**ON HYERS-ULAM STABILITY OF NONLINEAR**

**DIFFERENTIAL EQUATIONS**

Abstract: We investigate the stability of nonlinear differential equations of the form y(n)(x) = F(x, y(x), y′(x), . . . , y(n−1)(x)) with a Lipschitz condition by using a fixed point method. Moreover, a Hyers-Ulam constant of this differential equation is obtained **1.****Introduction**

In 1940, S. M. Ulam posed the following question concerning the stability of group homomorphisms before a Mathematical Colloquium: When can weassert that the solutions of an inequality are close to one of the exact solutionsof the corresponding equation?

A year later, D. H. Hyers dealt with ε-additive mapping by direct method, which gave a partial solution to the above question. The result was extended by T. Aoki [1], D. G. Bourgin [2]. We men- tion here that the interest of this topic has been increasing since it came into being, some other results concerning functional equations one can ﬁnd, e.g., in [4, 5, 6, 7, 8, 9] and some related information (e.g., ε-isometries, su- perstability of functional equations and the stability of diﬀerential expressions) we refer to [10, 11, 12, 13].

To the best of our knowledge, the ﬁrst one who pay attention to the sta- bility of diﬀerential equations is proved that the stability holds true for diﬀerential equation y′(x) = y(x). Then, a generalized result was given by S.-E. Takahasi, T. Miura and S. Miyajima , in which they investigated the stability of the Banach space valued linear diﬀerential equation of ﬁrst order (see also [14, 15]). A more general result on the linear diﬀerential equations of ﬁrst order of the form y′(t) + α(t)y(t) + β(t) = 0 was given by S.-M. Jung [16] and the stability of linear diﬀerential equations of second order was established by Y. Li et al. (see [17] . There are a number of results concerning the stability of thelinear ordinary diﬀerential equations, which prompts the question:Can we as-sort that all of the linear ordinary differential equations have the Hyers-Ulam stability

Indeed, P. Gavruta, S.-M. Jung and Y. Li [17] proved that the diﬀerential equation y′′ = 0 does not have the Hyers-Ulam stability on the whole domain. For some examples of diﬀerential equations which have the Hyers-Ulam stability on unbounded interval we refer the reader to which show that it is a very special case that the Hyers-Ulam stability holds true for general diﬀerential equations on the whole domain.

Recently, proved that the generalized Hyers-Ulam stability holds

for the case of general linear diﬀerential equations, and the stability of nonlinear diﬀerential equations

y′(x) = F ( x, y(x) )

with a Lipschitz condition on a local interval was investigated by a ﬁxed point method:

**2. Preliminaries**

**Definition 2.1:** We will say that the equation (3.1.11) has the Hyers – Ʋlam stability with the initial conditions (3.1.12) if there exists a positive constant K> 0 with the following property:

For every ε > 0, z ∈ C 2(I) where x is sufficiently large in R, if

|z′′ + p(x)z′ + (q(x) − α(x)) z| ≤ ε (2.1.1)

Then there exists some solution w ∈ (I) of the equation (3.1.13), such that

|z(x) − w(x)| ≤ K ε and satisfies the initial conditions

w(x0 ) = 0 = w′(x0 ) (2.1.2)

**Definition** 2.2: We say that equation has the Hyers -Ulam stability with initial conditions if there exists a positive constant K > 0 with the following property:

For every ε > 0, z ∈ C 2(I ) where x is sufficiently large in R, if

|z′′ + p(x)z′ + q(x)z − h(x)zβ e( )| ≤ ε (2.3)

Then there exists some solution w ∈ C 2(I) of the equation (3.1.13) and

w(x0 ) = w′(x0 ) = 0 (2.4)

Such that |z(x) − w(x)| ≤ KƐ .

**Definition 2.3** We will say that the equations (),() have the Hyers -Ulam asymptotic stability with the initial conditions (3.1.12) if the equation is stable in the sense of Hyers and Ulam and

**Lemma 2.4 :** *Let* 𝑢, : [0,∞) → [0,∞) *be integrable functions,*

*let* 𝑐> 0 *be a constant, and let* 𝑡0≥ 0 *be given. If* 𝑢*satisfies the*

*inequality*

𝑢 (𝑡) ≤𝑐 + (2.2.3)

*for all* 𝑡≥𝑡0*, then*

𝑢 (𝑡) ≤𝑐 exp ( (2.2.4)

for all 𝑡 ≥ 𝑡0.

**Proof:** It follows from that

 (2.2.5)

for all t≥t0. Integrating both sides of the last inequality from

t0 to t, we obtain

 In (2.2.6)

Or

 (2.2.7)

for each t≥t0, which together with (3.1.13) implies that

 (2.2.8)

for all t≥t0.

**3.1** **Hyers-Ulam stability of Non-linear differential equation of second order**

In this section we investigate the Hyers-Ulam stability of the following non- linear differential equation of second order

z"+ p(x) + q(x)z = h(x) |z| sgnz , ∈ (0, 1) (3.1.9)

with the initial conditions

z(x0) = 0 = z′(x0) (3.1.10)

where q ∈ C0(I) , , h , p ∈ C1(I), I = [x0, x] ⊆ R, x0> 0 , p(x) > 0, and h(x) is a bounded for all sufficiently large x in R. Moreover we proved the Hyers-Ulam stability of the linear differential equation of second order

z" + p(x)z′ + (q(x) −ᾳ(x)) z = 0 (3.1.11)

with the initial conditions

z(x0) = 0 = z′(x0) (3.1.12)

where (x) is a bounded function for all sufficiently large x in

It should be note here that we may assume that z > 0 in equation () because if z < 0 we set z = −u , u > 0. So we will consider in future the equation

 + p(x) + q(x)z = h(x) e , ∈ (0, 1)

 (3.1.13)

**Theorem** **3.1.1:**

Suppose |h(x)| ≤ A for all x ≥ x0, and that y ∈ C2(I) ,

such that satisfies the inequality

|y" + y − h(x) | ≤ , ∈ (0, 1) (3.1.14)

with the initial condition

y(x0) = 0 = y'(x0) (3.1.15)

If A <,for x≥ x0,then the equation

y" + y = h(x) , ∈ (0, 1) (3.1.16)

has the Hyers-Ulam stability with initial condition .

**Proof**. suppose that > 0, y ∈ C2(I) satisfies the inequation (3.1.14) with the initial conditions (3.1.15) and that M = .

We will show that there exists a function w(x) ∈ c2(I) satisfying the equation(3.1.16) and the initial condition (3.1.15) such that |z(x) − w(x)| ≤k.

−≤ y" + y − h(x) ≤ (3.1.17)

Multiply the inequality (3.1.14) by y' and then integrate we obtain

−2 y ≤y'2(x) + y2(x) – 2≤ 2y

From which we get that

y2(x) ≤ 2y + 2

Therefore

M ≤

Hence |y(x)| ≤k, for all x ≥ x0. Obviously, w0(x) = 0 satisfies the equation(3.1.16) and the zero initial condition (3.1.15) such that

|y(x) − w0(x)| ≤k

Thus the equation (3.1.16) has the Hyers-Ulam stability with initial condition(3.1.15).

**Corollary 3.1.1 :** Assume that h(x) and z(x) satisfy the conditions of Theo- rem 3.2, and the inequality (2.1.1) with the initial condition (3.3.12).

If A < , for x ≥ x0 and the integral

Converges then the equation (3.1.13) has the Hyers-Ulam stability with initial condition (3.1.12). Moreover, if the integral = ∞then the equation (3.1.15) has the Hyers-Ulam asymptotic stability with initial condition (3.1.12).

**Proof**: Suppose that z ∈ C2(I) satisfies the inequality (2.1.1) with the initial condition (3.1.12). Then from the 3.1.1it follows that the equation (3.1.16) has the Hyers- Ulam stability with initial condition (3.1.15), and according to the substitution used in Lemma 2.1 it follows that the equation (3.1.13) has the Hyers-Ulam stability with initial condition (3.1.12). Now if = ∞, then the equation (3.1.13) has the Hyers-Ulam asymptotic stability with initial condition (3.1.12).

Now we illustrate the Theorem by the following example.

**Example 3.1 :** Consider the equation

 z" + z' + z =

with the initial condition

z(x0) = 0 = z'(x0) (3.1.18)

If we set z(x) = in the the equation ( ) we obtain

y"(x) + y(x) = (3.1.19) We Iet y(x)= and estimate the difference

= (3.1.20)

Now we may choose the number x0 sufficiently large such that the inequality(3.1.20) will satisfy for any x ≥ x0 and for any0.

Hence y(x) = (x − x0)2 e−x is an approximate solution of the equation (3.1.17) satisfying the zero initial condition

y(x0) = 0 = y′(x0) (3.1.21)

Now we have

Therefore

M ≤k, where

It is clear that z0≡ 0 satisfies the zero initial condition and the inequality|y(x) − z0(x)| ≤k. Thus the equation (3.1.17) has the Hyers-Ulam stability.Moreover, since

,then it also is asymptotically stable inthe sense of Hyers and Ulam as x→∞. Now since the integral = ∞ , then by Lemma it follows that the equation (3.1.17) has the Hyers-Ulam stability with zero initial condition (3.1.18). Moreover the equation (3.1.17) isasymptotically stable in the sense of Hyers and Ulam as x →∞.

**3.2 Gronwall inequality and Hyers-Ulam stability**

In this section, we investigate the Hyers-Ulam stability of the

nonlinear differential equation

 u"(t)+F(t,u(t))=0. (3.3.22)

**3.2.1:**Given constants L> 0 and t0≥ 0, assume that F : [t0,∞) × R → (0,∞) is a function satisfyingF' (t, u(t))/F(t, u(t)) > 0 and F(t, 0) = 1 for all t≥t0 andu∈U(L;t0). If a function u : [t0 ,∞) → [0,∞) satisfiesu∈U(L; t0) and the inequality

 (3.2.23)

for all t≥t0 and for some > 0, then there exists a solution

u0 : [t0 ,∞) → [0,∞) of the differential equation (3.2.23) such

that

 (3.2.24)

for any t≥t0.

**Proof**:We multiply (3.2.23) with |u' (t)| to get

- (3.2.25)

for all t≥t0. If we integrate each termof the last inequalities

from t0 to t, then it follows from(ii) that

- (3.2.26)

for any t≥t0.

Integrating by parts and using (iii), the last inequalitiesyield

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 (3.2.27)

for all t≥t0. Then we have

(3.2.28)

for any t≥t0.

 Applying , we obtain

 (3.2.29)

for all t≥t0. Hence, it holds that |u(t)| ≤L for any t≥t0.

Obviously, u0(t) ≡ 0 satisfies (3.2.22) and u0∈U(L; t0) such that

 (3.2.30)

for all t≥t0.

In the following 3.2.1 , we investigate the Hyers-Ulam

stability of the Emden-Fowler nonlinear differential equation

of second order

u"(t) + h(t)u = 0 (3.2.31)

for the case where is a positive odd integer.

**3.2.2:**Given constants L> 0 and t0≥ 0, assume that ℎ : [t0,∞) → (0,∞) is a differentiable function. Let be an odd integer larger than 0. If a function u : [t0 ,∞) → [0,∞)

satisfies u∈U(L; t0) and the inequality

 (3.2.32)

for all t≥t0 and for some > 0, then there exists a solution

u0 : [t0 ,∞) → [0,∞) of the differential equation (3.2.31) such

that

1/ (3.2.33)

for any t≥t0, where := +1.

**Proof**. We multiply (3.2.32) with |u'(t)| to get

- (3.2.34)

for all t≥t0. If we integrate each termof the last inequalities

from t0 to t, then it follows from that

- (3.2.35)

for any t≥t0.

Integrating by parts and using , the last inequalities yield

- (3.2.36)

for all t≥t0. Applying , we obtain

 (3.2.37)

for all t≥t0, from which we have

 (3.2.38)

for all t≥t0. Hence, it holds that

1/ (3.2.39)

for any t≥t0, where we set = + 1. Obviously, (t) ≡ 0

satisfies (3.2.23) and ∈U(L; t0). Moreover, we get

1/ (3.2.40)

for all t≥t0.

**3.2.3:**Given constants L≥ 0, M> 0, and t0≥ 0,

assume that ℎ : [t0,∞) → [0,∞) is a function satisfying

c := Let be an odd integer larger than 0.If a function u∈U(L;M; t0) satisfies the inequality

 (3.2.41)

for all t≥t0 and for some > 0, then there exists a solution

 : [,∞) → R of the differential equation such that

 (3.2.42)

for any t≥t0.

**Proof**. We multiply (3.2.41) with |u' (t)| to get

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 (3.2.43)

for all t≥. If we integrate each termof the last inequalities from t0 to t, then it follows from that

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 (3.2.44)

for any t≥t0.

Integrating by parts and using and , the last

Inequalities yield

- (3.2.45)

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for all t≥t0. Then it follows from (iii') that

 (3.2.46)

For any t

for all t≥t0. Hence, it holds that

for any t≥t0. Obviously, u0 (t) ≡ 0 satisfies and u0∈

U(L; M; t0). Furthermore, we get

 for all t ≥ t0.

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