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SHIFTED CHEBYSHEV POLYNOMIALS OF THE THIRD KIND SOLUTION FOR SYSTEM OF NON-LINEAR FRACTIONAL DIFFUSION EQUATIONS

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Abstract

In this paper, an implementation of an efficient numerical method for solving the system of coupled non-linear fractional diffusion equations (NFD Es) is introduced. The proposed system has many applications such as porous media and plasma transport and others. The fractional derivative is described in the Caputo sense. The method is based upon a combination between the properties of the our scheme uses shifted Chebyshev polynomials of the third kind approximations and finite difference method (FDM). The proposed method reduces NFD Es to a system of ODEs, which solved using FDM. Numerical example is given to show the validity and the accuracy of the proposed algorithm.

Keywords: Coupled non-linear fractional diffusion equations; Caputo fractional derivatives; Shifted Chebyshev polynomials of third kind approximates; Finite difference method.

1. Introduction

Fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering [22]. Consequently, considerable attention has been given to the solutions of FDEs and integral equations of physical interest. In last decades, fractional calculus has drawn a wide attention from many physicists and mathematicians, because of its interdisciplinary application and physical meaning [21]. Fractional calculus deals with the generalization of differentiation and integration of non-integer order. Most FDEs do not have exact solutions, so approximate and numerical techniques ([8]-[14]) must be used. Several numerical methods to solve FDEs have been given such as, homotopy perturbation method [5], homotopy analysis method [6], collocation method ([29]-[31]) and finite difference method ([24], [28], [33]).

In recent decades, the Chebyshev polynomials are one of the most useful polynomials which are suitable in numerical analysis including polynomial approximation, integral and differential equations and spectral methods for partial differential equations and fractional order differential equations ([7], [15], [17]).

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and form the basis of the solution of differential equations [30]. Chebyshev polynomials are widely used in numerical computation. One of the advantages of using Chebyshev polynomials as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function $u(t)$ is infinitely differentiable. In [7], Khader introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials. In [21] the generalized Chebyshev polynomials were used to compute a spectral solution of a non-linear boundary value problems. Also, FDM plays an important rule in recent researches in this field. It has been shown that this procedure is a powerful tool for solving various kinds of problems. This technique reduces the problem to a system of non-linear algebraic equations [28].

Our fundamental goal of this work is to develop a suitable way to approximate the system of coupled non-linear fractional order diffusion equation using the shifted Chebyshev polynomials of the third kind with finite difference method together with Chebyshev collocation method.

The structure of this paper is arranged in the following way: In section 2, we introduce some basic definitions about Caputo fractional derivatives. In section 3, we introduce the mathematical formulation of the model. In section 4, we give some properties of Chebyshev polynomials of the third kind which are of fundamental importance in what follows and we derive an approximate formula for fractional derivatives using Chebyshev polynomials of the third kind expansion. In section 5, we give numerical example to solve the system of NF***D*** Es and show the accuracy of the presented method. Finally, in section 6, the paper ends with a brief conclusion.

2. Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

Definition 1.

The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\xi)}{(t-\xi)^{\alpha-m+1}} d\xi, & 0 \le m-1 < \alpha < m, \\ f^{(m)}(t), & \alpha = m \in \mathbb{N}. \end{cases}$$

Similar to integer-order differentiation, Caputo fractional derivative operator is linear

$$D^\alpha \left(c_1 p(t) + c_2 q(t)\right) = c_1 D^\alpha p(t) + c_2 D^\alpha q(t),$$

where c_1 and c_2 are constants. For the Caputo's derivative we have

$$D^{\alpha} C = 0, \qquad C \text{ is a constant}, \tag{1}$$

$$D^{\alpha} t^{n} = \begin{cases} 0, & \text{for } n \in N_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, & \text{for } n \in N_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \tag{2}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \ldots\}$.

For more details on fractional derivatives definitions and its properties see ([19], [26]).

3. Mathematical formulation of the model

Merkin and Needham [18] considered the reaction-diffusion travelling waves that can develop in a coupled system involving simple isothermal autocatalysis kinetics. They assumed that reactions took place in two separate and parallel regions, with, in I, the reaction being given by quadratic autocatalysis

$$F + G \rightarrow 2G(rate\, k_1 \rho \delta), \tag{3}$$

together with a linear decay step

$$G \rightarrow H(rate\, k_2 \delta), \tag{4}$$

where ρ and δ are the concentrations of reactant F and autocatalyst G, the $k_i (i = 1, 2)$ are the rate constants and H is some inert product of reaction. The reaction in region II was the quadratic autocatalytic step (3) only. The two regions were assumed to be coupled via a linear diffusive interchange of the autocatalytic species G. We shall consider a similar system as I, but with cubic autocatalysis

$$F + 2G \rightarrow 3G(rate\, k_3 \rho \delta^2), \tag{5}$$

together with a linear decay step

$$G \rightarrow H(rate\, k_4 \delta), \tag{6}$$

this leads to the system of equations (7).

The following boundary value problem on $0 < x < a$ and $t > 0$ for the dimensionless concentrations (u_1, v_1) in region I and (u_2, v_2) in region II of species F and G is considered

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} - u_1 v_1^2, \\ \frac{\partial v_1}{\partial t} &= \frac{\partial^2 v_1}{\partial x^2} + u_1 v_1^2 - k v_1 + \gamma(v_2 - v_1), \\ \frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} - u_2 v_2^2, \\ \frac{\partial v_2}{\partial t} &= \frac{\partial^2 v_2}{\partial x^2} + u_2 v_2^2 + \gamma(v_1 - v_2), \end{aligned} \tag{7}$$

with the boundary conditions

$$u_i(0,t) = u_i(a,t) = 1, \qquad v_i(0,t) = v_i(a,t) = 0, \qquad i = 1,2. \tag{8}$$

The constants k and γ represent the strength of the autocatalyst decay and the coupling between the two regions, respectively.

In this paper, we study the analytical approximate solution of the system of non-linear fractional diffusion equations of the form ([1], [15])

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= D^{\alpha}u_1 - u_1 v_1^2,\\
\frac{\partial v_1}{\partial t} &= D^{\alpha}v_1 + u_1 v_1^2 - k v_1 + \gamma(v_2 - v_1),\\
\frac{\partial u_2}{\partial t} &= D^{\alpha}u_2 - u_2 v_2^2,\\
\frac{\partial v_2}{\partial t} &= D^{\alpha}v_2 + u_2 v_2^2 + \gamma(v_1 - v_2),
\end{aligned} \tag{9}$$

with the boundary conditions (8). Where the symbol α refers to the Caputo fractional derivative.

Recently, several authors, for example ([4], [16], [23]) have investigated the fractional diffusion/wave equation and its special properties. The fractional diffusion and wave equations have important applications to mathematical physics. Fractional diffusion equation describes diffusion in special types of porous media [20]. It is also used to model anomalous diffusion in plasma transport. For more details on the proposed model see ([2], [3]).

In this paper, we use shifted Chebyshev polynomials of third kind and recall some important properties and its analytical form. Next we use these polynomials to approximate the numerical solution of (FDE) with the aid of the Chebyshev collocation method together with the finite difference method to convert the system of equations in algebraic equations that can be solved numerically.

4. Some properties of Chebyshev polynomials of the third kind

4.1. Chebyshev polynomials of the thirdkind

The Chebyshev polynomials $V_n(x)$ of the third kind ([17], [32]) are orthogonal polynomials of degree n in x defined on the $[-1,1]$

$$V_n = \frac{cos(n+\frac{1}{2})\Theta}{cos(\frac{\Theta}{2})}$$

where $x = cos\Theta$ and $\Theta \in [0, \pi]$. They can be obtained explicitly using the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, for the special case $\beta = -\alpha = 1/2$.

These are given by:

$$V_k(x) = \frac{2^{2k} P_k^{(-1/2,1/2)}(x)}{\binom{2k}{k}}. \tag{10}$$

Also, these polynomials $V_n(x)$ are orthogonal on $[-1, 1]$ with respect to the inner product:

$$< V_n(x), V_m(x) >= \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} V_n(x) V_m(x) = \begin{cases} \pi, & \text{for } n = m; \\ 0, & \text{for } n \neq m; \end{cases} . \tag{11}$$

where $\sqrt{\frac{1+x}{1-x}}$ is weight function corresponding to $V_n(x)$.

The polynomials $V_n(x)$ may be generated by using the recurrence relations

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad V_0(x) = 1, \quad V_1(x) = 2x - 1, \quad n = 1, 2, \ldots .$$

The analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n, using Eq. (10) and properties of Jacobi polynomials to obtain they are given as:

$$V_n(x) = \sum_{k=0}^{[\frac{2n+1}{2}]} (-1)^k (2)^{n-k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} (x+1)^{n-k}, \qquad n \in Z^+, \tag{12}$$

where $[\frac{2n+1}{2}]$ denotes the integer part of $(2n + 1)/2$.

4.2. The shifted Chebyshev polynomials of the third kind

In order to use these polynomials on the interval $[0, 1]$ we define the so called shifted Chebyshev polynomials of the third kind by introducing the change of variable $s = 2x - 1$. The shifted Chebyshev polynomials of the third kind are defined as $V_n^*(x) = V_n(2x - 1)$.

These polynomials are orthogonal on the support interval $[0, 1]$ as the following inner product:

$$< V_n^*(x), V_m^*(x) >= \int_0^1 \sqrt{\frac{x}{1-x}} V_n^*(x) V_m^*(x) = \begin{cases} \frac{\pi}{2}, & \text{for } n = m; \\ 0, & \text{for } n \neq m; \end{cases} . \tag{13}$$

where $\sqrt{\frac{x}{1-x}}$ is weight function corresponding to $V_n^*(x)$. and normalized by the requirement that $V_n^*(1) = 1$.

The polynomials $V_n^*(x)$ may be generated by using the recurrence relations

$$V_{n+1}^*(x) = 2(2x-1)V_n^*(x) - V_{n-1}^*(x), \quad V_0^*(x) = 1, \quad V_1^*(x) = 4x - 3, \quad n = 1, 2, \ldots .$$

The analytical form of the shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x given by:

$$V_n^*(x) = \sum_{k=0} (-1)^k (2)^{2n-2k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\Gamma(2n-2k+2)} (x)^{n-k}, \qquad n \in Z^+, \tag{14}$$

In a spectral method, in contrast, the function $g(x)$, square integrable in $[0, 1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as follows:

$$g(x) = \sum_{i=0}^{\infty} b_i V_i^*(x), \tag{15}$$

where b_i is a chosen sequence of prescribed basis functions. One then proceeds somehow to estimate as many as possible of the coefficients bi, thus approximating $g(x)$ by a finite sum of $(m + 1)$-terms such as:

$$g_m(x) = \sum_{i=0}^{m} b_i V_i^*(x), \tag{16}$$

where the coefficients b_i, $i = 0, 1, \dots$ are given by

$$b_i = \frac{1}{\pi} \int_{-1}^{1} g\left(\frac{x+1}{2}\right) V_i(x) \sqrt{\frac{1+x}{1-x}}\, dx, \tag{17}$$

where the coefficients b_i, $i = 0, 1, \dots$ are given by

$$b_i = \frac{2}{\pi} \int_{0}^{1} g(x) V_i^*(x) \sqrt{\frac{x}{1-x}}\, dx, \tag{18}$$

Theorem 1. *(Chebyshev truncation theorem) ([17], [25])*

The error in approximating $g(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$g_m(x) = \sum_{i=0}^{m} b_i V_i(x), \tag{19}$$

then

$$E_T(m) \equiv |g(x) - g_m(x)| \le \sum_{k=m+1}^{\infty} |b_i|, \tag{20}$$

for all $g(x)$, all m, and all $x \in [-1, 1]$.

The main approximate formula of the fractional derivative of $g_m(x)$ is given in the following theorem.

Theorem 2.

Let $g(x)$ be approximated by shifted Chebyshev polynomials of the third kind as (16) and also suppose $\alpha > 0$, then

$$D^{\alpha}(g_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} b_i N_{i,k}^{(\alpha)} x^{i-k-\alpha}, \tag{21}$$

where $N_{i,k}^{(\alpha)}$ is given by

$$N_{i,k}^{(\alpha)} = (-1)^k \frac{2^{2i-2k}(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\alpha)} \tag{22}$$

Proof. ([32]). □

5. Implementation of Chebyshev spectral method for solving system of NF*D*Es

In this section, we implement the shifted Chebyshev polynomials of the third kind method to solve numerically the system of coupled non-linear fractional diffusion equations. Test example is presented to validate the solution scheme.

We consider the system of coupled non-linear fractional diffusion equations (9) with the constants $a = 100$, $k = 0.2$, $\gamma = 0.1$ with different values of the time t and different values of α with the zeros initial conditions

$$u_1(x,0) = u_2(x,0) = v_1(x,0) = v_2(x,0) = 0. \tag{23}$$

In order to use the shifted Chebyshev polynomials of the third kind method, we approximate $u_1(x,t)$, $u_2(x,t)$, $v_1(x,t)$ and $v_2(x,t)$ with $m = 3$ as

$$\begin{aligned} u_1(x,t) &\simeq \sum_{i=0}^{3} u_{1,i}(t)\, V_i^*(x), \qquad & u_2(x,t) &\simeq \sum_{i=0}^{3} u_{2,i}(t)\, V_i^*(x), \\ v_1(x,t) &\simeq \sum_{i=0}^{3} v_{1,i}(t)\, V_i^*(x), \qquad & v_2(x,t) &\simeq \sum_{i=0}^{3} v_{2,i}(t)\, V_i^*(x). \end{aligned} \tag{24}$$

Substitution from Eqs.(24) and Theorem 1 in (9) we obtain

$$\sum_{i=0}^{3} \dot{u}_{1,i}(t)\, V_i^*(x) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{1,i}(t)\, N_{i,k}^{(\alpha)} x^{k-\alpha} - \sum_{i=0}^{3} u_{1,i}(t) V_i^*(x) \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x) \right)^2, \tag{25}$$

$$\begin{aligned} \sum_{i=0}^{3} \dot{v}_{1,i}(t)\, V_i^*(x) = {} & \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{1,i}(t)\, N_{i,k}^{(\alpha)} x^{k-\alpha} + \sum_{i=0}^{3} u_{1,i}(t) V_i^*(x) \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x) \right)^2 \\ & - (k+\gamma) \sum_{i=0}^{3} v_{1,i}(t) V_i^*(x) + \gamma \sum_{i=0}^{3} v_{2,i}(t) V_i^*(x), \end{aligned} \tag{26}$$

$$\sum_{i=0}^{3} \dot{u}_{2,i}(t)\, V_i^*(x) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{2,i}(t)\, N_{i,k}^{(\alpha)} x^{k-\alpha} - \sum_{i=0}^{3} u_{2,i}(t) V_i^*(x) \left(\sum_{i=0}^{3} v_{2,i}(t) V_i^*(x) \right)^2, \tag{27}$$

$$\sum_{i=0}^{3} \dot{v}_{2,i}(t)\, V_i^*(x) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{2,i}(t)\, N_{i,k}^{(\alpha)} x^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t) V_i^*(x) \left(\sum_{i=0}^{3} v_{2,i}(t) V_i^*(x) \right)^2$$
$$+ \gamma \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x) - \sum_{i=0}^{3} v_{2,i}(t) V_i^*(x) \right). \tag{28}$$

We now collocate Eqs.(25)-(28) at $(m+1-\lceil\alpha\rceil)$ points x_p, $p = 0, 1, \ldots, m-\lceil\alpha\rceil$ as

$$\sum_{i=0}^{3} \dot{u}_{1,i}(t)\, V_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{1,i}(t)\, N_{i,k}^{(\alpha)} x_p^{k-\alpha} - \sum_{i=0}^{3} u_{1,i}(t) V_i^*(x_p) \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x_p) \right)^2, \tag{29}$$

$$\sum_{i=0}^{3} \dot{v}_{1,i}(t)\, V_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{1,i}(t)\, N_{i,k}^{(\alpha)} x_p^{k-\alpha} + \sum_{i=0}^{3} u_{1,i}(t) V_i^*(x_p) \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x_p) \right)^2$$
$$-(k+\gamma) \sum_{i=0}^{3} v_{1,i}(t) V_i^*(x_p) + \gamma \sum_{i=0}^{3} v_{2,i}(t) V_i^*(x_p), \tag{30}$$

$$\sum_{i=0}^{3} \dot{u}_{2,i}(t)\, V_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{2,i}(t)\, N_{i,k}^{(\alpha)} x_p^{k-\alpha} - \sum_{i=0}^{3} u_{2,i}(t) V_i^*(x_p) \left(\sum_{i=0}^{3} v_{2,i}(t) V_i^*(x_p) \right)^2, \tag{31}$$

$$\sum_{i=0}^{3} \dot{v}_{2,i}(t)\, V_i^*(x_p) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{2,i}(t)\, N_{i,k}^{(\alpha)} x_p^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t) V_i^*(x_p) \left(\sum_{i=0}^{3} v_{2,i}(t) V_i^*(x_p) \right)^2$$
$$+ \gamma \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x_p) - \sum_{i=0}^{3} v_{2,i}(t) V_i^*(x_p) \right). \tag{32}$$

For suitable collocation points we use roots of shifted Chebyshev polynomials of the third kind $V^*_{m+1-\lceil\alpha\rceil}(x)$.

In this case, the roots x_p of shifted Chebyshev polynomials of the third kind $V_2^*(x)$, are

$$x_0 = 14.6447, \qquad x_1 = 85.3553.$$

Also, by substituting Eq.(24) in the boundary conditions (8) we can find

$$\sum_{i=0}^{3} (-1)^i u_{1,i}(t) = 1, \quad \sum_{i=0}^{3} u_{1,i}(t) = 1, \quad \sum_{i=0}^{3} (-1)^i v_{1,i}(t) = 0, \quad \sum_{i=0}^{3} v_{1,i}(t) = 0,$$
$$\sum_{i=0}^{3} (-1)^i u_{2,i}(t) = 1, \quad \sum_{i=0}^{3} u_{2,i}(t) = 1, \quad \sum_{i=0}^{3} (-1)^i v_{2,i}(t) = 0, \quad \sum_{i=0}^{3} v_{2,i}(t) = 0. \tag{33}$$

By using Eqs.(29)-(32) and (33) we obtain the following non-linear system of ODEs

$$\dot{u}_{1,0}(t) + {+}_1\, \dot{u}_{1,1}(t) + {+}_2\, \dot{u}_{1,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{1,i}(t)\, N_{i,k}^{(\alpha)} x_0^{k-\alpha} - \sum_{i=0}^{3} u_{1,i}(t) V_i^*(x_0) \left(\sum_{i=0}^{3} v_{1,i}(t) V_i^*(x_0) \right)^2,$$

(34)

$$\dot{v}_{1,0}(t) + {+}_1\,\dot{v}_{1,1}(t) + {+}_2\,\dot{v}_{1,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} v_{1,i}(t)\,N_{i,k}^{(\alpha)}x_0^{k-\alpha} + \sum_{i=0}^{3} u_{1,i}(t)V_i^*(x_0)\left(\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_0)\right)^2$$
$$-(k+\gamma)\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_0) + \gamma\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_0), \tag{35}$$

$$\dot{u}_{2,0}(t) + {+}_1\,\dot{u}_{2,1}(t) + {+}_2\,\dot{u}_{2,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} u_{2,i}(t)\,N_{i,k}^{(\alpha)}x_0^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t)V_i^*(x_0)\left(\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_0)\right)^2, \tag{36}$$

$$\dot{v}_{2,0}(t) + {+}_1\,\dot{v}_{2,1}(t) + {+}_2\,\dot{v}_{2,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} v_{2,i}(t)\,N_{i,k}^{(\alpha)}x_0^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t)V_i^*(x_0)\left(\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_0)\right)^2$$
$$+\gamma\left(\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_0) - \sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_0)\right), \tag{37}$$

$$\dot{u}_{1,0}(t) + {+}_{11}\,\dot{u}_{1,1}(t) + {+}_{22}\,\dot{u}_{1,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} u_{1,i}(t)\,N_{i,k}^{(\alpha)}x_1^{k-\alpha} - \sum_{i=0}^{3} u_{1,i}(t)V_i^*(x_1)\left(\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_1)\right)^2, \tag{38}$$

$$\dot{v}_{1,0}(t) + {+}_{11}\,\dot{v}_{1,1}(t) + {+}_{22}\,\dot{v}_{1,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} v_{1,i}(t)\,N_{i,k}^{(\alpha)}x_1^{k-\alpha} + \sum_{i=0}^{3} u_{1,i}(t)V_i^*(x_1)\left(\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_1)\right)^2$$
$$-(k+\gamma)\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_1) + \gamma\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_1), \tag{39}$$

$$\dot{u}_{2,0}(t) + {+}_{11}\,\dot{u}_{2,1}(t) + {+}_{22}\,\dot{u}_{2,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} u_{2,i}(t)\,N_{i,k}^{(\alpha)}x_1^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t)V_i^*(x_1)\left(\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_1)\right)^2, \tag{40}$$

$$\dot{v}_{2,0}(t) + {+}_{11}\,\dot{v}_{2,1}(t) + {+}_{22}\,\dot{v}_{2,3}(t) = \sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i} v_{2,i}(t)\,N_{i,k}^{(\alpha)}x_1^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}(t)V_i^*(x_1)\left(\sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_1)\right)^2$$
$$+\gamma\left(\sum_{i=0}^{3} v_{1,i}(t)V_i^*(x_1) - \sum_{i=0}^{3} v_{2,i}(t)V_i^*(x_1)\right), \tag{41}$$

$$u_{1,0}(t) - u_{1,1}(t) + u_{1,2}(t) - u_{1,3}(t) = 1, \tag{42}$$

$$u_{1,0}(t) + u_{1,1}(t) + u_{1,2}(t) + u_{1,3}(t) = 1, \tag{43}$$

$$u_{2,0}(t) - u_{2,1}(t) - u_{2,2}(t) - u_{2,3}(t) = 1, \tag{44}$$

$$u_{2,0}(t) + u_{2,1}(t) + u_{2,2}(t) + u_{2,3}(t) = 1, \tag{45}$$

$$v_{1,0}(t) - v_{1,1}(t) + v_{1,2}(t) - v_{1,3}(t) = 0, \tag{46}$$

$$v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t) + v_{1,3}(t) = 0, \tag{47}$$

$$v_{2,0}(t) - v_{2,1}(t) + v_{2,2}(t) - v_{2,3}(t) = 0, \tag{48}$$

$$v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t) + v_{2,3}(t) = 0, \tag{49}$$

where $+_1 = V_1^*(x_0)$, $+_2 = V_3^*(x_0)$, $+_{11} = V_1^*(x_1)$, $+_{22} = V_3^*(x_1)$.

Now, to use finite difference method [24] for solving the system (34)-(49), we use the notations $t_n = n\Delta t$ to be the integration time $0 \le t_n \le T$, $\Delta t = T/N$, for $n = 0, 1, ..., N$. Define $u_{i,k}^n = u_{i,k}(t_n)$, $v_{i,k}^n = v_{i,k}(t_n)$, $i = 1, 2$, $k = 0, 1, 2, 3$. Then the system (34)-(49), is discretized and takes the following form

$$\frac{u_{1,0}^{n+1} - u_{1,0}^n}{\Delta t} + +_1 \frac{u_{1,1}^{n+1} - u_{1,1}^n}{\Delta t} + +_2 \frac{u_{1,3}^{n+1} - u_{1,3}^n}{\Delta t} = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{1,i}^{n+1} N_{i,k}^{(\alpha)} x_0^{k-\alpha} - \sum_{i=0}^{3} u_{1,i}^{n+1} V_i^*(x_0) \left(\sum_{i=0}^{3} v_{1,i}^{n+1} V_i^*(x_0) \right)^2, \tag{50}$$

$$\begin{aligned}\frac{v_{1,0}^{n+1} - v_{1,0}^n}{\Delta t} + +_1 \frac{v_{1,1}^{n+1} - v_{1,1}^n}{\Delta t} + +_2 \frac{v_{1,3}^{n+1} - v_{1,3}^n}{\Delta t} &= \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{1,i}^{n+1} N_{i,k}^{(\alpha)} x_0^{k-\alpha} + \sum_{i=0}^{3} u_{1,i}^{n+1} V_i^*(x_0) \left(\sum_{i=0}^{3} v_{1,i}^{n+1} V_i^*(x_0) \right)^2 \\ &\quad - (k + \gamma) \sum_{i=0}^{3} v_{1,i}^{n+1} V_i^*(x_0) + \gamma \sum_{i=0}^{3} v_{2,i}^{n+1} V_i^*(x_0),\end{aligned} \tag{51}$$

$$\frac{u_{2,0}^{n+1} - u_{2,0}^n}{\Delta t} + +_1 \frac{u_{2,1}^{n+1} - u_{2,1}^n}{\Delta t} + +_2 \frac{u_{2,3}^{n+1} - u_{2,3}^n}{\Delta t} = \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} u_{2,i}^{n+1} N_{i,k}^{(\alpha)} x_0^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}^{n+1} V_i^*(x_0) \left(\sum_{i=0}^{3} v_{2,i}^{n+1} V_i^*(x_0) \right)^2, \tag{52}$$

$$\begin{aligned}\frac{v_{2,0}^{n+1} - v_{2,0}^n}{\Delta t} + +_1 \frac{v_{2,1}^{n+1} - v_{2,1}^n}{\Delta t} + +_2 \frac{v_{2,3}^{n+1} - v_{2,3}^n}{\Delta t} &= \sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} v_{2,i}^{n+1} N_{i,k}^{(\alpha)} x_0^{k-\alpha} + \sum_{i=0}^{3} u_{2,i}^{n+1} V_i^*(x_0) \left(\sum_{i=0}^{3} v_{2,i}^{n+1} V_i^*(x_0) \right)^2 \\ &\quad + \gamma \left(\sum_{i=0}^{3} v_{1,i}^{n+1} V_i^*(x_0) - \sum_{i=0}^{3} v_{2,i}^{n+1} V_i^*(x_0) \right),\end{aligned} \tag{53}$$

$$\frac{u_{1,0}^{n+1}-u_{1,0}^{n}}{\Delta t}+{+}_{11}\frac{u_{1,1}^{n+1}-u_{1,1}^{n}}{\Delta t}+{+}_{22}\frac{u_{1,3}^{n+1}-u_{1,3}^{n}}{\Delta t}$$
$$=\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i}u_{1,i}^{n+1}N_{i,k}^{(\alpha)}x_1^{k-\alpha}-\sum_{i=0}^{3}u_{1,i}^{n+1}V_i^*(x_1)\left(\sum_{i=0}^{3}v_{1,i}^{n+1}V_i^*(x_1)\right)^2, \tag{54}$$

$$\frac{v_{1,0}^{n+1}-v_{1,0}^{n}}{\Delta t}+{+}_{11}\frac{v_{1,1}^{n+1}-v_{1,1}^{n}}{\Delta t}+{+}_{22}\frac{v_{1,3}^{n+1}-v_{1,3}^{n}}{\Delta t}$$
$$=\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i}v_{1,i}^{n+1}N_{i,k}^{(\alpha)}x_1^{k-\alpha}+\sum_{i=0}^{3}u_{1,i}^{n+1}V_i^*(x_1)\left(\sum_{i=0}^{3}v_{1,i}^{n+1}V_i^*(x_1)\right)^2 \tag{55}$$
$$-(k+\gamma)\sum_{i=0}^{3}v_{1,i}^{n+1}V_i^*(x_1)+\gamma\sum_{i=0}^{3}v_{2,i}^{n+1}V_i^*(x_1),$$

$$\frac{u_{2,0}^{n+1}-u_{2,0}^{n}}{\Delta t}+{+}_{11}\frac{u_{2,1}^{n+1}-u_{2,1}^{n}}{\Delta t}+{+}_{22}\frac{u_{2,3}^{n+1}-u_{2,3}^{n}}{\Delta t}$$
$$=\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i}u_{2,i}^{n+1}N_{i,k}^{(\alpha)}x_1^{k-\alpha}+\sum_{i=0}^{3}u_{2,i}^{n+1}V_i^*(x_1)\left(\sum_{i=0}^{3}v_{2,i}^{n+1}V_i^*(x_1)\right)^2, \tag{56}$$

$$\frac{v_{2,0}^{n+1}-v_{2,0}^{n}}{\Delta t}+{+}_{11}\frac{v_{2,1}^{n+1}-v_{2,1}^{n}}{\Delta t}+{+}_{22}\frac{v_{2,3}^{n+1}-v_{2,3}^{n}}{\Delta t}$$
$$=\sum_{i=\lceil\alpha\rceil}^{3}\sum_{k=\lceil\alpha\rceil}^{i}v_{2,i}^{n+1}N_{i,k}^{(\alpha)}x_1^{k-\alpha}+\sum_{i=0}^{3}u_{2,i}^{n+1}V_i^*(x_1)\left(\sum_{i=0}^{3}v_{2,i}^{n+1}V_i^*(x_1)\right)^2 \tag{57}$$
$$+\gamma\left(\sum_{i=0}^{3}v_{1,i}^{n+1}V_i^*(x_1)-\sum_{i=0}^{3}v_{2,i}^{n+1}V_i^*(x_1)\right),$$

$$u_{1,0}^{n+1}-u_{1,1}^{n+1}+u_{1,2}^{n+1}-u_{1,3}^{n+1}=1, \tag{58}$$

$$u_{1,0}^{n+1}+u_{1,1}^{n+1}+u_{1,2}^{n+1}+u_{1,3}^{n+1}=1, \tag{59}$$

$$u_{2,0}^{n+1}-u_{2,1}^{n+1}-u_{2,2}^{n+1}-u_{2,3}^{n+1}=1, \tag{60}$$

$$u_{2,0}^{n+1}+u_{2,1}^{n+1}+u_{2,2}^{n+1}+u_{2,3}^{n+1}=1, \tag{61}$$

$$v_{1,0}^{n+1}-v_{1,1}^{n+1}+v_{1,2}^{n+1}-v_{1,3}^{n+1}=0, \tag{62}$$

$$v_{1,0}^{n+1}+v_{1,1}^{n+1}+v_{1,2}^{n+1}+v_{1,3}^{n+1}=0, \tag{63}$$

$$v_{2,0}^{n+1}-v_{2,1}^{n+1}+v_{2,2}^{n+1}-v_{2,3}^{n+1}=0, \tag{64}$$

$$v_{2,0}^{n+1}+v_{2,1}^{n+1}+v_{2,2}^{n+1}+v_{2,3}^{n+1}=0. \tag{65}$$

This system presents the numerical scheme of the proposed problem (9) and represents non-linear system of algebraic equations. Solving this system using the Newton iteration method yields the

numerical solution of the coupled non-linear fractional diffusion equations (9).
At $n = 0$, we will evaluate the values of

$$\begin{bmatrix} u_{1,0}^{0} & u_{1,1}^{0} & u_{1,2}^{0} & u_{1,3}^{0} \\ u_{2,0}^{0} & u_{2,1}^{0} & u_{2,2}^{0} & u_{2,3}^{0} \\ v_{1,0}^{0} & v_{1,1}^{0} & v_{1,2}^{0} & v_{1,3}^{0} \\ v_{2,0}^{0} & v_{2,1}^{0} & v_{2,2}^{0} & v_{2,3}^{0} \end{bmatrix}$$

using the initial conditions (23). Therefore, we can obtain the solutions (for $n = 1, 2, ..., N$)

$$\begin{bmatrix} u_{1,0}^{n} & u_{1,1}^{n} & u_{1,2}^{n} & u_{1,3}^{n} \\ u_{2,0}^{n} & u_{2,1}^{n} & u_{2,2}^{n} & u_{2,3}^{n} \\ v_{1,0}^{n} & v_{1,1}^{n} & v_{1,2}^{n} & v_{1,3}^{n} \\ v_{2,0}^{n} & v_{2,1}^{n} & v_{2,2}^{n} & v_{2,3}^{n} \end{bmatrix}$$

Using the numerical scheme (50)-(65).

Numerical results

In this section, we implement the proposed method to solve the coupled non-linear system of fractional diffusion equations (9) with the constants $\gamma = 0.1, k = 0.2$ and $a = 100$. The obtained approximate solutions by means of the proposed method are shown in figures 1-5. Where in figures 1-3, we presented the behavior of the approximate solution with different values of $\alpha(\alpha = 2, 1.8,$ and 1.6, respectively), with time step $\Delta t = 0.05,$ and $m = 5$ with final time $t = 2$.

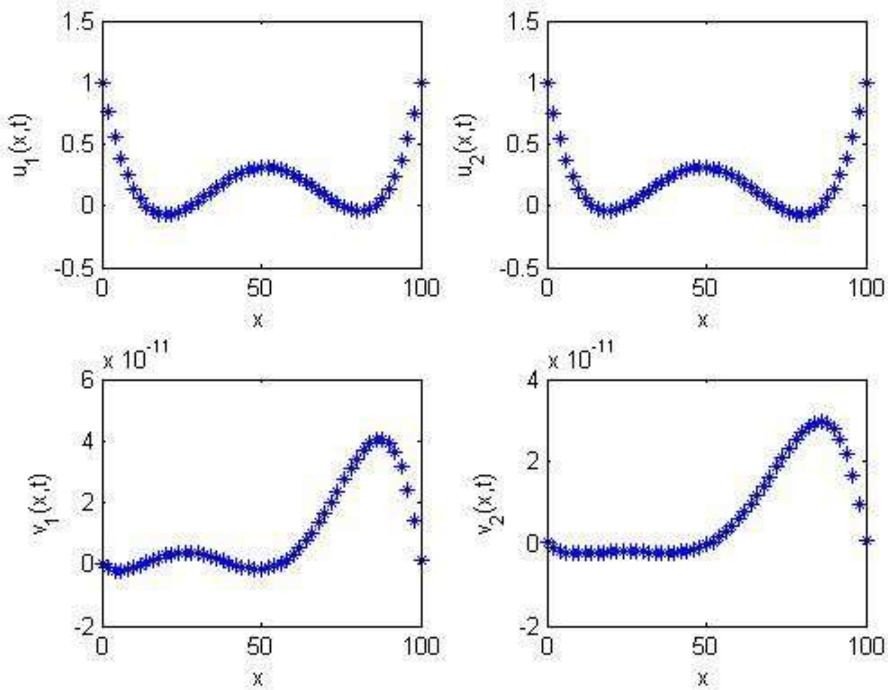

Figure 1. The behavior of the approximate solution at $t = 2$ for (9) with $\alpha = 2$, $\gamma = 0.1$, $k = 0.2$ and $a = 100$.

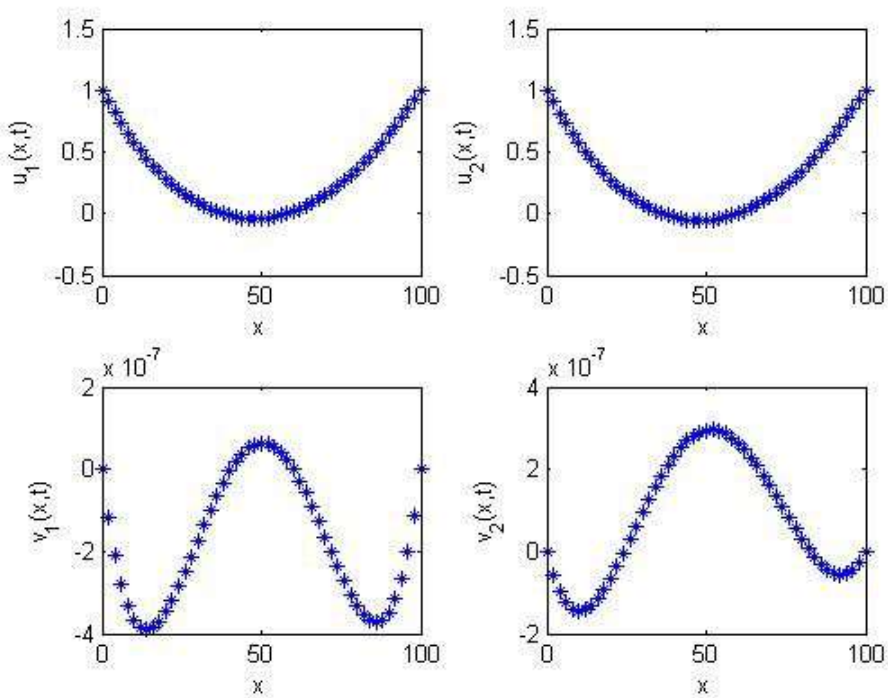

Figure 2. The behavior of the approximate solution at $t = 2$ for (9) with $\alpha = 1.8$, $\gamma = 0.1$, $k = 0.2$ and $a = 100$.

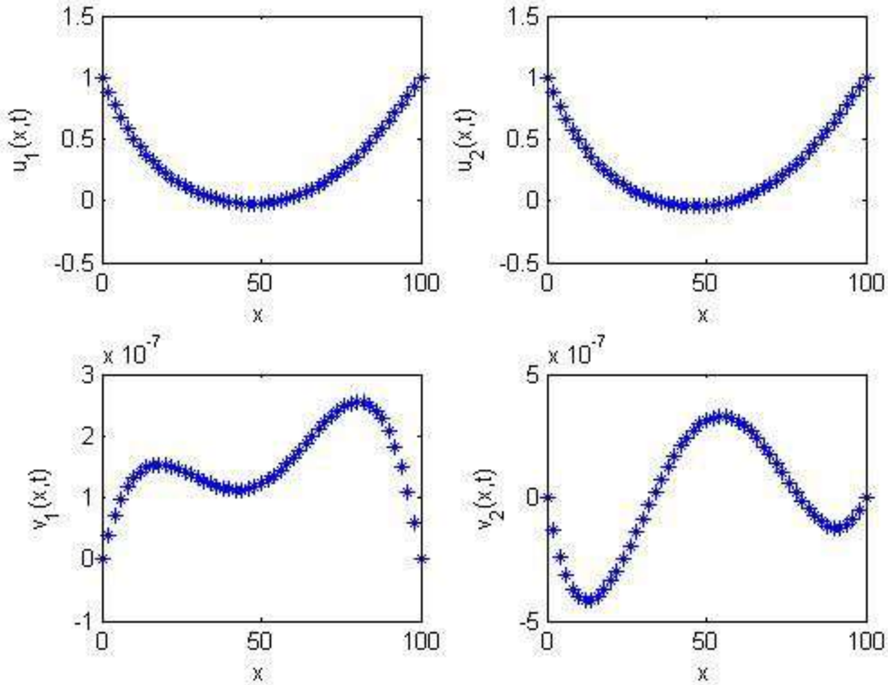

Figure 3. The behavior of the approximate solution at $t = 2$ for (9) with $\alpha = 1.6$, $\gamma = 0.1$, $k = 0.2$ and $a = 100$.

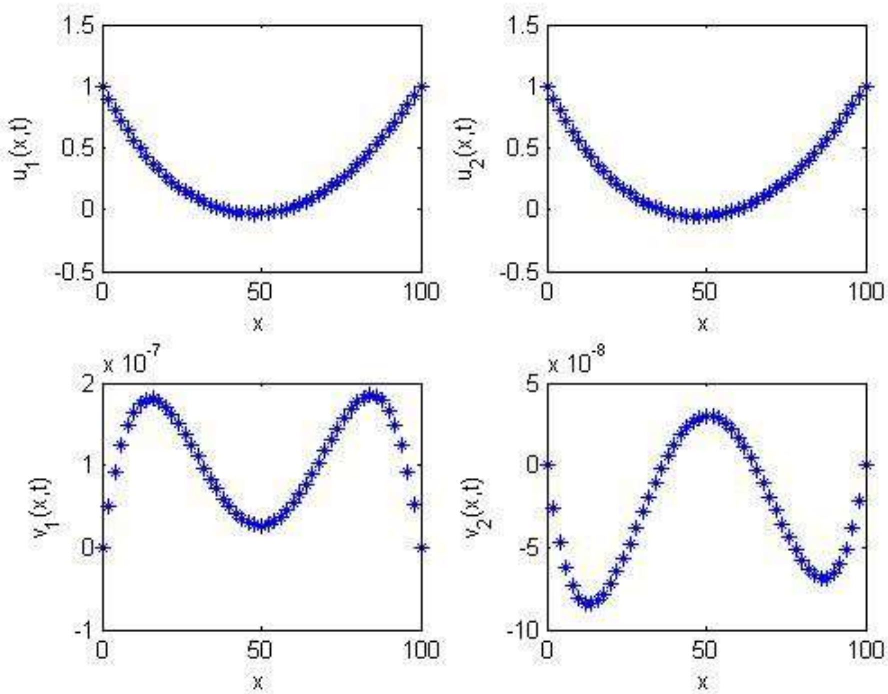

Figure 4. The behavior of the approximate solution at $t = 5$ for (9) with $\alpha = 1.75$, $\gamma = 0.1$, $k = 0.2$ and $a = 100$.

Also, in figures 4-5, we presented the behavior of the approximate solution with different values of t(t = 4 and 5, respectively), with time step $\Delta t = 0.05$, and $m = 5$ with $\alpha = 1.75$. From these figures, it seen that u_1, u_2, v_1 and v_2 decrease with increase in t and with decrease in the fractional values of α. This confirms the physical behavior of the proposed system.

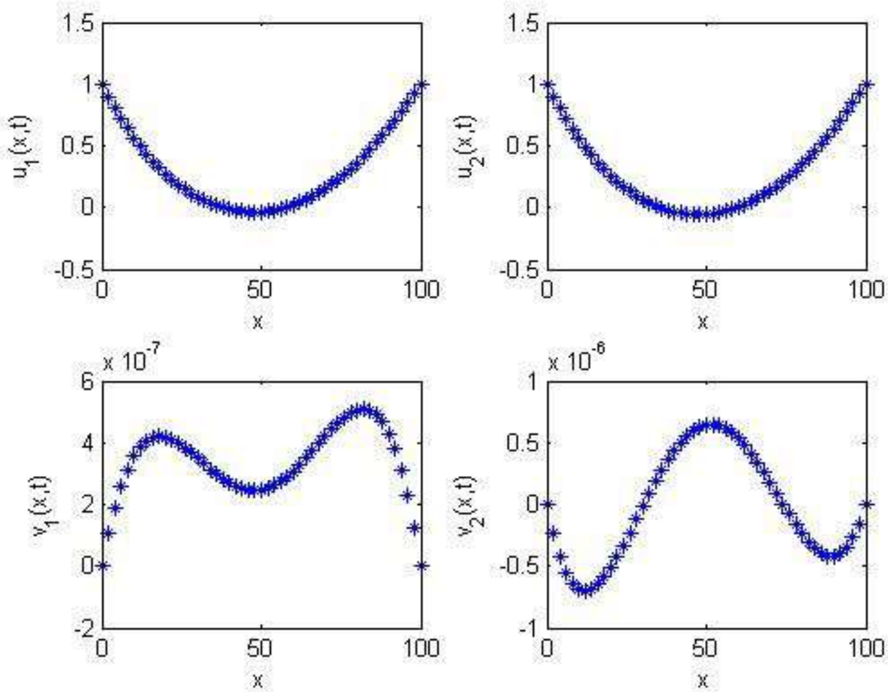

Figure 5. The behavior of the approximate solution at $t = 7$ for (9) with $\alpha = 1.75$, $\gamma = 0.1$, $k = 0.2$ and $a = 100$.

6. Conclusion and remarks

In this article, the shifted Chebyshev polynomials of the third kind method is implemented for solving the system of coupled non-linear fractional diffusion equations. The fractional derivative is considered in the Caputo sense. The properties of the shifted Chebyshev polynomials of the third kind are used to reduce the proposed problem to the solution of a system of ODEs which is solved by using FDM. Special attention is given to study the convergence analysis and to estimate an upper bound of the error of the derived formula and the approximate solution. From the behavior of the obtained numerical solutions using the suggested method we can see that the physical behavior of the proposed system is confirmed. So, we can show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms in the series (24). All computations in this paper are done using Matlab 12b.

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