جمهوريه العراق

وزاره التعليم العالي والبحث العلمي

جامعه القادسية/كليه التربية

قسم الرياضيات

SOME PROPERTIES OF IKEDA MAP

رسالة مقدمة إلى

قسم الرياضيات – كلية التربية – جامعة القادسية ، وكجزء من متطلبات نيل درجة البكالوريوس في علوم الرياضيات

من قبل

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١٤٣٨هـ ٢٠١٧

بسم الله الرحمن الرحيم

((قَالَ إِنِّي عَبْدُ اللَّهِ آَتَانِيَ الْكِتَابَ وَجَعَلَنِي نَبِيًّا (٣٠) وَجَعَلَنِي
مُبَارَكًا أَيْنَ مَا كُنْت وَأَوْصَانِي بِالصَّلَاةِ وَالزَّكَاةِ مَا دُمْتُ حَيًّا
(٣١) وَبَرًّا بِوَالِدَتِي وَلَمْ يَجْعَلْنِي جَبَّارًا شَقِيًّا))

صدق الله العظيم

سورة مريم (٣٠-٣١)

الإهداء

الى من كلله الله بالهيبه والوقار...

الى من علمني العطاء بدون انتظار ..

الى من احمل اسمه بكل افتخار ارجو من الله ان يمد في عمرك لترى ثماراً قد حان قطفها

بعد طول انتظار (الى والدي العزيز)

الى من ارضعتني الحب والحنان

الى رمز الحب

الى القلب الناصع بالبياض (الى ولدتي الحبيبة)

واخيراً اهدي هذا البحث المتواضع الى الشموع التي تضيء الأخرين الى

شهداء الحشد الشعبي المقدس اهدي لهم ثمره جهدي المتواضع

الشكر والتقدير

الحمد لله كثيراً كما امر والصلاة والسلام على سيدنا محمد (صلى) عبده ورسوله وإلى اهل بيته ائمة الهدى ومصابيح الدجى والذين اذهب الله عنهم الرجس وطهرهم تطهيرا ولى من اتبع الهدى .

لابد لنا ونحن نخطو خطواتنا الاخيرة في الحياة الجامية من وقفه نقودها الى اوام قضيناها في رحاب الجامعه و اساتذتنا الكرام الذين قدموا لنا الكثير وقبل ان نمضي نقدم لهم اسمى ايات الشكر والتقدير الى الذين حملوا اقدس رسالة الى جميع اساتذتي الافاضل.

وعرفانا مني بالجميل اتقدم بفائق الشكر والتقدير والامتنان الى اساتذتي الفاضله ((وفاء هادي عبد الصاحب)) الذي كان دعمها اللامحدود وتذليل كافه الصعوبات التي واجهتني من خلال متابعه البحث خطوه بخطوه فجزاها الله عني اوفر الجزاء

كما اتقدم بالشكر الى اهلي الذين ساندوني وقدمولي المساعدة والتوجيهات حتى اصل هذه المرحلة

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Introduction

The dynamical system theory is the branch of mathematics devoted to the motion of system evolved according to simple ruler. It was developed originally in the 17 th century by newton to model the motions of their system evolving under the ruler of his new theory universal gravitation (2). Chaos theory was not known by this name until 1975 being paper y.Li youk which include the first use of word chaos in the context of dynamical system (1).

The goal of our work is study the dynamics and chaotic properties of the Ikeda map.

This work consists of two chapters:

In chapter one we recall the Basic definition which we needed.

And in chapter two we studied some properties of Ikeda map and proved properties chaotic of Ikeda

Abstract

In the work we introduced Basic definitions which we needed through this work we prove some necessary properties of the Ikeda map .and we find chaotic properties (under some conditions) of Ikeda a map. One of the main characteristics of chaotic system is that the Ikeda map has sensitive depended on initial condition . we prone this map has positive lyapunov exponent (under some conditions).

Chapter one

Definition basic

And

some properties of

Ikeda map

Definition(1-1)[2]:-

Any $p=\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ for which $f_1(P) = P_1, f_2(P) = P_2, f_3(P) = P_3$ is called **a fixed point**.

Definition(1-2)[3]:-

Let V be a subset of R^3 and $v_0 = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be any element in V consider

$F:V\to R^3$ a map. Furthermore assume that the first partials of the coordinate maps f_1 , f_2 and f_3 of F exist at v_0, **the differential of F at v_0** is the linear map $DF(v_0)$ defined on R^3 by :

$$DF(v_0)=\begin{bmatrix} \frac{\partial f_1(v_0)}{\partial x} & \frac{\partial f_1(v_0)}{\partial y} & \frac{\partial f_1(v_0)}{\partial t} \\ \frac{\partial f_2(v_0)}{\partial x} & \frac{\partial f_2(v_0)}{\partial y} & \frac{\partial f_2(v_0)}{\partial t} \\ \frac{\partial f_3(v_0)}{\partial x} & \frac{\partial f_3(v_0)}{\partial y} & \frac{\partial f_3(v_0)}{\partial t} \end{bmatrix}$$

For all v_0 in V. The determinant of $DF(v_0)$ is called the **Jacobian** of F at v_0 and is denoted by $J = det\ DF(v_0)$.

Definition(1-3) [3]:- :-

Let $F: R^3 \to R^3$ be a map and $v_0 \in R^3$. If $|J\,F(v_0)| < 1$ then F is called **area contracting** at v_0, $|J\,F(v_0)| > 1$ then F is called **area expanding** at v_0.

Definition(1-4) [4]:- :-

A map $F:R^n \to R^n$ is called a **diffeomorpism** provided it is:

1. One-to-one.

2. Onto.

3. C^∞

4. its inverse $F^{-1}: R^n \to R^n$ is C^∞.

Definition(1-5) [2]:-:

Suppose that A is a 2×2 matrix. The real number $\lambda$is an **eigenvalue** of A provided that there is a nonzero v in R^2such that $Av=\lambda v$. In this case v is an eigenvector of A (relative to λ) .

Definition(1-6) [3]:-:-

Let $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be a fixed point of F, then $\begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$ is attracting fixed point. If and only if then is a disk centered of $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ such that $F^n \begin{bmatrix} x \\ y \\ t \end{bmatrix} \to \begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$ as $n \to \infty$ for every $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ in the disk centered of $\begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$.by contrast $\begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$is repelling fixed point if and only if there is a disk centered at $\begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$ such that $\left\| F\begin{pmatrix} u \\ v \\ w \end{pmatrix} - F\begin{pmatrix} x_0 \\ y_0 \\ t_0 \end{pmatrix} \right\| > \left\| \begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ t_0 \end{pmatrix} \right\|$, for every $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ in the disk

for which $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \neq \begin{bmatrix} x_0 \\ y_0 \\ t_0 \end{bmatrix}$

Proposition(1-7) :-

If $a \neq 1$ and $b \neq 1$ the Ikeda map $I_{a,b}$ has unique fixed point

Proof:-

By the definition of fixed point we get:

$$\begin{bmatrix} 1 + ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$$

$bt = t$ since $b \neq 1$ then $t = 0$ since $axsin(t) + aycos(t) = y$ then $y = 0$ therefore $1 + \text{axcost} - \text{aysint} = x$. Hence $x = \frac{1}{1-a}$

such that $a \neq 1$ then $\begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$ is the fixed point ■

Proposition(1-8) :-

1. If $a = 1$, $b = 1$ and $t \neq m\pi$ then Ikeda map has infinite fixed points**.**

2. If $a \neq 0, 1$, $b = 1$, $t \neq m\pi$ and $t \neq \cos^{-1}(\frac{a^2+1}{2a})$ then Ikeda map has infinite fixed points.

Proof:

1. By the definition of fixed point, we get:

$$\begin{bmatrix} 1 + x\cos t - y\sin t \\ x\sin t + y\cos t \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$$

Then $t = t$, so $x\,sint + y\,cost = y$ then $x\,sint = y(1 - cost)$ since $t \neq m\pi$ therefore $x = \frac{y\,(1-cost)}{sint}$ so $1 + \left(\frac{y\,(1-\cos t)}{\sin t}\right)\cos t - y\sin t = \frac{y\,(1-\cos t)}{\sin t}$, then $\frac{\sin t + y\cos t(1-\cos t) - y\sin^2 t - y + y\cos t}{\sin t} = 0$,

$$sint + 2ycost - 2y = 0 \text{ therefore } y = \frac{\sin t}{2(1-\cos t)} \text{ then}$$

$x = \frac{\sin t\,(1-\cos t)}{2(1-\cos t)} \cdot \frac{1}{\sin t} = \frac{1}{2}$, then the fixed point $\begin{pmatrix} \frac{1}{2} \\ \frac{\sin t}{2(1-\cos t)} \\ t \end{pmatrix}$, $\forall t \in R$.

2. By the definition of fixed points, we get:- $\begin{bmatrix} 1 + a\,x\cos t - a\,y\sin t \\ a\,x\sin t + ay\cos t \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$,

then $t = t$, so $a\,x\,sint + a\,y\,cost = y$ then $a\,x\,sint = y - a\,y\,cost$ since $a \neq 0$, and $t \neq m\pi$ therefore $x = \frac{y\,(1-a\cos t)}{a\sin t}$, so

$1+\frac{y\,cost\,(1-a\,cost)}{sint}$ – $\mathrm{a\,y\ sint} = \frac{\mathrm{y(1-cost)}}{\mathrm{a\,sint}}$ then $\mathrm{a\,sint + a\,y\,cost} +$ $\mathrm{a^2\,y\,cos^2\,t - a^2\ y\,sint^2\,t - y + a\,y\,cost = 0}$,since $t \neq \cos^{-1}\left(\frac{1+a^2}{2a}\right)$ therefore $y = \frac{a\,sint}{1+a^2-2\,a\,cost}$ then $x = \frac{1-a\,cost}{1+a^2-2a\,cost}$ .So Ikeda map has infinite fixed points $\begin{pmatrix} \frac{1-a\,cost}{1+a^2-2a\,cost} \\ \frac{a\,sint}{1+a^2-2\,a\,cost} \\ t \end{pmatrix}$ $\forall t \in R$ ■

Remark(1-9) :

1. If $a = 0, b = 1$ and $t = m\pi$ then Ikeda map has infinite fixed point.

2. If $a = 0, b \neq 1$ and $t \neq m\pi$ then Ikeda map has unique fixed point.

3.If $a = 1, b = 1$ and $t = m\pi$ then if m is even number we get Ikeda map has no fixed point and if m is odd number we get Ikeda map have infinite fixed points.

4. If $a \neq 1, b = 1$ and $t = m\pi$ then Ikeda map has infinite fixed points.

5. If $a \neq 1, b \neq 1$ and $t = m\pi$ then if m is even number we get Ikeda map has unique fixed point and if m is odd number we get Ikeda map has unique fixed point.

6. If $a = 1, b \neq 1$ and $t \neq m\pi$ then Ikeda map has no fixed point.

7. If $a = 1$, $b \neq 1$ and $t = m\pi$ then Ikeda map has no fixed point.

proof:

By the definition of fixed point we get :

1. $I_{0,1}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ t \end{bmatrix}$, It is easy to show $I_{0,1}$ has infinite fixed point .

2. $I_{0,b}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, it is easy to show $I_{0,b}$ has unique fixed point .

3. since $a = 1$ *and* $b = 1, t = m\pi$ then $I_{1,1}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1\pm x\\\pm y\\t\end{bmatrix}$ .If m is even number then $I_{1,1}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1+x\\y\\t\end{bmatrix}$ therefore $I_{1,1}$ has no fixed point and If m is odd number then

$I_{1,1}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1-x\\-y\\t\end{bmatrix}$ therefore $t = t$, so $y = -y$ then $2y = 0$ therefore $y = 0$. *And* $x = 1 - x$ *then* $2x = 1$ therefore

$x = \frac{1}{2}$. hence $\begin{bmatrix}\frac{1}{2}\\0\\t\end{bmatrix}$ is a fixed point of Ikeda map, hence $I_{a,b}$ have infinite fixed points .

4. since $a \neq 1$ *and* $b = 1$ then $I_{a,1}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1\pm ax\\\pm a\,y\\t\end{bmatrix}$, if $I_{a,1}\begin{bmatrix}x\\y\\t\end{bmatrix}=\begin{bmatrix}1+ax\\a\,y\\t\end{bmatrix}$

then $t = t$ so $y = a\,y$ then $y - a\,y = 0$ therefore $y = 0$ so $x = 1 + a\,x$ then $x(1-a) = 1$ hence $x=\frac{1}{1-a}$ then $\begin{bmatrix}\frac{1}{1-a}\\0\\t\end{bmatrix}$ is a fixed point , hence $I_{a,b}$ have infinite fixed point .and if

$I_{1,1}\begin{bmatrix}x\\y\\t\end{bmatrix} = \begin{bmatrix}1-ax\\-a\,y\\t\end{bmatrix}$ then $t = t$ so $y = -ay$ therefore $y = 0$ so $x = 1 - a\,x$ then $x=\frac{1}{1+a}$ hence $\begin{bmatrix}\frac{1}{1+a}\\0\\t\end{bmatrix}$ is a fixed point , hence

$I_{a,b}$ have infinite fixed points.

5. since $a \neq 1, b \neq 1$ and $t = m\pi$ then $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 \pm a\,x \\ \pm a\,y \\ bt \end{bmatrix}$, if m is even number then $I_{a,1}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 + ax \\ a\,y \\ bt \end{bmatrix}$ $then\ bt = t$, therefore t=0 so $y = a\,y$ then $y - a\,y = 0$so $y = 0$therefore $x = 1 + a\,x$ then $x(1-a) = 1$, hence x=$\frac{1}{1-a}$then $I_{a,b}\begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$ is a fixed point , hence $I_{a,b}$ have unique fixed point.

If m is odd even then $I_{a,1}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 - ax \\ -a\,y \\ bt \end{bmatrix}$ so $t = bt$ then $t = 0$ so $y = -ay$ then $y(1+a) = 0$ therefore $y = 0$, so $x = 1 - ax$ then $x + ax = 1$ thereforex=$\frac{1}{1+a}$then$\begin{bmatrix} \frac{1}{1+a} \\ 0 \\ 0 \end{bmatrix}$is the fixed point , hence $I_{a,b}$ is the unique fixed point.

6. If a=1 and b≠1 then $I_{a,b}\begin{pmatrix} x \\ y \\ t \end{pmatrix}$, It is easy to show $I_{a,b}$ has no fixed point

7. Similarity prove (6) ■

<u>**proposition(1-10) :-**</u>

The Jacobian of Ikeda map $I_{a,b}$ isa^2b.

Proof:

The differential matrix of Ikeda is

$$DI_{a,b}(v_0)=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial t}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial t}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial t}\end{bmatrix}=\begin{bmatrix}a\cos t & -a\sin t & -ax\sin t-ay\cos t\\ a\sin t & a\cos t & ax\cos t-ay\sin t\\ 0 & 0 & b\end{bmatrix}$$ then

$$J=\det DI_{a,b}\ (v_0)=b\det\begin{bmatrix}a\cos t & -a\sin t\\ a\sin t & a\cos t\end{bmatrix}=ba^2\ \blacksquare$$

<u>Proposition (1-11) :-</u>

Let $I_{a,b}$ be the Ikeda map :-

(1) $I_{a,b}$ is area contracting map if $|a|<1$ and $|b|<1$**.**

(2) $I_{a,b}$ is area expanding map if **:-**

(i)$|b|>1,\ b\neq 0$ and $|a|^2>\frac{1}{|b|}$ or

(ii)$|a|>1$ and $|b|>\frac{1}{|a|^2}$.

Proof:-

1) If $|a|<1$ and $|b|<1$ then$|J|=\ |ba^2|\ <1$ so the Jacobian of Ikeda map is least than 1 so from definition area contracting.

2)

i. If $b \neq 0$ since $J = |det(D\, I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}| = |a^2 b| = |a|^2|b|$, by hypothesis

$|b| > 1$ and so $J > \frac{1}{|b|} \cdot |b| > 1$ Then $I_{a,b}$ is an area expending map.

ii. Similarity proof (i) ■

<u>Proposition (1-12) :-</u>

If $a \neq 0$, $b \neq 0$ then $I_{a,b}$ is onto.

Proof:-

Case(1):-If $t \neq (\frac{1}{2} + m)\pi$, $m \in Z$

Let $\begin{bmatrix} v \\ w \\ s \end{bmatrix}$ be any element in R^3 such that $\begin{bmatrix} v \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1 + ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix}$.

Then $v = 1 + axcost - aysint, since\ a \neq 0\ and\ t \neq (m + \frac{1}{2})\pi\ then$

$x = \frac{v-1+aysint}{acost}$ so $w = axsint + aycost \quad then$

$$w = a\frac{v-1+aysint}{acost} sin(t) + a\,ycos(t) \quad therefore$$

$w = vtan(t) - tan(t) + \left(aysin(t)tan(t)\right) + aycos(t)$ Then

$w - v\tan(t) + \tan(t) = y\,(a\tan(t)\sin(t) + a\cos(t))$ therefore

$y = \frac{w - v\tan(t) + \tan(t)}{(ay\tan(t)\sin(t) + a\cos(t))}$ then $bt = s$, since $b \neq 0$ therefore $t = \frac{s}{b}$

then there exist $\begin{pmatrix} \frac{v-1+\left(\frac{w-v\tan\left(\frac{s}{b}\right)+\tan\left(\frac{s}{b}\right)}{\tan\left(\frac{s}{b}\right)\sin\left(\frac{s}{b}\right)+\cos\left(\frac{s}{b}\right)}\right)\sin(\frac{s}{b}}{\cos(t)} \\ \frac{w-v\tan\left(\frac{s}{b}\right)+\tan\left(t\left(\frac{s}{b}\right)\right)}{\left(a\tan\left(\frac{s}{b}\right)\sin(\frac{s}{b})+a\cos\left(\frac{s}{b}\right)\right)} \\ \frac{s}{b} \end{pmatrix} \in R^3$ such that $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} v \\ w \\ s \end{bmatrix}$ then

$I_{a,b}$ is onto.

Case(2):-

If $\neq 0\ , b \neq 0\ and\ t = (m + \frac{1}{2})\pi$. Let $\begin{bmatrix} v \\ w \\ s \end{bmatrix}$ be any element in R^3 such that

$\begin{bmatrix} v \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1-ay \\ ax \\ bt \end{bmatrix}$, $s = bt$ then $t = \frac{s}{b}$ so $w = xa$ since $a \neq 0$ then $x = \frac{w}{a}$ so

$v = 1 - a\, y$ then $y = \frac{1-v}{a}$ such that $a \neq 0$ there exists

<u>Proportion (1-13)</u> :-

The Ikeda map is C^∞

Proof:-

Note that $\frac{\partial f_1(x,y,t)}{\partial x} = a\cos t$, $\frac{\partial^2 f_1(x,y,t)}{\partial x^2} = 0$ .... $\frac{\partial^n f_1(x,y,t)}{\partial x^n} = 0$,

$\frac{\partial f_1(x,y,t)}{\partial y} = -a\sin t$, $\frac{\partial^2 f_1(x,y,t)}{\partial y^2} = 0$,... , $\frac{\partial^n f_1(x,y,t)}{\partial y^n} = 0$ For all $n \in N$ and $n \geq 2$.

$\frac{\partial f_1(x,y,t)}{\partial t} = -a\sin t - ay\cos t$, $\frac{\partial^2 f_1(x,y,t)}{\partial t^2} = -ax\cos t + ay\sin t$, For all n∈N, $\frac{\partial\ f_2(x,y,t)}{\partial x} = a\sin t$

, $\frac{\partial^2 f_2(x,y,t)}{\partial x^2} = 0$ …, $\frac{\partial^n f_2(x,y,t)}{\partial x^n} = 0$ For all n∈N and n≥2…, $\frac{\partial\ f_2(x,y,t)}{\partial y} = a\cos t$ …..

$\frac{\partial^n f_2(x,y,t)}{\partial y^n} = 0$ For all n∈N and $n \ge 2$. $\frac{\partial f_2(x,y,t)}{\partial t} = ax\cos t - ay\sin t$,

$\frac{\partial^2 f_2(x,y,t)}{\partial t^2} = -ax\sin t - ay\cos t$ For all n∈N

$\frac{\partial\ f_3(x,y,t)}{\partial x} = 0$, …, $\frac{\partial\ f_3(x,y,t)}{\partial y} = 0$ ….. $\frac{\partial\ f_3(x,y,t)}{\partial t} = b$,…, $\frac{\partial^n f_3(x,y,t)}{\partial t^n} = 0$ for all n∈ N and n≥2.Then the partial derivatives exist and are continuous then $I_{a,b}$ is C^∞■

Proposition (1-14) :-

If $a \neq 1$ and $b \neq 1$ then the eigenvalues of Ikeda map at the fixed point . $is\ \lambda_{1,2} = a, \lambda_3 = b$

Proof :-

$\mathrm{Det}(DI_{a,b}(v) - \lambda I) = \det\begin{bmatrix} a-\lambda & 0 & 0 \\ 0 & a-\lambda & 0 \\ 0 & 0 & b-\lambda \end{bmatrix} = (a-\lambda)(a-\lambda)(b-\lambda) = 0$ Then $\lambda_{1,2} = a$, $\lambda_3 = b$ ■

Proposition(1-15) :-

Let $I_{a,b}$ be Ikeda map and a≠0 , b≠0 then

1.$If\ |a| < 1\ and\ |b| < 1$ then the fixed point of Ikeda map is attracting fixed point.

2. $If\ |a| > 1\ and\ |b| > 1$ then the fixed point of Ikeda map is repelling fixed point

3.$If\ |a| > 1\ and\ |b| < 1$ then the fixed point of Ikeda map is saddle fixed point.

4. $If\ |a| < 1\ and\ |b| > 1$ then the fixed point of Ikeda map is saddle fixed point.

Proof:-

By propositions (1-14) and definition (1-5)then the proposition satisfied■

Chapter two

Properties chaotic of Ikeda map

2-1 Sensitive Dependence on Initial Condition of

Ikeda Map:-

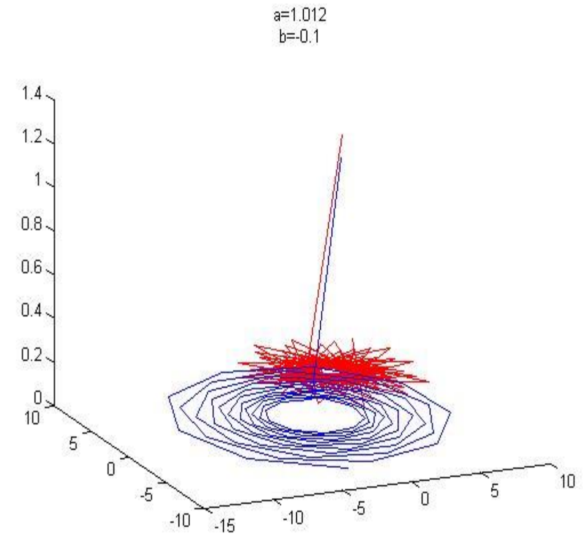
Chaos is character ited by a sensitive dependence of system dynamical variables on the initial conditions . trajectories starting with slightly different initial conditions locally diverge from each other at an exponential rates to provide a rigorous characterization as well as a way of measuring sensitive dependence on initial conditions.

Definition (2-2-1):-

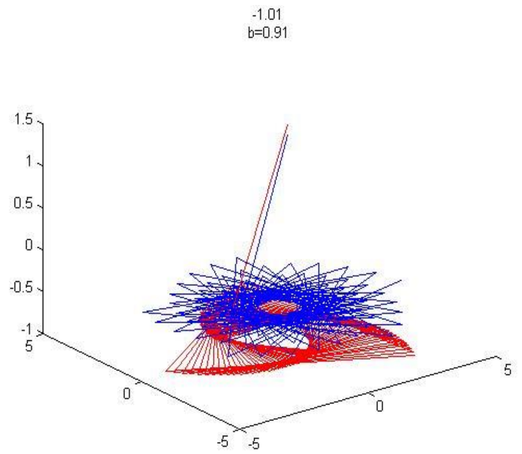
Let (X,d) be a metric space .A map f: (X,d) →(X,d) is said to be sensitive dependence on initial conditions if there exit $\varepsilon > 0$ such that for any $x_0 \in x$ and any open set U⊆ X containing x_0 there exists $y_0 \in U$ and n∈ Z^+such that

$d\ (f^n\ (x_0), f^n(y_0)) > \in$ that is $\exists \in > 0, \forall\ x\ , > 0, \exists\ y\ \in B_0(x), \exists\ n \in N, d\big(f^n(x_0), f^n(y_0)\big) > \varepsilon$.

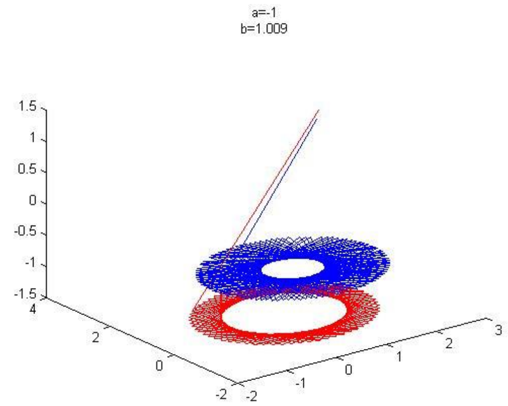
now , we draw some figures to The Ikeda map to show or approve the sensitivity dependence to initial condition



Figure(1.2) a = -1.012 , b = 0.1 with initial points (1.7,1.3,1.2) and (1.8,1.4,1.2)

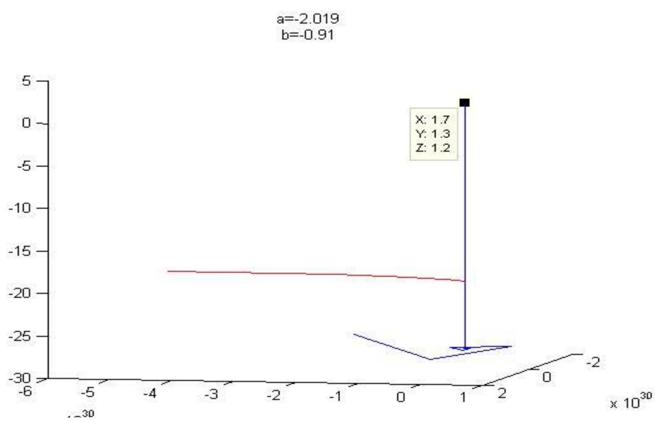


Figure(1.3) a = -1.01 , b = 0.91 with initial point (1.7,1.3,1.2) and (1.8,1.4,1.2)



Figure(1.4) a = -1, b = -1.009 with initial point (1.7,1.3,1.2) and (1.8,1.4,1.2)

o Sensitive Dependence on Initial Condition of Ikeda Map:-



The sensitive dependence on initial conditions is one of the basic ideas in several definitions of chaos, for example Gulick defined the map as chaotic if it satisfies sensitive dependence on initial condition or has positive Lyapunov exponent .

2-2The lyapunov Exponents of Ikeda map:

The Lyapunov exponents give the average exponential rate of divergence or convergence of nearly orbital in the phase – space .in system exhibiting exponential orbital divergence, small initial differences which we may not be able to resolve get magnified rapidly leading to less of predictability any system containing at least one positive Lyapunov exponent and it is defined to be chaotic with the magnitude of the exponent reflecting the time scale on which dynamics system become unpredictable.

Definition (2-2-1)[3]

Let F: X→X be continuous differential map, where X is any metric space. Then all x in X in direction V the Lyapunov exponent was defined of a map F at X by L(x,v)= $\lim_{n\to\infty}\frac{1}{n}\, ln\left|\left| DF_x^n v\right|\right|$ whenever the limit exists in higher dimensions for example in R^n the map F will have n Lyapunov exponents, say

$L_1^{\pm}(x,v_1), L_2^{\pm}(x,v_2), \dots, L_n^{\pm}(x,v_n)$, for a maximum Lyapunov exponent that is

$$L_{\pm}(x,v) = Max\left\{L_1^{\pm}(x,v_1), L_2^{\pm}(x,v_2), L_3^{\pm}(x,v_3), \dots, L_n^{\pm}(x,v_n)\right\},$$

where v=(v_1 ,v_2,…,v_n)

Proposition (2-2-2):-

Let $I_{a,b}$: $R^3 \rightarrow R^3$ be the Ikeda map if either |a|>1 or |b|>1 then the Ikeda map has positive Lyapunov exponents.

Proof:-

If |a|<1 and |b|>1 by proposition $|\lambda_{1,2}|=|a|$, if |a|<1 since $X_{1,2}\left(\begin{pmatrix}x\\y\\t\end{pmatrix}, v1,2\right) = \lim_{n\rightarrow\infty}\frac{1}{n}\ln\left|\left|DI_{a,b}\begin{pmatrix}x\\y\\t\end{pmatrix}, V1,2\right|\right| < 0$, but If |b|>1

Then $X_3\left(\begin{pmatrix}x\\y\\t\end{pmatrix}, v_3\right) = \lim_{n\rightarrow\infty}\frac{1}{n} ln\left\|\left|DI_{a,b}\begin{pmatrix}x\\y\\t\end{pmatrix}, v_3\right|\right\| > 0$ so $Lv=\max\{x_1^{\pm}(x,v_1), x_2^{\pm}(x,v_2), x_3^{\pm}(x,v_3)\}$ then Lv>0 So in the same way, we can prove if |b|<1 and |a|>1 then Lyapunov exponent of Ikeda map is positive. Finally, it is clear if |a|>1 and |b|>1 then L(v)>0 ■

Then by definition of Gulik we prove that Ikeda map is chaotic :-.

The sensitive dependence on initial conditions is one of the basic ideas in several definitions of chaos

Definition (2-2-3) [1]:-:-

A map f is **chaotic** if it satisfies at least one of the following conditions:-

1. F has a positive Lyapunov exponent at each point in its domain that is not eventually periodic.

2. F has sensitive dependence on initial conditions on its domain.

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