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On Expansive Random Operators over a Uniform Random Dynamical Systems

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Abstract

In this paper the random dynamical system and random sets in uniform space are defined and some essential properties of these two concepts are proved. Also the expansivity of uniform random operator is studied.

Keywords: random dynamical system, uniform random dynamical system, uniform random set, expansive random dynamical system, random operator, expansive random operator

1. Introduction

Random dynamical systems (RDS's) ascend in the modeling of many phenomena in physics, climatology economics, biology, etc., and the random effects frequently reproduce essential properties of these phenomena before just to reward for the faults in deterministic models. The history of study of RDS's energies back bone to Ulam and von Neumann in 1945 [11] and it has succeeded since the 1980s due to the detection that the solutions of stochastic ordinary differential equations profit a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. In this paper the random dynamical system studied when the "phase space" is a uniform space rather than metric space. In the following we shall state some definitions and results about uniform space that are needed in this work. Throughout this paper $\Delta_X$denote to the diagonal in $X \times X$, i.e., the set $\Delta_X = \{(x,x): x \in X\}$. U^{-1}If $U \subset X \times X$then we define $U^{-1}$by $U^{-1} = \{(x,y): (y,x) \in U\}$. If $U^{-1} = U$, the we say that U is symmetric.The set$U \cap U^{-1}$is symmetric for any $U \subset X \times X$. The composite $U \circ V$ of two subsets U and V of $X \times X$is defined by

$U \circ V = \{(x,y): \exists z \in X \ni (x,z) \in U \text{and } (z,y) \in V\}$.

Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$be uniform spaces. A mapping $f: X \to Y$ is said to be **uniform continuous** [9,12] if for each $V \in \mathcal{U}_Y$, there exists $U \in \mathcal{U}_X$ such that $(x,y) \in U$ implis that $(f(x), f(y)) \in V$. If f is bijective and both f and f^{-1} are uniform continuous, the $f$is called a **uniform isomorphism** (uniform equivalence) and$X$ and Y are called uniformly isomorphic (uniformly equivalent)[12]. Every uniformly continuous mapping is continuous and hence every uniform isomorphism is homeomorphism[12].Let$(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ are uniform spaces. Then the product of $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ is a uniform space $(Z,\mathcal{U}_Z)$ with the underlying set $Z = X \times Y$ and the uniformity $\mathcal{U}_Z$on$Z$ whose base involves the sets

$$\mathcal{U}_Z = \left\{\left((x,y),(x',y')\right) \in Z \times Z: (x,x') \in U, (y,y') \in V\right\},$$

where $U \in \mathcal{U}_X$ and $V \in \mathcal{U}_Y$. The uniformity$\mathcal{U}_Z$is calledthe **product** of $\mathcal{U}_X$and$\mathcal{U}_Y$[12]and is written as $\mathcal{U}_Z = \mathcal{U}_X \times \mathcal{U}_Y$.Let $(X,\mathcal{U}_X)$ be a uniform space and $(\Omega,\mathcal{F})$ be a measurable space. A single- value mapping $T: \Omega \times X \to X$ is a random operator[6] if $T(.,x): \Omega \to X$ is **measurable** and it is (uniform) **continuous** if $T(\omega,.): X \to X$ is (uniform) continuous.We denote the nth iterate $T(\omega, T(\omega, T(\omega, \dots, T(\omega,x), \dots,)$ of a random operator $T: \Omega \times X \to X$by $T^n(\omega,x)$. The letter I denotes the random mapping $I: \Omega \times X \to X$ defined by $I(\omega,x) = x$and $T^0 = I$.A subset S of a topological group $\mathbb{G}$ is said to be right (left) syndetic [5,10] if there exists a compact subset K of $\mathbb{G}$ such that $\mathbb{G} = KS$ ($\mathbb{G} = SK$).Unless otherwise state, by "syndetic set" we mean " left syndetic set".

In Sec.2 the notion of uniform random dynamical system is introduced and some essential results are proved. In Sec. 3 the concept of uniform random set is introduced and some essential properties of such set are proved. In Sec.4 the expansivity of random operator in uniform random dynamical system is studied.

2. Uniform Random Dynamical System

Here we shall introduce the notion of random dynamical system over a uniform space.

Definition 2.1[2,3,4] The 5-tuple $(\mathbb{G},\Omega,\mathcal{F},\mathbb{P},\theta)$ is called a **metric dynamical system** (Shortly MDS) if $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, $(\mathbb{G},*)$ be a locally compact group and

i. $\theta: \mathbb{G} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{G}) \otimes \mathcal{F}, \mathcal{F})$ −measurable,
ii. $\theta(e,\omega) = Id_\Omega$,
iii. $\theta(t * s, \omega) = \theta(t, \theta(s,\omega))$ and
iv. $\mathbb{P}(\theta(t)F) = \mathbb{P}(F)$, for every $F \in \mathcal{F}$ and every $\in \mathbb{G}$.

Note that we write $\theta: \mathbb{G} \times \Omega \to \Omega$ either in the form $\theta(t,\omega)$(as a function of two variable or in the form $\theta_t\omega$.

Definition 2.2[2,3,4]A ***measurable random dynamical system*** on the measurable space $(X,\mathcal{B}(X))$ over an MDS $(\mathbb{G},\Omega,\mathcal{F},\mathbb{P},\theta)$ with time is a mapping $\varphi: \mathbb{T} \times \Omega \times X \to X$, with the following properties:

i. Measurability,φ is $\mathcal{B}(\mathbb{G}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$ − measurable.
ii. Cocycle property: The mappings $\varphi(t,\omega) \coloneqq \varphi(t,\omega,\cdot): X \to X$ form a cocycle over $\theta(\cdot)$, i. e. they satisfy

$\varphi(e,\omega) = id_X$for all ω ∈ Ω,e is the identity element of $\mathbb{G}$ (2.1)

$\varphi(t * s,\omega) = \varphi(t,\theta(s)\omega) \circ \varphi(s,\omega)$for all $s,t \in \mathbb{G}$, ω ∈ Ω. (2.2)

If there is no ambiguity the RDS is denoted by (θ,φ) rather than $(\mathbb{G},\Omega,X,\theta,\varphi)$.

In the following definition we shall define a random dynamical system on a uniform space.

Definition 2.3 A ***uniform random dynamical system***(URDS) on a uniform space $(X,\mathcal{U}_X)$ over the MDS $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is a measurable RDS which satisfies in addition the following properties:

1. For each $t \in \mathbb{G}$, $\varphi(t,\omega): X \longrightarrow X$is uniformly continuous with respect to the uniformity $\mathcal{U}_X$on X. (2.3)
2. For every $V \in \mathcal{U}_X$, there exists $W \in \mathcal{U}_{\mathbb{G}}$so that $(t_1,t_2) \in W$ implies $(\varphi(t_1,\omega)x, \varphi(t_2,\omega)x) \in V$ for all$x \in X$; i.e.,

$(t_1,t_2) \in W$implies$(\varphi(t_1,\omega),\varphi(t_2,\omega)) \in V^X$ (2.4)

The cocycleφ is called uniform cocycle.

Definition 2.4 If $\varphi_1: \mathbb{G} \times \Omega \times X_1 \longrightarrow X_1$and $\varphi_2: \mathbb{G} \times \Omega \times X_2 \longrightarrow X_2$are cocycles where $\mathbb{G}$ is a locally compact group ,then a **map of cocycles**, or **cocycle map**, $h: \varphi_1 \longrightarrow \varphi_2$is a map $h: \Omega \times X_1 \longrightarrow X_2$satisfies

(i) $h(\cdot,x):\Omega \longrightarrow X_2$ is measurable for all $x \in X_1$;
(ii) $h(\omega,\cdot):X_1 \longrightarrow X_2$is uniform continuous for all $\omega \in \Omega$;
(iii)$\varphi_2(t,\omega)\circ h(\omega) = h(\theta_t\omega)\circ\varphi_1(t,\omega)$for all $t \in \mathbb{G}$.

The cocycleφ_1 is called a ***subsystem*** of φ_2 if h is an embedding, that is, a homeomorphism of X_1 onto its image in X_2. The cocycle φ_2 is called factor of φ_1 if h is a quotient map, that is, a surjection on X_2 inducing the quotient topology from X_1.

Theorem 2.5 Let (θ,φ) be a uniform random dynamical system on X, let $\alpha:\mathbb{G}\to\mathbb{G}$ be a uniform continuous automorphism of the additive group $\mathbb{G}$, $\xi:\Omega\to\Omega$ be bimeasurable and let $h:X\to Y$ be a uniform isomorphism. Then $\sigma:\mathbb{G}\times Y\to Y$ defined by $\sigma \coloneqq \xi\theta(\alpha\times\xi)^{-1}$and $\psi:\mathbb{G}\times\Omega\times Y\to Y$ defined by $\psi \coloneqq h\varphi(\alpha\times\xi\times h)^{-1}$ are form a uniform random dynamical system on Y. We call (σ,ψ) the uniform random dynamical system induced from (θ,φ) by the triple (α,ξ,h).

Proof. To show that $\sigma \coloneqq \xi\theta(\alpha\times\xi)^{-1}$ is a group action of $\mathbb{G}$on X.

$\sigma(e,x) = \xi\theta(\alpha\times\xi)^{-1}(e,\omega)$
$= \xi\theta(\alpha^{-1}\times\xi^{-1})(e,\omega)$
$= \xi\theta(\alpha^{-1}(e)\times\xi^{-1}(\omega))$
$= \xi\theta(e\times\xi^{-1}(\omega))$
$= \xi\xi^{-1}(\omega) = \omega = id_\Omega$.
$\sigma(t*s,\omega) = \xi\theta(\alpha\times\xi)^{-1}(t*s,\omega)$
$= \xi\theta[\alpha^{-1}\times\xi^{-1}(t*s,\omega)]$
$= \xi\theta(\alpha^{-1}(t*s),\xi^{-1}(\omega))$
$= \xi\theta(\alpha^{-1}(t)*\alpha^{-1}(s),\xi^{-1}(\omega))$
$= \xi[\theta\left(\alpha^{-1}(t),\theta\big(\alpha^{-1}(s),\xi^{-1}(\omega)\big)\right)]$
$= \xi[\theta\big(\alpha^{-1}(t),\theta(\alpha^{-1}\times\xi^{-1})(s,\omega)\big)]$
$= \xi[\theta\big(\alpha^{-1}(t),\xi^{-1}\xi\theta(\alpha^{-1}\times\xi^{-1})(s,\omega)\big)]$
$= \xi\theta(\alpha^{-1}\times\xi^{-1})\big(t,\xi\theta(\alpha^{-1}\times\xi^{-1})(s,\omega)\big)$
$= \sigma\big(t,\xi\theta(\alpha^{-1}\times\xi^{-1})(s,\omega)\big)$
$= \sigma\big(t,\sigma(s,\omega)\big)$.

Thus $\sigma \coloneqq \xi\theta(\alpha\times\xi)^{-1}$ is a group action of $\mathbb{G}$on Ω. The measurability of $\sigma:\mathbb{G}\times Y\to Y$ follows from the fact that α,ξ and θ are measurable. Now we have α is uniform is uniform isomorphism and $\mathcal{V}$ is uniform continuous, then $\mathcal{V}':\mathbb{G}'\longrightarrow\mathbb{G}'$ is uniform continuous since $\mathcal{V}' = \alpha\mathcal{V}\alpha^{-1}$.

To show that ψ is cocycle.

$\psi(e,\omega,y) = h\varphi(\alpha\times\xi\times h)^{-1}(e,\omega,y)$
$= h\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(e,\omega,y)$
$= h\varphi(\alpha^{-1}(e),\xi^{-1}(\omega),h^{-1}(y))$
$= h\varphi(e,\xi^{-1}(\omega),h^{-1}(y))$
$= hh^{-1}(y) = y$
$\psi(t*s,\omega,y) = h\varphi(\alpha\times\xi\times h)^{-1}(t*s,\omega,y)$
$= h\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(t*s,\omega,y)$
$= h\varphi(\alpha^{-1}(t*s),\xi^{-1}(\omega),h^{-1}(y))$
$= h\varphi(\alpha^{-1}(t),\theta_{\alpha^{-1}(s)}\xi^{-1}(\omega),h^{-1}h\varphi(\alpha^{-1}(s),\xi^{-1}(\omega),h^{-1}(y)))$
$= h\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(t,\theta_s\omega,h\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1}(s,\omega,y)))$
$= \psi(t,\theta_s\omega,\psi(s,\omega,y)$

Now, To show that $\psi(\cdot,\omega,y):\mathbb{G}\to Y$ is uniform continuous. Let $V'\in\boldsymbol{\mathcal{U}}_Y$, then $V \coloneqq h^{-1}\times h^{-1}(V')\in\boldsymbol{\mathcal{U}}_X$. By hypothesis there exists $W\in\boldsymbol{\mathcal{U}}_{\mathbb{G}}$ so that

$(t_1,t_2)\in W$implies$(\varphi(t_1,\omega)x,\varphi(t_2,\omega)x)\in V$ for all$x\in X$. (2.5)

Set $W' \coloneqq \alpha\times\alpha(W)\in\boldsymbol{\mathcal{U}}_{\mathbb{G}'}$. Let $(s_1,s_2)\in W'$and $y\in Y$. Then
$(\alpha^{-1}(s_1),\alpha^{-1}(s_2))\in W$and $h^{-1}(y)\in X$.

By (2.5) we have $(\alpha^{-1}(s_1),\alpha^{-1}(s_2)) \in W$ implies
$(\varphi(\alpha^{-1}(s_1),\omega,h^{-1}(y)),\varphi(\alpha^{-1}(s_2),\omega,h^{-1}(y))) \in V$
$\Rightarrow (\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(s_1,\omega,y),\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(s_2,\omega,y) \in V$
$\Rightarrow (h\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(s_1,\omega,y),h\,\varphi(\alpha^{-1}\times\xi^{-1}\times h^{-1})(s_2,\omega,y) \in V' \Rightarrow$
$(\psi(s_1,\omega,y),\psi(s_2,\omega,y)) \in V'$.

Thus $\psi(\cdot,\omega,y)\colon \mathbb{G}\to Y$ is uniform continuous. In a similar way we can show that $\psi(t,\omega,)\colon Y\to Y$ is uniform continuous. Therefore (σ,ψ) is uniform random dynamical system.

Theorem 2.6 [7]The product of two RDS's is an RDS.

Proof. Let $(\mathbb{G}_1,\Omega_1,X_1,\theta_1,\varphi_1),(\mathbb{G}_2,\Omega_2,X_2,\theta_2,\varphi_2)$ be two RDS. Wedefine the product of the given dynamical systems as follows:

$(\mathbb{G}_1\times\mathbb{G}_2,\Omega_1\times\Omega_2,X\times Y,\theta_1\times\theta_2,\varphi_1\times\varphi_2)$,

where

$\mathbb{P} \coloneqq \mathbb{P}_1\otimes\mathbb{P}_2\colon \mathcal{F}_1\otimes\mathcal{F}_2 \longrightarrow [0,1]$

defined by $\mathbb{P}(A_1\times A_2) \coloneqq \mathbb{P}_1\otimes\mathbb{P}_2(A_1\times A_2) \coloneqq \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$,

$\theta \coloneqq \theta_1\times\theta_2\colon G_1\times G_2\times\Omega_1\times\Omega_2\colon\to\Omega_1\times\Omega_2$

define by $\theta\big((g,q),(\omega,\varpi)\big) \coloneqq (\theta_1(g,\omega),\theta_2(q,\varpi))$,

and

$\varphi \coloneqq \varphi_1\times\varphi_2\colon G_1\times G_2\times\Omega_1\times\Omega_2\times X\times Y\to X\times Y$,

defined by $\varphi\big((g,q),(\omega,\varpi),(x,y)\big) \coloneqq (\varphi_1(g,\omega,x),\varphi_2(q,\varpi,y))$.

Proof. First, the triple $(\Omega_1\times\Omega_2,\mathcal{F}_1\otimes\mathcal{F}_2,\mathbb{P}_1\otimes\mathbb{P}_2)$ is probability space [7,Theorem 2.16]. To show that θ is a metric dynamical system. Note that $\mathbb{G}_1\times\mathbb{G}_2$ is locally compact topological groups. Thusthere exists at least one regular Haar measure on $\mathbb{G}_1\times\mathbb{G}_2$.

(i) $\theta\big((e,e'),(\omega,\varpi)\big) = \big(\theta_1(e,\omega),\theta_2(e',\varpi)\big) = (\omega,\varpi) = Id_{\Omega_1\times\Omega_2}$.

(ii) $\theta\big((g,q)*(g',q'),(\omega,\varpi)\big) = \theta\big((g*g',q*q'),(\omega,\varpi)\big)$
$= \big(\theta_1(g*g',\omega),\theta_2(q*q',\varpi)\big)$
$= (\theta_1(g,\theta_1(g',\omega)),\theta_2(q,\theta_2(q',\varpi)))$
$= \theta\big((g,q),(\theta_1(g',\omega),\theta_2(q',\varpi))\big)$
$= \theta\big((g,q),\theta((g',q'),(\omega,\varpi))\big)$.

(iii) Since $\mathcal{B}(G_1)\otimes\mathcal{F}_1\otimes\mathcal{B}(G_2)\otimes\mathcal{F}_2 = \mathcal{B}(G_1\times G_2)\otimes\mathcal{F}_1\otimes\mathcal{F}_2$, then θ is $(\mathcal{B}(G_1\times G2\otimes\mathcal{F}1\otimes\mathcal{F}2,\mathcal{F}1\otimes\mathcal{F}2-$measurable.

(iv) Let $\mathbb{P} \coloneqq \mathbb{P}_1\otimes\mathbb{P}_2$, then
$\mathbb{P}(\theta(h,A)) = \mathbb{P}_1\big(\theta_1(g,A_1)\big)\mathbb{P}_2(\theta_2(q,A_2))$
$= \mathbb{P}_1(A_1)\mathbb{P}_2(A_2) = \mathbb{P}(A)$.

Then θ is an MDS.Now, to show that φ is cocycle.

(i') $\varphi(\tilde{e},\tilde{\omega},\tilde{x}) = \varphi\big((e,e'),(\omega,\varpi),(x,y)\big)$.
$= (\varphi_1(e,\omega,x),\varphi_2(e',\varpi,y))$
$= (x,y) = id_{X\times Y}$.

(ii') $\varphi\left(\tilde{g},\theta_{\tilde{q}}\tilde{\omega},\varphi\big(\tilde{h},\tilde{\omega},\tilde{x}\big)\right) = \varphi(\tilde{g},\theta_{\tilde{h}}\tilde{\omega},(\varphi_1(g,\omega,x),\varphi_2(q,\varpi,y)))$
$= \Big(\varphi_1\big(g,\theta_1(g',\omega),\varphi_1(g',\omega,x)\big),\varphi_2\big(q,\theta_2(q',\varpi),\varphi_2(q',\varpi,y)\big)\Big)$
$= \big(\varphi_1(g*g',\omega,x),\varphi_2(q*'q',\varpi,y)\big)$
$= \varphi\big(\tilde{g}\,\tilde{*}\,\tilde{h},\tilde{\omega},\tilde{x}\big)$.

(iii') Since φ_1 and φ_2 are continuous, then so is their product $\varphi_1\times\varphi_2$. Thus from the above discussion we get that the pair (θ,φ) form an RDS and this complete the proof

Theorem 2.7 The Cartesian product of two URDS's is also URDS.

Proof. Suppose that $(\mathbb{G}_1,\Omega_1,X_1,\theta_1,\varphi_1)$ and $(\mathbb{G}_2,\Omega_2,X_2,\theta_2,\varphi_2)$ are two URDS's with uniformities$\mathcal{U}_{X_1}$and $\mathcal{U}_{X_2}$ respectively. Since the Cartesian product of two RDS's is also RDS, it is sufficient to satisfy (i) and (ii) of Definition 2.2.Let $(t_1,t_2)\in\mathbb{G}_1\times\mathbb{G}_2$. Since $\varphi_i(t_i,\omega_i)\colon X_i\longrightarrow X_i$is uniformly continuous with respect to the uniformity $\mathcal{U}_{X_i}$on X_i, $i = 1,2$, then $\varphi_1(t_1, \omega_1) \times \varphi_2(t_2, \omega_2): X_1 \times X_2 \longrightarrow X_1 \times X_2$ is uniformly continuous with respect to the uniformity$W \coloneqq \mathcal{U}_{X_1} \times \mathcal{U}_{X_2}$ on $X_1 \times X_2$. Now, let $V \in W$, then for every$\left((x_1^1, x_2^1), (x_1^2, x_2^2)\right) \in V$, there exist$U_1 \in \mathcal{U}_{X_1}$ and $U_2 \in \mathcal{U}_{X_2}$such that $(x_1^1, x_1^2) \in U_1$ and $(x_2^1, x_2^2) \in U_2$. By hypothesis there exists$A_i \in \mathcal{U}_{\mathbb{G}_i}$ so that $(t_i^1, t_i^2) \in A_i$ implies $(\varphi_i(t_i^1, \omega_i)x_i, \varphi(t_i^2, \omega_i)x_i) \in U_i$ for all$x_i \in X_i$,$i = 1,2$.Set

$A \coloneqq \left\{\left((t_1^1, t_2^1), (t_1^2, t_2^2)\right): (t_1^1, t_1^2) \in A_1, (t_2^1, t_2^2) \in A_2\right\}$,

then $A \in \mathcal{U}_{\mathbb{G}_1} \times \mathcal{U}_{\mathbb{G}_2}$.Let $(x_1, x_2) \in X_1 \times X_2$,

$$(\varphi_1 \times \varphi_2\left((t_1^1, t_2^1), (\omega_1, \omega_2)\right)(x_1, x_2), \varphi_1 \times \varphi_2\left((t_1^2, t_2^2), (\omega_1, \omega_2)\right)(x_1, x_2))$$
$$= ((\varphi_1(t_1^1, \omega_1)x_1, \varphi_2(t_2^1, \omega_2)x_2), (\varphi_1(t_1^2, \omega_1)x_1, \varphi_2(t_2^2, \omega_2)x_2))$$
$$= ((\varphi_1(t_1^1, \omega_1)x_1, \varphi_1(t_1^2, \omega_1)x_1), (\varphi_2(t_2^1, \omega_2)x_2, \varphi_2(t_2^2, \omega_2)x_2)) \in U_1 \times U_2$$
$$\subset V.$$

From the above discussion we get the result.

3. Uniform Random Set

In this section the concept of uniform random set is introduced and some essential properties are proved.

Definition 3.1 Let $(X, \mathcal{U}_X)$ be a uniform space and $(\Omega, \mathcal{F})$ be a measurable space. The set-valued map $A: \Omega \longrightarrow \mathcal{B}(X \times X)$ is said to be uniform random set if

$\breve{\Omega} \coloneqq \{\omega: A(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$for every $U \in \mathcal{U}_X$.

We say that A is closed (open, compact) uniform random set if it is closed (resp. open, compact) in the product $X \times X$ (with the product of the uniform topology).

Remark 3.2:

(1) If $A(\omega) \in \mathcal{U}_X$, then $\breve{\Omega} = \Omega$.

(2) The set-valued map $A: \Omega \longrightarrow \mathcal{B}(X \times X)$ defined by $A(\omega) = \Delta$ is uniform random set. In fact it is closed uniform random set.

Theorem 3.3 Let $(X, \mathcal{U}_X) \cong_\Phi (Y, \mathcal{U}_Y)$ where X, Y are compact spaces and $(\Omega, \mathcal{F}) \cong_\xi (\Xi, \mathcal{G})$. If the set-valued map $A: \Omega \longrightarrow \mathcal{B}(X \times X)$ is uniform random set then so is the set-valued map $B: \Xi \longrightarrow \mathcal{B}(Y \times Y)$ defined by

$B(\varpi) \coloneqq \Phi \times \Phi(A\left(\xi^{-1}(\varpi)\right))$for every $\varpi \in \Xi$.

Proof. First, we need to show that $B(\varpi) \in \mathcal{B}(Y \times Y)$. Since Φ is uniform isomorphism on a compact space, then it is homeomorphism and consequently it is bimeasurable. Thus $\Phi \times \Phi$ is bimeasurable. Therefore $B(\varpi) \in \mathcal{B}(Y \times Y)$. Now, to show

$\breve{\Xi} \coloneqq \{\varpi: B(\varpi) \cap V \neq \emptyset\} \in \mathcal{G}$for every $V \in \mathcal{U}_Y$.

Let $V \in \mathcal{U}_Y$. Since Φ is uniform isomorphism, then there exists $U \in \mathcal{U}_X$ such that $\Phi \times \Phi(U) = V$. Now

$$B(\varpi) \cap V \neq \emptyset \Leftrightarrow \Phi \times \Phi(A\left(\xi^{-1}(\varpi)\right)) \cap \Phi \times \Phi(U) \neq \emptyset$$
$$\Leftrightarrow \Phi \times \Phi[A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U}] \neq \emptyset$$
$$\Leftrightarrow A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset.$$

Thus $\breve{\Xi} \coloneqq \{\varpi: B(\varpi) \cap V \neq \emptyset\} = \{\varpi: A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset\}$. Since ξ is surjective, then there exists $\omega \in \Omega$ such that $\omega \coloneqq \xi^{-1}(\varpi)$. Since A is uniform random set, then

$\left\{\omega: A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset\right\} = \{\xi^{-1}(\varpi): A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset\} \in \mathcal{F}$.

Since ξ is bimeasurable, then

$\xi\left(\{\xi^{-1}(\varpi): A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset\}\right) = \left\{\varpi: A\left(\xi^{-1}(\varpi)\right) \cap \mathrm{U} \neq \emptyset\right\} \in \mathcal{G}$.

Thus $\breve{\Xi} \coloneqq \{\varpi: B(\varpi) \cap V \neq \emptyset\} \in \mathcal{G}$. Therefore $B(\varpi)$ is uniform random set and this complete the proof.

Theorem 3.4 If $\{A_n : n \geq 1\}$ be a sequence of uniform random closed set, then so are $Cl(\bigcup_{n\geq1} A_n)$and $\bigcap_{n\geq1} A_n$.

Proof Let$(X, \mathcal{U}_X)$ be a uniform space and $(\Omega, \mathcal{F})$ be a measurable space.If $U \in \mathcal{U}_X$, then $\{\omega : Cl(\bigcup_{n\geq1} A_n(\omega))(\omega) \cap U \neq \emptyset\} = \bigcup_{n\geq1}\{\omega : A_n(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$.

This means that $Cl(\bigcup_{n\geq1} A_n)$ is uniform random closed set. To show that $\bigcap_{n\geq1} A_n$ is uniform random closed set observe that

$\bigcap_{n\geq1} A_n(\omega) \cap U = \bigcap_{n\geq1}(A_n(\omega) \cap U)$,

for every $U \in \mathcal{U}_X$ (in fact for any set U) so that

$\{\omega : \bigcap_{n\geq1} A_n(\omega) \cap U \neq \emptyset\} = \bigcap_{n\geq1}\{\omega : A_n(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$.

Corollary 3.5 If A and B are two uniform random closed sets, then so are $A \cup B$and $A \cap B$.

$\breve{\Omega} := \{\omega : A(\omega) \cap U \neq \emptyset\} \in \mathcal{F}$for every $U \in \mathcal{U}_X$.

Theorem3.6 Let $(X, \mathcal{U}_X)$ be a uniform space $\xi : X \times X \longrightarrow \mathbb{R}$ be a continuous function and $R(\omega)$ be a random variable. If the set $\xi_{R(\omega)} := \{(x, y) : \xi(x, y) \leq R(\omega)\}$ is nonempty for all $\omega \in \Omega$, then it is uniform random closed set.

Proof. Since ξ is continuous, then $\xi_{R(\omega)}$ is closed set in $X \times X$. Now, to show that $\{\omega : \xi_{R(\omega)} \cap$ *U≠∅* is measurable for every *U∈UX*. It is sufficient to show *ω:ξR(ω)∩U≠∅c=ω:ξR(ω)∩U=∅* is measurable. The measurability of $\{\omega : \xi_{R(\omega)} \cap U = \emptyset\}$ follows from the relation

$\{\omega : U \subset X \times X / \xi_{R(\omega)}\} = \{\omega : R(\omega) < s, for\ any\ s \in \xi(U)\}$.

Since

$X \times X / \xi_{R(\omega)} = \xi^{-1}(\mathbb{R}) / \xi^{-1}((-\infty, R(\omega)]) = \xi^{-1}((R(\omega), \infty))$ (3.1)

we have that $U \subset X \times X / \xi_{R(\omega)}$ if and only if $\xi(U) \subset (R(\omega), \infty)$.This implies (3.1) and therefore

$\{\omega : U \subset X \times X / \xi_{R(\omega)}\} = \bigcap_{n\in\mathbb{N}}\{\omega : R(\omega) < s_n\}$,

where$s_n \in \xi(U)$ and $s_n \longrightarrow \inf \xi(U)$ as $n \longrightarrow \infty$. This completes the proof.

4. Uniform Expansive Random Operator

In this section we study the expansivity of random operator defined on uniform random dynamical system. In fact we shall give the generalization of the notion of expansive random dynamical system defined in [4] and [7].

Definition 4.1 [4] An RDS (θ, φ) is said to be expansive if for every $x, y \in X$with $x \neq y$, there exist tempered random variable δ(called an expansivity characteristic variable) and a full measure subset $\widetilde{\Omega}$ of Ω such that for every $\omega \in \widetilde{\Omega}$ we have

$d(\varphi(n, \omega)x, \varphi(n, \omega)y) > \delta(\theta(n)\omega)$.

Definition 4.2 [7]RDS(θ, φ) is said to be $\{g_n\}$ −expansive if for every $x, y \in X$with $x \neq y$, there exist$\{g_n\}$ −tempered random variable δ(called an expansivity characteristic variable) and a full measure subset $\widetilde{\Omega}$ of Ω such that for every $\omega \in \widetilde{\Omega}$ we have

$d(\varphi(g_n, \omega)x, \varphi(g_n, \omega)y) > \delta(\theta(g_n)\omega)$.

Now, we shall introduce our new definition:

Definition 4.3 Let (θ, φ) be a uniform random dynamical system. The random operator $T : \Omega \times X \rightarrow X$ is said to be uniform expansive if for every $x, y \in X$with $x \neq y$, there exista uniform random set U (called an expansivity characteristic), syndetic subset S of $\mathbb{G}$ and a full measure subset $\widetilde{\Omega}$ of Ω such that

$(T^n(\omega, \varphi(t, \omega)x), T^n(\omega, \varphi(t, \omega)y)) \notin U(\theta_t\omega)$

for every $t \in S$ and every $\omega \in \widetilde{\Omega}$ and some positive integer n.

Remark 4.4: If we put in Definition 4.3 $n = 0$, we get Definition 4.1.

If we put in Definition 4.3 $n = 0$ and replace t by a divergent sequence $\{g_n\}$, we get Definition 4.2.

Notation 4.5 Let (θ,φ) be a uniform random dynamical system and $T:\Omega\times X\to X$ be a uniform random operator for every uniform random set U , every $t\in S$ and every $\omega\in\widetilde{\Omega}$ define a set $\Gamma_\delta(x)$ by

$\Gamma_\delta(x) := \{y\in X: (T^n(\omega,\varphi(t,\omega)x), T^n(\omega,\varphi(t,\omega)y))\in U(\theta_t\omega)\}$.

Theorem 5.6 Let (θ,φ) be a uniform random dynamical system. The uniform random operator $T:\Omega\times X\to X$ is uniform expansive if and only if for every uniform random set U we have$\Gamma_\delta(x)=\{x\}$.

Proof Suppose that $T:\Omega\times X\to X$ is uniform expansive. Assume contrary that $\Gamma_\delta(x)$ is not singleton set, then there exists $y\in\Gamma_\delta(x)$with $x\neq y$. Thus we have $x,y\in X$with $x\neq y$,then by hypothesis there exista uniform random set U, syndetic subset S of $\mathbb{G}$ and a full measure subset $\widetilde{\Omega}$ of Ω such that

$(T^n(\omega,\varphi(t,\omega)x), T^n(\omega,\varphi(t,\omega)y))\notin U(\theta_t\omega)$

for every $t\in S$ and every $\omega\in\widetilde{\Omega}$ and some positive integer n.But that is a contradiction. Thus $\Gamma_\delta(x)=\{x\}$. Conversely, suppose that $\Gamma_\delta(x)=\{x\}$. Assume contrary that $T:\Omega\times X\to X$ is not expansive, then there exist $x,y\in X$with $\neq y$, for every syndetic subset S of $\mathbb{G}$ and every full measure subset $\widetilde{\Omega}$ of Ω we have

$(T^n(\omega,\varphi(t,\omega)x), T^n(\omega,\varphi(t,\omega)y))\in U(\theta_t\omega)$

for every $t\in S$ and every $\omega\in\widetilde{\Omega}$ and every positive integer n.Thus $y\in\Gamma_\delta(x)$ and this contradict the fact that $\Gamma_\delta(x)=\{x\}$.

Definition4.7 Let (θ,φ) be a uniform random dynamical system. The uniformity $\boldsymbol{U}$is said to be random separated if for every $x,y\in X$with $x\neq y$, there exists a uniform random set $U\in\boldsymbol{U}$ such that $(x,y)\notin U(\theta_t\omega)$ for every $t\in\mathbb{G}$.

Theorem 4.8. Let (θ,φ) be a uniform random dynamical system such that $(X,\boldsymbol{\mathcal{U}})$ be random separated uniform space and $T:\Omega\times X\to X$ be a uniform random operator. If T^n not constant for some $n\geq 1$, then T is uniform expansive on X.

Proof. Suppose that T^n not constant for some $n\geq 1$. Let $x,y\in X$with $x\neq y$. Thus $T^n(\omega,x)\neq T^n(\omega,y)$ for all $\omega\in\Omega$. Since X is random separated then there exists a uniform random set U such that

$(T^n(\omega,x), T^n(\omega,y))\notin U(\theta_t\omega)$.

Since $\varphi(t,\omega)x\neq\varphi(t,\omega)y$, then

$(T^n(\omega,\varphi(t,\omega)x), T^n(\omega,\varphi(t,\omega)y))\notin U(\theta_t\omega)$

for every $t\in\mathbb{G}$ and $\omega\in\Omega$.This means that T is uniform expansive.

Theorem 4.9. Let (θ,φ), (σ,ψ) be two uniform random dynamical systems, and $T_1:\Omega\times X\to X, T_2:\Xi\times Y\to Y$ be equivariant topologically conjugate via

$(\mu,\psi):(G_1,X,\theta_1)\to(G_2,Y,\theta_2)$.

T_1isuniform expansive, if and only if T_2 is uniform expansive.

Proof. Suppose that T_1 is expansive. Let $y_1,y_2\in Y$, with $y_1\neq y_2$. Since Φ is bijective then there exists $x_1,x_2\in X$ such that $y_1=\Phi(x_1), y_2=\Phi(x_2)$ and this implies that $x_1=\Phi^{-1}(y_1), x_2=\Phi^{-1}(y_2)$. Since Φ is bijective, then so is Φ^{-1} and consequently $\Phi^{-1}(y_1)\neq\Phi^{-1}(y_2)$. By hypothesis there existsa uniform random set U (called an expansivity characteristic), syndetic subset S of $\mathbb{G}$ and a full measure subset $\widetilde{\Omega}$ of Ω such that

$$(T^n(\omega,\varphi(t,\omega)x_1), T^n(\omega,\varphi(t,\omega)x_2))\notin U(\theta_t\omega) \quad (4.1)$$

for every $t\in S$ and every $\omega\in\widetilde{\Omega}$ and some positive integer n.Since $\Phi(\omega)^{-1}$ is uniform continuous, then there exists uniform random set V such that

$$(y_1,y_2)\in V(\sigma_{\tilde{t}}\varpi)\text{implies } (\Phi(\omega)^{-1}y_1, \Phi(\omega)^{-1}y_2)\in U(\theta_t\omega) \quad (4.2)$$

By using (4.1) and the contrapositive of (4.2) we get

$(\Phi(\omega)T_1^n(\omega,\varphi(t,\omega)x_1), \Phi(\omega)T_1^n(\omega,\varphi(t,\omega)x_2))\notin V(\sigma_{\tilde{t}}\varpi)$

This implies that

$(T_2^n(\varpi,\psi(\tilde{t},\varpi)\Phi(\omega)x_1), T_2^n(\varpi,\psi(\tilde{t},\varpi)\Phi(\omega)x_2))\notin V(\sigma_{\tilde{t}}\varpi)$

Consequently,

$(T_2^n(\varpi,\psi(\tilde{t},\varpi)y_1), T_2^n(\varpi,\psi(\tilde{t},\varpi)y_2))\notin V(\sigma_{\tilde{t}}\varpi)$

This mean that T_2 is uniform expansive. The converse also follows analogously.

Definition4.10 Let (θ, φ) be a uniform random dynamical system and $f_k: \Omega \times X \to X, k \in \mathbb{N}$, be a sequence of uniform random operator then $F \coloneqq \{f_k\}_{k=1}^{\infty}$ is said to be expansive in the iterative way if there exist a uniform random set U, asyndeticsubset $S$of $\mathbb{G}$ and a subset of full measure $\widetilde{\Omega} \subseteq \Omega$such that for any $x, y \in X$with $x \neq y$ and positive integers n, k such that

$$(F_k^n(\omega, \varphi(t, \omega)x), F_k^n(\omega, \varphi(t, \omega)y)) \notin U(\theta_t\omega) ,$$

for all $t \in S$ and $\omega \in \Omega$.

Definition 4.11 Let (θ, φ) be a uniform random dynamical system and $f_k: \Omega \times X \to X, k \in \mathbb{N}$, be a sequence of uniform random operator then $F \coloneqq \{f_k\}_{k=1}^{\infty}$ is said to be expansive in the iterative way if there exist a uniform random set U, asyndeticsubset $S$of $\mathbb{G}$ and a subset of full measure $\widetilde{\Omega} \subseteq \Omega$such that for any $x, y \in X$with $x \neq y$ and positive integers n, k such that

$$(f_k^n(\omega, \varphi(t, \omega)x), f_k^n(\omega, \varphi(t, \omega)y)) \notin U(\theta_t\omega) ,$$

for all $t \in S$ and $\omega \in \Omega$.

Theorem 4.12 Let $(\theta, \varphi), (\sigma, \psi)$ be two uniform random dynamical systems and $F \coloneqq \{f_k\}_{k=1}^{\infty}$, $H \coloneqq \{h_k\}_{k=1}^{\infty}$ be two sequences of uniform random operator on $\Omega \times X, \Xi \times Y$ respectively. If there exists an equivariant uniform isomorphism $\Phi: X \to Y$ such that F and $\mathcal{G}$ are h −conjugate, then F is expansive on X in the iterative (or successive) way if and only if $\mathcal{G}$ is expansive on Y.

Proof. Suppose $F \coloneqq \{f_k\}_{k=1}^{\infty}$ has expansive on X in the iterative way. There exists there exists a uniform random set U, asyndeticsubset $S$of $\mathbb{G}$ and a subset of full measure $\widetilde{\Omega} \subseteq \Omega$such that for any $x, \tilde{x} \in X$with $x \neq y$ and positive integers n, k such that

$$(F_k^n(\omega, \varphi(t, \omega)x), F_k^n(\omega, \varphi(t, \omega)\tilde{x})) \notin U(\theta_t\omega) ,$$

for all $t \in S$ and $\omega \in \widetilde{\Omega}$.

Since Φ^{-1} is uniform continuous therefore there exists a uniform random set Vsuch that for any $y_1, y_2 \in Y$ with

$$(y_1, y_2) \in V(\sigma_{\tilde{t}}\varpi), (\Phi(\omega)^{-1}y_1, \Phi(\omega)^{-1}y_2) \in U(\theta_t\omega) ,$$

for all $t \in S$ and $\omega \in \Omega$, where $x_1 = \Phi(\omega)^{-1}y_1, x_2 = \Phi(\omega)^{-1}y_1$. Hence for any $y_1 = \Phi(\omega)x_1, y_2 = \Phi(\omega)x_2 \in Y$, we have

$$(x_1, x_2) \notin U(\theta_t\omega) \Longrightarrow (y_1, y_2) \notin V(\sigma_{\tilde{t}}\varpi) \tag{4.3}$$

Since

$$(F_k^n(\omega, \varphi(t, \omega)x), F_k^n(\omega, \varphi(t, \omega)\tilde{x})) \notin U(\theta_t\omega) ,$$

for all $t \in S$ and $\omega \in \Omega$. Now we use (4.3) and observe that for all $t, s \in \mathbb{G}$

$$(\Phi(\theta_t\omega)F_k^n(\omega, \varphi(t, \omega)x), \Phi(\theta_t\omega)F_k^n(\omega, \varphi(t, \omega)\tilde{x})) \notin V(\sigma_s\varpi).$$

Since

$\Phi(\theta_t\omega) \circ F_k^n = H_k^n \circ \Phi(\omega)$ and $\Phi(\omega)$ is equivariant, we have

$$(H_k^n(\varpi, \psi(s, \varpi)\Phi(\omega)x), H_k^n(\varpi, \psi(s, \varpi)\Phi(\omega)\tilde{x})) \notin V(\sigma_s\varpi)$$

and hence

$$(H_k^n(\varpi, \psi(s, \varpi)y), H_k^n(\varpi, \psi(s, \varpi)\Phi(\omega)\tilde{y})) \notin V(\sigma_s\varpi)$$

for all $s \in \mathbb{G}$ and all $\varpi \in \Xi$. This establishes $H \coloneqq \{h_k\}_{k=1}^{\infty}$ is expansive in the iterative way. The converse statement can be proved similarly.

The case of expansive in the successive way also follows analogously.

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