neq 0$, since I is c-BCK-ideal, thus $x \in I$, consequently $x \in J$, then J is c-BCK-ideal .

Remark 2-23:- The converse of proposition above needs not be true in general as in the following example.

Example 2-24:- Let $X = \{ 0, 1, 2, 3, 4 \}$ and a binary operation $*$ is defined

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	1	1	0	0
4	4	4	4	4	0

Then $(X, 0, *)$ is BCK-algebra

let $I = \{ 0, 4 \}$ is closed with respect to $J = \{ 0, 1, 3, 4 \}$ and J is c-BCK-ideal

but I is not c-BCK-ideal.

Proposition 2-25:- Let f is isomorphism from a BCK-algebra X to a BCK-algebra Y . If I is closed with respect to J in X , then $f(I)$ is closed with respect to $f(J)$ in Y .

Proof :

Let $a * b \in f(J)$, $\forall b \in f(I)$, $b \neq 0$ such that $a = f(x)$, $b = f(y)$

When $x, y \in J$, then $f(x) * f(y) \in f(J)$, $f(y) \in f(I)$, $f(y) \neq 0$

Thus $f(x * y) \in f(J)$, $\forall f(y) \in f(I)$, $f(y) \neq 0$ Since f is one to one, then $x * y \in J$, $\forall y \in I$, $y \neq 0$ Since I closed with respect to, then $x * y \in I$

Thus $f(x * y) \in f(I)$, then $f(x) * f(y) \in f(I) \Rightarrow a * b \in f(I)$

Then $f(I)$ is closed with respect to $f(J)$.

Proposition 2-26:- Let f is epimorphism from a BCK-algebra X into a BCK-algebra Y . If I is closed with respect to J in Y . Then $f^{-1}(I)$ is closed with respect to $f^{-1}(J)$ in X .

Proof :

Let $x * y \in f^{-1}(J)$, $\forall y \in f^{-1}(I)$, $y \neq 0$, since $f(x * y) \in J$, $\forall y \in f^{-1}(I)$, $y \neq 0$

Since f is onto, then $f(x * y) \in J$, $\forall f(y) \in I$, $f(y) \neq 0$

As I is closed with respect to J , then $f(x * y) \in I$, thus $x * y \in f^{-1}(I)$

Then $f^{-1}(I)$ is closed with respect to $f^{-1}(J)$.

Proposition 2-27:- Let I be closed with respect to J in BCK-algebra X and I be a c-BCK-ideal. If J is closed c-BCK-ideal then I is closed c-BCK-ideal.

Proof :

Let $x, y \in I$. Since $I \subseteq J$ then $x, y \in J$

Since J is closed c-BCK-ideal, then $x * y \in J$, thus

$x * y \in J$, $\forall y \in I$, as I is closed with respect to J thus $x * y \in I$

Then I is closed c-BCK-ideal (since I is c-BCK-ideal).

Remark 2-28:- The converse of proposition above needs not be true in general as in the following example.

Example 2-29:- In example (2-5), let $I = \{0, 1\}$ is closed with respect to $J = \{0, 1, 3\}$, and I is closed c-BCK-ideal but J is not closed c-BCK-ideal, since $1, 3 \in J$ but $3 * 1 = 2 \notin J$.

Definition 2-30:- Let I be a c-BCK-ideal of a BCK-algebra X . Then X is said to be implicative with respect to I if satisfied the condition $x * (x * y) = x$ for all $y \in I$ such that $y \neq 0$ & $x \in X$.

Example 2-31:- Let $X = \{0, 1\}$ and a binary operation $*$ is defined by

*	0	1
0	0	0
1	1	0

Let $I = \{0, 1\}$, then X be implicative with respect to I .

Proposition 2-32:- Let X be a BCK-algebra and I is c-BCK-ideal if X is implicative with respect to I , then I is ideal.

Proof: Let $x * y \in I$, $y \in I$ such that $y \neq 0$

Since $(x * (x * y)) * y = 0 \in I$, $\forall y \in I$, $y \neq 0$ Since I is c-BCK-ideal,

then $x * (x * y) \in I$ Since $x * (x * y) = x$, thus $x \in I$, then I is ideal.

Proposition 2-33:- Let X be a commutative BCK-algebra has at least two elements. If X is implicative with respect to I . Then $X = I$ has only two elements.

Proof: Let $x, y \in X$ such that $x \neq 0$, $y \neq 0$, since X is implicative with respect to I , then

$x * (x * y) = x$ & $y * (y * x) = y$ But X is commutative, then $x = y$.

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