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By

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بِسْمِ اللَّـهِ الرَّحْمَـٰنِ الرَّحِيمِ

(اللَّهُ نُورُ السَّمَاوَاتِ وَالْأَرْضِ مَثَلُ نُورِهِ كَمِشْكَاةٍ فِيهَا مِصْبَاحٌ الْمِصْبَاحُ فِي زُجَاجَةٍ الزُّجَاجَةُ كَأَنَّهَا كَوْكَبٌ دُرِّيٌّ يُوقَدُ مِنْ شَجَرَةٍ مُبَارَكَةٍ زَيْتُونَةٍ لَا شَرْقِيَّةٍ وَلَا غَرْبِيَّةٍ يَكَادُ زَيْتُهَا يُضِيءُ وَلَوْ لَمْ تَمْسَسْهُ نَارٌ نُورٌ عَلَى نُورٍ يَهْدِي اللَّهُ لِنُورِهِ مَنْ يَشَاءُ وَيَضْرِبُ اللَّهُ الْأَمْثَالَ لِلنَّاسِ وَاللَّهُ بِكُلِّ شَيْءٍ عَلِيمٌ)

صَدَقَ اللهُ الْعَلِيُّ الْعَظِيْمُ

[سورة النور:35]

Supervisor's Certification

I certify that the Graduation Research which is entitled " **On Coherent Modules** " by Ekhlass Abd AlWahed Abd Alreda was made under my supervision at the University of Al-Qadisiyah/ College of Education/ Department of Mathematics as a partial fulfillment of the requirements for the degree of Bachelor Of Science in Mathematics.

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Date: / /2017

Abstract

A left R-module M is called coherent if it is finitely generated and every finitely generated left submodule of M is finitely presented. A ring R is left coherent, if it is a coherent left module over itself [3]. In this work, we give survey of some known properties and results of left coherent modules and ring and rewrite proofs, with more details, for some of them.

Content

Introduction:

In 1960, coherent rings and modules first appear in the literature by Chase [4], still without being mentioned by name. Coherent rings appear as such named, in 1961 by Bourbaki [2]. A ring R is called left coherent ring if every finitely generated left ideal of R is finitely presented [2]. The concept of coherent rings extended to coherent modules. A left R-module M is called coherent if it is finitely generated and every finitely generated left submodule of M is finitely presented (see for example [3]). Hence a ring R is left coherent, if it is a left coherent module over itself.

This work consists of two sections. In section one, we introduce some basic concepts which we will need in the second section.

In section two, we give survey of some known properties and results of left coherent modules and ring and rewrite proofs, with more details, for some of them. For examples: we rewrite the proof with some details of the result: every finitely generated module over a Noetherian ring R is a coherent R-module. Proposition(2-8) states that every left semi hereditary ring is a coherent ring. Proposition (2-17) gives a characterization of coherent modules and which states that: let M be a finitely generated left R-module. Then M is coherent if and only if $\text{ann}_R(b)$ is a finitely generated left ideal of R for any $b \in M$, and the intersection of any two finitely generated submodules of M is finitely generated. Finally, we rewrite a proof, with some details, of Theorem (2-24) which states that: Let N be a finitely generated submodule of finitely generated left R-module M, then M is coherent if and only if N and M/N are coherent R-modules.

Section One

Basic Concepts

Section One: Basic Concepts

This section introduces some basic concepts which are relevant to this work.

Definition (1-1): (see [6]) Let R be a ring. A left R-module is a set M together with:

(1) A binary operation $+$ on M under which M is an abelian group.

(2) A mapping $.: R \times M \to M$ (is called a module multiplication) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies

(a) $(r+s)\,m = rm + sm$, for all $r, s \in R, m \in M$.

(b) $(rs)\,m = r(sm)$, for all $r, s \in R, m \in M$.

(c) $r(m+n) = rm + rn$, for all $r \in R$, $m, n \in M$.

If the ring R has an identity element 1_R and

(d) $1.m = m$, for all $m \in M$, then M is said to be a unitary left R-module.

Examples (1-2):

(1) Every left ideal $(I, +, .)$ of a ring $(R, +, .)$ is a left R-module.

(2) Every ring $(R, +, .)$ is a left and right R-module.

(3) Every Abelian group is $\mathbb{Z}$-module.

(4) Let R be a ring with 1 and let $n \in \mathbb{Z}^{+}$. Define $R^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in R$, for all $i\}$. Make R^n into a left R-module by component wise addition and multiplication by elements of R as follows:

$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ and

$r(a_1, a_2\ , \dots , a_n) = (ra_1, ra_2\ , \dots , ra_n)$, for all $r \in R$ and $(a_1, a_2\ , \dots , a_n), (b_1, b_2\ , \dots , b_n) \in R^n$.

Definition (1-3): (see [6]) Let R be a ring and let M be a left R-module. A left R-submodule of M is a subgroup N of M such that $r \bullet n \in N$, for all $r \in R$, and for all $n \in N$, where $\bullet$ is the module multiplication defined on M. We will use $N \hookrightarrow M$ to denote that N is a submodule of M.

Examples (1-4):

(1) Every left R-module M contains at least two submodules M and 0.

(2) Let R be a ring with 1_R , then the left submodules of R as a left

R-module are exactly the left ideals of a ring $(R, +, .)$.

(3) Let $\mathbb{Z}$ be a left $\mathbb{Z}$-module. The left submodules of a left $\mathbb{Z}$-module $\mathbb{Z}$ are $< n >$ for all $n \in \mathbb{Z}$.

(4) The left submodules of a left $\mathbb{Z}$-module Z_6 are $Z_6, < 2 >, < 3 >$ and $< 0 >$.

(5) Let F be a field. Then the left submodules of a left F-module are F and $\{0\}$.

Definition (1-5): (see [1]) If X is a subset of a left R-module M, then $< X >$ will denote the intersection of all the submodules of M that contain X. This is called the submodule of M generated by X, while the elements of X are called generators of $< X >$.

Proposition (1-6): (see [1]) Let X be a subset of a left R-module M. Then $< X >$ is the smallest left submodule of M that contains a subset X.

Definition (1-7): (see [1])

(1) A left R-module M is said to be finitely generated if it generated by a finite subset X, that is $M =< X >$.

(2) A left R- module M is said to be cyclic if it generated by a subset $X = \{a\}$ contains one element only, that is $M =< \{a\} >$.

Examples (1-8):

(1) Every left R- module M has a generated set M.

(2) Every ring R with identity 1 is a cyclic left R-module, since ${}_RR=< 1 >$.

(3) Let $M=Z_{24}$ as Z_{24} -module. Then $< 6,12 > =< Z24 \cap< 2 > \cap < 3 > \cap < 6 > = \{0,6,12,18\} = < 6 >$. Also, $Z_{24} =< 1 > =< 5 > =< 7 >$ as Z_{24} – module.

Definition (1-9): (see [1]) A submodule N of a left R-module M is said to be a direct summand of M if there is a submodule K of M such that $M = N \oplus K$. In other word, there is a submodule K of M such that $M = N + K$ and $N \cap K = 0$.

Examples (1-10):

(1) Let $M = Z_6$ as a left $\mathbb{Z}$-module. Then the direct summand of M are $M, 0, N_1 =< 2 >= \{0,2,4\}$ and $N_2 =< 3 >= \{0,3\}$.

(2) Let F be a field. Then the direct summands of ${}_F F$ are F and 0.

(3) Let $M = Z_{30}$ as a left $\mathbb{Z}$-module. Then the direct summands of M are $M, 0, <2>, <15>, <3>, <10>, <5>$, and $<6>$.

(4) Let $M = \mathbb{Z}$ as a left $\mathbb{Z}$-module. Then the direct summands of M are <0> and $\mathbb{Z}$.

Proposition (1-11): (see [6]) Let R be a ring , Let M be a left R-module and let N be a left submodule of M. The (additive, abelian) quotient group M/N can be made into a left R-module by defining a module multiplication $\bullet: R\times(M/N)\rightarrow M/N$ by $r\bullet(x+N)=(rx)+N$, for all $r \in M/N$.

Definition (1-12): (see [6]) The left R-module M/N is defined in Proposition (1-11) is called quotient (or factor) module.

Definition (1-13): (see [6]) Let N and M be left R-modules.

(1) A function $f:N\rightarrow M$ is said to be a left R-homomorphism if for all $a, b \in N$ and $r \in R$, then $f(a+b)=f(a)+f(b)$ and $f(ra) = r f(a)$.

(2) A left R-homomorphism is a monomorphism if it is injective and is an epimorphism if it is surjective. A left R-module homomorphism is an isomorphism if it is both injective and surjective. The modules N and M are said to be isomorphic, denoted by $N \cong M$, if there is left isomorphism $\alpha: N \rightarrow M$.

(3) If $f:N\rightarrow M$ is a left R-homomorphism, let $\ker(f)=\{n\in N \mid f(n)=0\}$ (the Kernel of f) and let $\operatorname{im}(f)=\{m\in M \mid m=f(n)$ for some $n \in N\}$ (the image of f).

Examples (1-14):

(1) Let M be a left R-module and let N be a left submodule of M. The inclusion mapping $i_N : N \to M$ defined by $i_N(n) = n$, for all $n \in N$ is a left R- monomorphism.

(2) Let M be a left R-module. The identity mapping $I_M: M \to M$ defined by $I_M(n) = n$ for all $n \in M$ is a left R-isomorphism .

(3) Let M be a left R-module and let N be a left submodule of M. Then the natural mapping $\pi: M \to M/N$ defined by $\pi(x) = x + N$ for all $x \in M$ is a left R-epimorphism with kernel N.

Definition (1-15): (see [1]) A sequence $M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ of left R-modules and left R-homomorphism is said to be exact at M if im(f) =ker(g).

Definition (1-16): (see [1]) A sequence of left R-modules and left homomorphisms of the form:

$$S: \ldots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to \ldots, \quad n \in Z,$$

is said to be an exact sequence if it is exact at M_n between a pair of R-homomorphisms for each $n \in Z$.

Proposition (1-17): (see [1]) Let A, B and C be left R-modules. Then:

(1) The sequence $0 \to A \xrightarrow{f} B$ is exact at A if and only if f is a monomorphism.

(2) The sequence $B \xrightarrow{g} C \to 0$ is exact at C if and only if g is an epimorphism.

Corollary (1-18): (see [1]) The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of left R-modules is exact if and only if f is monomorphism, g is epimorphism, and im(f)=ker(g).

Definition (1-19): (see [1]) The exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a short exact sequence.

Examples (1-20):

(1) Let N a submodule of a left R-module M. Then the sequence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ is a short exact sequence, where i is the inclusion mapping and π is the natural mapping.

(2) Let $f{:}M{\to}N$ be a left R-homomorphism. Then:

(a) The sequence $0 \to \ker(f) \xrightarrow{i} M \xrightarrow{\pi} M/\ker(f) \to 0$ is a short exact sequence.

(b) The sequence $0 \to F(M) \xrightarrow{i} N \xrightarrow{\pi} N/F(M) \to 0$ is a short exact sequence.

Theorem (1-21): (see [1]) Let F be a left R-module. Then F is free left R-module if and only if $F \cong R^{(I)}$ as a left R-module, for some index I.

Lemma (1-22): (see [6]) A left R-module M is finitely generated left R-module if and only if there is a left R-epimorphism $\alpha: R^{(I)} \to M$ for some $n \in \mathbb{Z}^{+}$.

Lemma (1-23): (see [1]) Let I be an index set. Then a free left R-module $R^{(I)}$ is finitely generated if and only if I is finite set. In other word: a free left R-module $R^{(I)}$ is finitely generated if and only if $R^{(I)} = R^n$, for some $n \in \mathbb{Z}^+$.

Definition (1-24): (see [1]) A left R-module M is said to be finitely presented if there is an exact sequence $0 \to K \xrightarrow{f} F \xrightarrow{g} M \to 0$ of left R-modules, where F is finitely generated and free and K is finitely generated.

Proposition (1-25): (see [1]) Let $n \in Z^+$ and let N be a left submodule of a left R-module R^n . If N is finitely generated, then the left R-module R^n/N is finitely presented.

Corollary (1-26): (see [1]) Let M be a left R-module. Then M is finitely presented if and only if $M \cong R^n/N$ for some $n \in \mathbb{Z}^+$ and for some finitely generated left submodule N of a left R-module R^n.

Examples (1-27):

(1) For every $n, m \in \mathbb{Z}^+$ the $\mathbb{Z}$-module $\mathbb{Z}^m/< n, 0,0, \ldots ,0 >$ is finitely presented.

(2) For every $n \in \mathbb{Z}^+$ the $\mathbb{Z}$-module Z_n is finitely presented.

Theorem (1-28): (see [1]) The following statements are equivalent for a left R-module M:

(1) M is is finitely presented.

(2) There exists an exact sequence $R^m \xrightarrow{\propto} R^n \xrightarrow{\beta} M \to 0$ of a left R-modules for some $n.m \in \mathbb{Z}^+$.

(3) There exists an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ of a left R-modules, where F_1 and F_0 are finitely generated free R-modules.

Corollary (1-29): (see [1]) Every finitely presented left R-module is finitely generated.

Proposition (1-30): (see [1]) Every finitely generated free left R-module is finitely presented.

Definition (1-31): (see [6]) A left R-module M is said to be Noetherian if every left submodule of M is finitely generated. A ring R is said to be left Noetherian if R is a left Noetherian R-module.

Theorem (1-32): (see [6])

(1) A left R-module M is Noetherian if and only if every submodule of M is finitely generated.

(2) If M is finitely generated left module over a left Noetherian ring R, then M is a Noetherian R-module.

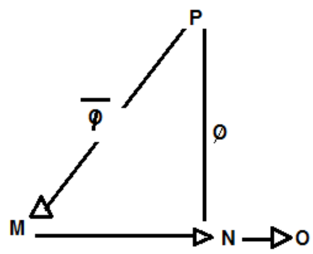
<u>Examples (1-33):</u>

(1) The ring of integers $\mathbb{Z}$ is a Noetherian ring.

(2) The ring of integers modulo n $(Z_n, +_n, \cdot_n)$ is a Noetherian ring.

(3) Let F be a field, then F is a Noetherian ring.

<u>Definition (1-34):</u> (see [10]) Let P, M and N be left R-modules, then P is projective relative to M (or P is M-projective) if and only if for each epimorphism $\sigma : M \to N$ and each homomorphism $\varphi: P \to N$, there is an R-homomorphism $\bar{\varphi}: P \to M$ such that the following diagram commutes, that is $\sigma\bar{\varphi} = \varphi$. A module P is said to be projective if P is M-projective, for all R-module M.



<u>Proposition (1-35):</u> (see [10]) An R-module M is projective if and only if it is isomorphic to direct summand of a free R-module.

<u>Lemma (1-36):</u> (see [9]) If M is a finitely presented left R-module and the sequence $0 \to K \to L \to M \to 0$ is exact with L is a finitely generated left R-module, then K is finitely generated.

Section Two

On Coherent Modules

Section Two: On Coherent Modules

In this section, we will give survey of some properties and results of Jacobson radical of modules.

Definition (2-1): (see [5]) A left R-module M is said to be **Coherent** if it is finitely generated and every finitely generated submodule of M is finitely presented.

Definition (2-2): (see [5]) A ring R is said to be **Coherent** if it is a coherent R-module, (i.e., if every finitely generated ideal of R is finitely presented).

Proposition (2-3): (see [5]) Every finitely generated module over a Noetherian ring R is a coherent R-module.

Proof. Let M be a finitely generated module over a Noetherian ring R. Let N be any finitely generated submodule of M. By Lemma (1-22), there is an epimorphism $\alpha: R^n \to N$ for some $n \in \mathbb{Z}^+$. By the First Isomorphism Theorem, $R^n/\ker(\alpha) \cong N$. Since R is a Noetherian ring it follows from Theorem (1-32)(2) that R^n is a Noetherian R-module. By Theorem (1-32)(1), $\ker(\alpha)$ is a finitely generated submodule of an R-module R^n. Hence $(R^n / \ker(\alpha))$ is a finitely presented R-module and this implies that N is a finitely presented R-module. Thus every finitely generated submodule of M is finitely presented and hence M is a coherent R-module. □

<u>Corollary (2-4):</u> (see [5]) Every Noetherian ring is a coherent ring.

<u>Proof.</u> Let R be a Noetherian ring. Since R is a cyclic module (generated by *1*), thus R is a finitely generated module over a Noetherian ring R. By Proposition (2-3), R is a coherent R-module and hence R is a coherent ring. □

<u>Corollary (2-5):</u> (see [5]) If R is Noetherian ring, then the polynomial ring $R[x]$ is a coherent ring.

<u>Proof.</u> Suppose that R is a Noetherian ring. By Hilbert Basis Theorem, $R[x]$ is a Noetherian ring and by Corollary (2-4) we have that $R[x]$ is a coherent ring. □

A ring R is said to be principal ideal if every ideal of R is a cyclic.

<u>Corollary (2-6):</u> (see [5]) Every principal ideal ring is a coherent ring.

<u>Proof.</u> Let R be a principal ideal ring, then every ideal of R is a cyclic and Theorem (1-32)(1) implies that R is a Noetherian ring. By Corollary (2-4), R is a coherent ring. □

<u>Examples (2-7):</u>

(1) The ring of integer $\mathbb{Z}$ is a coherent ring.

<u>Proof.</u> Since every ideal of $\mathbb{Z}$ is a cyclic, $\mathbb{Z}$ is a Noetherian ring and hence from Corollary (2-4) we have that $\mathbb{Z}$ is a coherent ring. □

(2) The ring of integer module n $(Z_n, +_n, \cdot_n)$ is a coherent ring.

Proof. Since $(Z_n, +_n, \cdot_n)$ is a Noetherian ring it follows from Corollary (2-4) that $(Z_n, +_n, \cdot_n)$ is a coherent ring. □

(3) The rings $\mathbb{Z}[x]$ and $Z_n[x]$ are coherent rings.

(4) Every finitely generated Abelian group is a coherent $\mathbb{Z}$-module.

(5) Z_n is a coherent $\mathbb{Z}$-module, $n \in \mathbb{Z}^+$.

Definition (2-8): (see [1]) A ring R is said to be left semihereditary if every finitely generated left ideal of R is projective.

Proposition (2-9): (see [1, Problem 10(b), p. 167]) Every left semihereditary ring is a coherent ring.

Proof. Let R be a left semi hereditary ring is a coherent ring and let I be a finitely generated left ideal of R. By Lemma (1-22), there is an epimorphism $\alpha: R^n \to N$ for some $n \in \mathbb{Z}^+$. The sequence $\sum: 0 \to \ker(\alpha) \xrightarrow{i} R^n \xrightarrow{\alpha} I \to o$ is a short exact sequence. Since R is a left semihereditary ring (by hypothesis), I is a projective left R-module. By Proposition (1-35), $\ker(\alpha)$ is a direct summand of R^n. Since R^n is finitely generated left R-module, $\ker(\alpha)$ is finitely generated.

Hence I is finitely presented left ideal of R. Thus every finitely generated left ideal of R is finitely presented and hence R is a coherent ring. □

A ring R is called regular (in the sense of von Neumann) if for every $a \in R,$ there is $b \in R \ni a = aba$ (see [9]).

Corollary (2-10): (see [9]) Every regular ring is a left coherent ring.

Proof. Let R be a regular ring. By [6, Exercise 13, p. 38], every finitely generated left ideal of R is a direct summand of a left R-module R.

Since R is a free left R-module, every finitely generated left ideal of R is projective (by Proposition (1-35)). Hence R is a left semihereditary ring.

By Proposition (2-9), R is a coherent ring.□

Proposition (2-11): (see [7])

(1) Let M be a left R-module and let $a \in M$. Then Ra is finitely presented left R-module if and only if $\text{ann}_R(a)$ is a finitely generated left ideal of R.
(2) If R is a left coherent ring, then the left ideal $\text{ann}_R(a)$ is finitely generated, for any $a \in R$.

Proof. **(1)** Let $a \in M$. Define $\alpha: R \to R\ by\ \alpha(r) = ra,\ \forall r \in R$. It is clear that α is a left epimorphism and $\ker(\alpha) = \{r \in R |\ \alpha(r) = 0\} = \{r \in R |\ ra = 0\} = \text{ann}_R(a)$. Thus the sequence $0 \to ann_R(a) \overset{i}{\to} R \overset{\alpha}{\to} Ra \to 0$ is exact. If Ra is a finitely presented left R-module, then Lemma (1-36) implies that $\text{ann}_R(a)$ is finitely generated. Conversely, if $\text{ann}_R(a)$ is finitely generated, then Ra is finitely presented by the definition of finitely presented.

(2) Suppose that R is a left coherent ring and let $a \in R$. Since Ra is a finitely generated left ideal of R it follows that Ra is finitely presented left ideal of R. By (1), $\text{ann}_R(a)$ is a finitely generated left ideal of R.□

In the following example, we give a non-coherent ring.

Example (2-12): (see [10]) Let $R = Z_2[x_1, x_2, \ldots]$ and let I be the ideal of a ring R generated by $x_1^2, x_1x_2, x_1x_3 \ldots,$ (i.e., $I =< x_1^2, x_1x_2, x_1x_3 \ldots . >$). The ring R/I is not left coherent. Assume that the ring R/I is left coherent. Since $\operatorname{ann}_{R/I}(x_1 + I)$ is generated by $\{x_1 + I, x_2 + I, \ldots\}$ (by [10, p.220]) and hence $ann_{R/I}(x_1 + I)$ is not finitely generated ideal of R/I and this contradicts with Proposition (2-11)(2). Thus R/I is not left coherent ring.

Lemma (2-13): Let *A*,*B* be submodules of a left *R*-module *M*.

1) Assume that *A* and *B* are finitely presented modules, then *A*+*B* is finitely presented iff $A \cap B$ is finitely generated.

2) Assume that *A* is finitely presented module and *B*=*Rb* for some $b \in M$. Then *A*+*B* is finitely presented iff the left ideal $\{r \in R | \, rb \in A\}$ of *R* is finitely generated.

Proof. See [7, Lemma 4.55, p.141].□

Proposition (2-14): (see [7]) Let *M* be a finitely generated left *R*-module. Then *M* is coherent if and only if $\operatorname{ann}_R(b)$ is a finitely generated left ideal of *R* for any $b \in M$, and the intersection of any two finitely generated submodule of *M* is finitely generated.

Proof:($\Longrightarrow$) Suppose that *M* is a coherent left *R*-module and let $b \in M$. Since *M* is a coherent module, *Rb* is a finitely presented left *R*-module. By Proposition (2-11)(1), $\operatorname{ann}_R(b)$ is a finitely generated left ideal of *R*. Let *A*,*B* be finitely generated submodules of *M*, thus *A*+*B* is a finitely generated left submodule of M. Since M is a coherent R-module, A,B and $A+B$ are finitely presented R-modules. By Lemma (2-13)(1), $A \cap B$ is finitely generated.

($\Longleftarrow$) We shall show, by induction on n, that any submodule $Ra_1 + Ra_2 + \ ... + Ra_n + Rb \subseteq M$ is finitely presented.

If $n = 0$, then ann$_R$*(b)* is a finitely generated left ideal of R (by hypothesis). By Proposition (2-11)(1), Rb is finitely presented left R-module. For $n \geq 1$, let $A = Ra_1 + Ra_2 + \ ... + Ra_n$ and let $B = Rb$. Then B is finitely presented as above. By the inductive hypothesis, B is finitely presented. By hypothesis, $A \cap B$ is finitely generated. By Lemma (2-13)(1), $A + B$ is finitely presented and hence $Ra_1 + Ra_2 + \ ... + Ra_n + Rb$ is finitely presented. Thus M is a coherent module.□

Corollary (2-15): (see [7]) A ring R is a left coherent if and only if ann$_R$*(b)* is a finitely generated left ideal of R for any $b \in R$, and the intersection of any two finitely generated left ideals in R is finitely generated.

Proof. By Proposition (2-14).□

Corollary (2-16): (see [7]) A domain R is left coherent if and only if the intersection of any two finitely generated left ideals in R is finitely generated.

Proof. Since R is an integral domain (by hypothesis) it follows that ann$_R$*(a)=0*, for any $0 \neq a \in R$. Thus the result follows from Corollary (2-15).□

A left module M over a commutative domain R is said to be torsion free if, for $m \in M$ and $r \in R$, $rm = 0 \Rightarrow m = 0$ or $r = 0$ [7, p.44].

Corollary (2-17): (see [7]) Let M be a finitely generated torsion-free module over a commutative domain R. Then M is coherent if and only if the intersection of any two finitely generated submodules of M is finitely generated.

Proof. Since M is a torsion-free, $\mathrm{ann}_R(a)=0$ for any $0 \neq a \in M$. Thus the result follows from Proposition (2-14).□

Proposition (2-18): (see [7]) A finitely generated left R-module M is coherent if and only if the following two conditions hold:

(1) The intersection of any two finitely presented submodules of M is finitely generated.

(2) $\mathrm{ann}_R(b)$ is a finitely generated left ideal of R for any $x \in M$.

Proof. (⟹) Suppose that a left R-module M is coherent. Thus M is finitely generated by the definition of coherent module. Let A,B be any two finitely presented submodules of M. Thus A,B are finitely generated submodules of M. By Proposition (2-14), $A \cap B$ is finitely generated. Also, from Proposition (2-14), we have that $\mathrm{ann}_R(x)$ is a finitely generated left ideal of R for any $x \in M$.

(⟸) Let A be a finitely generated submodule of M. If A is cyclic, then $A = Ra$, for some $a \in M$ and hence $A \cong \frac{R}{\mathrm{ann}_R(a)}$.

By hypothesis, $\mathrm{ann}_R(a)$ is finitely generated and hence A is finitely presented.

If $A = < a_1, a_2, \dots, a_n >$, then $A = Ra_1 + Ra_2 + \dots + Ra_n$. Since Ra_1 and Ra_2 are cyclic it follows from above that Ra_1 and Ra_2 are finitely presented. By hypothesis, $Ra_1 \cap Ra_2$ is finitely generated. By Lemma (2-13)(1), $Ra_1 + Ra_2$ is finitely presented. By induction, we can prove that *A* is finitely presented. Hence every finitely generated submodule of *M* is finitely presented and hence *M* is a coherent *R*-module.□

<u>Lemma (2-19):</u> Let *N* be a submodule of a left *R*-module *M*. Then

1) If *M* is finitely presented and *N* is finitely generated, then *M/N* is finitely presented.

2) If *N* and *M/N* are finitely presented, then *M* is finitely presented.

<u>Proof.</u> See [8, Theorem 2.3, P.81].□

<u>Theorem (2-20):</u> (see [9]) Let *N* be a finitely generated submodule of a finitely generated left *R*-module *M*. Then *M* is coherent if and only if *N* and *M/N* are coherent.

<u>Proof:</u> (⟹) Suppose that *M* is a coherent left *R*-module and let *N* be a finitely generated submodule of *M*. Let *A*,*B* be any two finitely generated submodules of *N*, thus *A*,*B* are finitely generated submodules of *M*. Since *M* is coherent it follows from Proposition (2-14) that $A \cap B$ is finitely generated. Let $a \in N$, thus $a \in M$. Since *M* is coherent it follows from Proposition (2.14) that $\operatorname{ann}_R(a)$ is finitely generated. Again by Proposition (2-14), *N* is a coherent *R*-module. Let *B* be a finitely generated submodule of *M/N*, thus $= \frac{A}{N}$, where *A* is a finitely generated

submodule of M with $N \hookrightarrow A$. Since M is coherent, A is finitely presented and hence B is finitely presented (by Lemma (2-19)(1)). Thus M/N is a coherent module.

($\Longleftarrow$) Suppose that N and M/N are coherent modules. Let A be a finitely generated submodule of M. Then $A \cap N$ is finitely generated (by [8, Theorem 2.2, p.81]). Since N is a coherent module (by hypothesis), $A \cap N$ is finitely presented. Since $A + N$ is finitely generated, $\frac{A+N}{N}$ is finitely generated. Since $\frac{A+N}{N} \hookrightarrow \frac{M}{N}$ and M/N is coherent it follows that $\frac{A+N}{N}$ is finitely presented. Since $\frac{A+N}{N} \cong \frac{A}{A \cap N} N$ (by the Third Isomorphism Theorem), thus $\frac{A}{A \cap N}$ is finitely presented. By Lemma (2-19)(2)), A is finitely presented and hence M is coherent.□

<u>Corollary (2-25):</u> (see [9]) Let N and M be finitely generated left R-modules. If the sequence $0 \to N \overset{\alpha}{\to} M \overset{\beta}{\to} K \to 0$ is exact, then M is coherent if and only if N and K are coherent.

<u>Proof</u>: ($\Longrightarrow$) Suppose that M is a coherent module. Since $\alpha(N) \hookrightarrow M$ it follows from Theorem (2-20) that $\alpha(N)$ and $M/\alpha(N)$ are coherent left R-modules. Since the sequence $0 \to N \overset{\alpha}{\to} M \overset{\beta}{\to} K \to 0$ is exact, thus $N \cong \alpha(N)$ and $\frac{M}{\ker(\beta)} = \frac{M}{\alpha(N)} \cong K$ and hence N and K are coherent modules.

($\Longleftarrow$) Suppose that N and K are coherent R-modules. Since the sequence $0 \to N \overset{\alpha}{\to} M \overset{\beta}{\to} K \to 0$ is exact, thus $N \cong \alpha(N) \hookrightarrow N$ and $\frac{M}{\ker(\beta)} = \frac{M}{\alpha(N)} \cong K$ are coherent modules. By Theorem (2.20), M is a coherent R-module.□

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