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SS-Injective Modules and Related Concepts

A thesis

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بسم الله الرحمن الرحيم

﴿يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات
والله بما تعملون خبير﴾

صدق الله العظيم

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Publications

1. Some results from chapter two were published in the Al-Qadisiyah Journal of computer science and mathematics, Vol (8), No (2), 2016, under the title: SS-Flat Modules.

2. Some results from chapters two and three will be a part of the preprint "SS-Injective Modules and Rings".

Abstract

In this work, we introduce and study the concept of ss-injective modules as a generalization of both soc-injective and small injective modules. Also, we introduce and study the concept of ss-flat modules as a dual notion of ss-injective modules. The notion strongly ss-injective modules is defined by using ss-injectivity as a generalization of strongly soc-injective modules. Various characterizations of these modules and rings are given. By using ss-injectivity, we provide many other new characterizations of semisimple rings, quasi-Frobenius rings, Artinian rings and universally mininjective rings. Several results in the literature are improved and extended by some results of this thesis.

List of Symbols

Symbol	Description
R	an associative ring with identity
${}_RM$ (resp. M_R)	a left (resp. right) R-module M
Mod-R	the class of right R-modules
$A \subseteq B$	A is a subset of B
$A \subsetneq B$	A is a proper subset of B
$N \hookrightarrow M$	N is a submodule of M
$N \subseteq^{ess} M$	N is an essential submodule of M
$N \subseteq^{max} M$	N is a maximal submodule of M
$N \subseteq^{\oplus} M$	N is a direct summand of M
$N \ll M$	N is a small submodule of M
fg	the composition function of g and f
$\prod_{i \in I} M_i$	the direct product of the modules M_i
$\bigoplus_{i \in I} M_i$	the direct sum of the modules M_i
M^I	the direct product of I copies of M
$M^{(I)}$	the direct sum of I copies of M
$E(M)$	the injective hull of M
$\mathrm{soc}(M)$	the socle of a module M
S_r (resp. S_ℓ)	the right (resp. left) socle of a ring R
S_2^r	the right second socle of a ring R
$J(M)$	the Jacobson radical of a module M
J	the Jacobson radical of a ring R
$Z(M)$	the singular submodule of a module M
Z_r (resp. Z_ℓ)	the right (resp. left) singular ideal of a ring R
$Z_2(M)$	the second singular submodule of a module M
Z_2^r	the right second singular ideal of a ring R
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the ring of integer numbers
$\mathbb{Q}$	the group of all rational numbers
$\mathbb{Z}_n$	the ring of all integers modulo n
$\mathbb{Z}_{p^\infty}$	the Prüfer group at the prime p
$\mathrm{Hom}_R(M, N)$	the group of all homomorphisms from M into N
$\mathrm{End}(M)$	the ring of all endomorphisms of M
$\ker(\alpha)$ (resp. $\mathrm{im}(\alpha)$)	the kernel (resp. image) of a homomorphism α

Symbol	Description
$\text{coker}(\alpha)$	the cokernel of a homomorphism α
I_M	the identity homomorphism $I_M: M \longrightarrow M$
f^*	the map $f^*: \text{Hom}_R(B, D) \longrightarrow \text{Hom}_R(A, D)$ which is defined by $f^*(g) = gf$, where $A, B, D \in$ Mod-R and $f: A \longrightarrow B$ is a right R-homomorphism
M^+	the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$
M^d	the dual module $\text{Hom}_R(M, R)$
$\lim\limits_{\longrightarrow} M_i$	the direct limit of the direct system of modules M_i
$M \otimes_R N$	the tensor product over R of M_R and ${}_RN$
$\text{Ext}^n(-,-)$	the nth cohomology group derived from $\text{Hom}_R(-,-)$ using left projective resolutions
$\text{Tor}_n(-,-)$	the nth homology group derived from the tensor product over R using left projective resolutions
SSI	the class of all ss-injective right R-modules
SSF	the class of all ss-flat left R-modules
$r_R(X)$	the set $\{r \in R \mid xr = 0$, for all $x \in X\}$, where X is a subset of a right R-module M
$l_R(X)$	the set $\{r \in R \mid rx = 0$, for all $x \in X\}$, where X is a subset of a right R-module M
$r(X)$ (resp. $l(X)$)	$r(X) = r_R(X)$ (resp. $l(X) = l_R(X)$), where X is a subset of R
$l_M(X)$	the set $\{m \in M \mid mr = 0$, for all $r \in X\}$, where X is a subset of R

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Introduction

Throughout this thesis, R stands for an associative ring with identity and all modules are unitary. A right R-module M is said to be injective, if for every right R-monomorphism $f: N \longrightarrow K$ (where N and K are right R-modules) and every right R-homomorphism $g: N \longrightarrow M$, there exists a right R-homomorphism $h: K \longrightarrow M$ such that $g = hf$ [6]. Injective module has been studied widely, and various generalizations for this module were given, for examples, principally injective module [24], quasi-injective module [24], s-injective module [36], and mininjective module [23].

I. Amin, M. Yousif and N. Zeyada in [2] introduced soc-injective modules as a generalization of injective modules. A right R-module M is called soc-N-injective (where N is a right R-module) if every R-homomorphism $f: K \longrightarrow M$ extends to N, where K is a semisimple submodule of N. A module M is called soc-injective, if M is soc-R-injective and it is called strongly soc-injective, if M is soc-N-injective for all $N \in$ Mod-R. In [27, 30], the notion of small injectivity was discussed as a generalization of injective modules, a right R-module M is called small injective if every R-homomorphism from a small right ideal of R into M can be extended from R_R to M. Also, in [35], self-injective rings were generalized by the concept simple J-injective rings. A ring R is called right simple J-injective if, for any small right ideal I and any R-homomorphism $f: I_R \longrightarrow R_R$ with simple image can be extended to R.

In this thesis, we introduce the concept of ss-injective modules as a generalization of both soc-injective modules and small injective modules. A right R-module M is said to be ss-N-injective (where N is a right R-module), if every right R-homomorphism $f: K \longrightarrow M$ has an extension from N into M for every semisimple small submodule K of N. A module M is said to be ss-injective (resp. strongly ss-injective) if it is ss-R-injective (resp. ss-N-injective, for all $N \in$ Mod-R). A ring R is said to be right ss-injective (resp. strongly right ss-injective), if a right R-module R is ss-injective (resp. strongly ss-injective).

Also, we introduce the notion of ss-flat modules as a dual concept of ss-injective modules. A left R-module M is said to be ss-flat, if $\mathrm{Tor}_1^R(R/(S_r \cap J), M) = 0$. Min-flat modules were introduced in [19] as the dual concept of the mininjective modules. The concept ss-flatness is stronger than min-flatness and weaker than flatness.

This thesis consists of three chapters. In chapter one, we state some basic concepts and results which are related to our work.

Chapter two is divided into three sections. In section 1, we introduce the concept of ss-N-injective modules as a generalization of soc-N-injective modules (where N is a right R-module); specially, the concept of ss-R-injective (briefly, ss-injective) modules is a generalization of soc-injective and small injective module. We give examples to show that the ss-injectivity is distinct from that soc-injectivity, small injectivity, principally injectivity, and mininjectivity (see Example 2.1.2). Some elementary properties of ss-injective modules are given, for example, we show that the class of ss-injective right R-modules is closed under direct products, finite direct sums, summands, and isomorphisms. Also we prove that if $M \in$ Mod-R, then $\mathrm{Soc}(M) \cap J(M) = 0$ if and only if every simple submodule of M is ss-M-injective. Thus, we obtain that a ring R is right universally mininjective if and only if every simple right ideal is ss-injective. Many results are provided in terms of ss-injectivity. For example, if M is a projective right R-module, then every quotient of an ss-M-injective right R-module is ss-M-injective if and only if every sum of two ss-M-injective submodules of a right R-module is ss-M-injective if and only if $\mathrm{soc}(M) \cap J(M)$ is projective. In Theorem 2.1.20, we prove that if every simple singular right R-module is ss-injective, then $r(a) \subseteq^{\oplus} R_R$ for every $a \in S_r \cap J$ and S_r is projective. By using Lemma 2.1.31, we have establish the connection between ss-injectivity and other injectivities (see Corollary 2.1.32).

In section 2, we study ss-injective rings and give some properties and characterizations of its. For example, in Proposition 2.2.1, we state some characterizations of ss-injective rings. The results [2, Proposition 4.6 and Theorem 4.12] have been improved by ss-injectivity (see Proposition 2.2.14 and Corollary 2.2.15). We also give an example to show that the result [38, Theorem 2.12] is not true and we rewrite [38, Theorem 2.12] in a correct version by using ss-injectivity.

We introduce in section 3 the concept of ss-flat modules as a dual of ss-injective modules. We show that the three classes of modules: ss-flat modules, min-flat modules, and flat modules are different (see Example 2.3.2). We also show that the classes of all ss-flat left R-modules and all ss-injective right R-modules are form an example of almost dual pair (see Corollary 2.3.4). The notion min-coherent rings was introduced by L. Mao [19]. In this section, we generalize the concepts of coherent rings to ss-coherent rings. A ring R is said to be right ss-coherent ring if it is a right min-coherent and $S_r \cap J$ is a finitely generated. Some characterizations of ss-coherent and min-coherent rings are given (see Theorem 2.3.9, Theorem 2.3.10, and Corollary 2.3.11). For a right min-coherent ring R, we prove that every ss-flat left R-module is flat if and only if every ss-injective right R-module is FP-injective if and only if every ss-injective pure injective right R-module is injective. Some equivalence statements of ss-injective ring are given by using ss-flatness and ss-injectivity; for example, if R is a right ss-coherent, then R is a right ss-injective ring if and only if every left R-module has a monic ss-flat preenvelope if and only if every right R-module has epic ss-injective cover if and only if every injective left R-module is ss-flat if and only if every flat right R-module is ss-injective.

Chapter three is divided into two sections. In section 1, examples are given to distinguish strongly ss-injectivity from ss-injectivity and strongly soc-injectivity (see Example 3.1.2). We prove that a right R-module M is strongly ss-injective if and only if every small submodule A of any right R-module N, every R-homomorphism $\alpha: A \longrightarrow M$ with semisimple image extends to N, for all $N \in$ Mod-R. In particular, if every simple right R-module is strongly ss-injective, then R is a semiprimitive ring, but not conversely (see Example 3.1.11). If R is a right perfect ring, we show that the class of strongly ss-injective right R-modules and the class of strongly soc-injective right R-modules are equal (see Theorem 3.1.13). Also, we extend the results ([2, Theorem 3.3, Theorem 3.6 and Proposition 3.7]) by the following results: a ring R is right Artinian if and only if any direct sum of strongly ss-injective right R-modules is injective, and a ring R is QF ring if and only if every strongly ss-injective right R-module is projective.

In section 2, we show that every strongly ss-injective ring is right simple J-injective ring, but not conversely (see Example 3.2.8). Also, we prove that a ring R is QF if and only if R is strongly ss-injective and right noetherian with essential right socle. I. Amin, M. Yousif and N. Zeyada [2] proved that if a ring R is left perfect and two-sided strongly soc-injective, then R is QF ring. We extend their result by Corollary 3.2.15. Finally, we improve Corollary 3.2.15 as follows: a ring R is QF if and only if R is two-sided strongly ss-injective, left Kasch, and J is left t-nilpotent (see Theorem 3.2.16).

Chapter one

Section one

1.1 Preliminaries

In this section, we recall some basic concepts of modules and rings and we state some of their properties and results which are related to our work.

Definition 1.1.1 [16, p. 106]. A submodule N of a right R-module M is said to be small (resp. essential) in M, notationally $N \ll M$ (resp. $N \subseteq^{ess} M$), if every submodule K of M with $K + N = M$ (resp. with $K \cap N = 0$) implies $K = M$ (resp. implies $K = 0$).

Lemma 1.1.2 [16, Lemma 5.1.3 and 5.1.5, p. 108 and 109]. The following holds for right R-modules M and N:

(1) If $A \hookrightarrow B \hookrightarrow M \hookrightarrow N$ and $B \ll M$, then $A \ll N$.
(2) If $A_i \ll M$, $i = 1, 2, \dots, n$, then $\sum_{i=1}^{n} A_i \ll M$.
(3) If $A \ll M$ and $\alpha \in \mathrm{Hom}_R(M, N)$, then $\alpha(A) \ll N$.
(4) If $A \hookrightarrow B \hookrightarrow M \hookrightarrow N$ and $A \subseteq^{ess} N$, then $B \subseteq^{ess} M$.
(5) If $A \subseteq^{ess} N$ and $\alpha \in \mathrm{Hom}_R(M, N)$, then $\alpha^{-1}(A) \subseteq^{ess} M$.

Definition 1.1.3 [16, p. 107]. A right R-module M is called semisimple if every submodule of M is a direct summand. A ring R is called semisimple if R is semisimple as right (or left) R-module; equivalently, if every right (or left) R-module is semisimple.

Example 1.1.4.

(1) $< 2 >$ is small submodule in the $\mathbb{Z}_8$-module $\mathbb{Z}_8$, but not semisimple $\mathbb{Z}_8$-module.
(2) $< 2 >$ is simple right ideal of $\mathbb{Z}_6$, but not small submodule of the $\mathbb{Z}$-module $\mathbb{Z}_6$.
(3) The $\mathbb{Z}$-module $< 2 > \oplus < 2 >$ is semisimple small submodule of the $\mathbb{Z}$-module $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, but not simple submodule.

Definition 1.1.5 [6, p. 16 and 27] and [16, p. 144]. Let X be a subset of a right R-module M, the right (resp. left) annihilator of X in R is defined by $r_R(X) =$

$\{r \in R|\ xr = 0$, for all $x \in X\}$ (resp. $l_R(X) = \{r \in R|\ rx = 0$, for all $x \in X\}$. If $M = R$, we write $r_R(X) = r(X)$ and $l_R(X) = l(X)$. Similarly, we define $l_S(X)$ where $S = \text{End}(M)$.

Definition 1.1.6 [16, p. 214]. The Jacobson radical (resp. the socle) of a right R-module M is denoted by $J(M)$ (resp. $\text{soc}(M)$) and defined by $J(M) = \sum_{A \ll M} A = \bigcap_{B \subseteq max_M} B$ (resp. $\text{soc}(M) = \sum_{\substack{A \text{ is simple} \\ in\ M}} A = \bigcap_{B \subseteq ess_M} B$).

Remark 1.1.7. Let M and N be right R-modules, then:

(1) If $\alpha \in \text{Hom}_R(M, N)$, then $\alpha\big(J(M)\big) \hookrightarrow J(N)$ and $\alpha\big(\text{soc}(M)\big) \hookrightarrow \text{soc}(N)$ (see [16, Theorem 9.1.4, p. 214]).
(2) Let $m \in M$, we have $mR \ll M$ if and only if $m \in J(M)$ (see [16, Corollary 9.1.3, p. 214]).
(3) If M is finitely generated, then $J(M) \ll M$ (see [16, Theorem 9.2.1, p.218]).
(4) $MJ \hookrightarrow J(M)$ (see [16, Theorem 9.2.1, p. 218]).
(5) If $N \hookrightarrow M$ and $J(M/N) = 0$, then $J(M) \hookrightarrow N$ (see [16, Theorem 9.1.4, p. 214]).

Definition 1.1.8 [25, p. 96]. A ring R is said to be local if R/J is a division ring; equivalently, J is a maximal right (or left) ideal of R.

Definition 1.1.9 [6, p. 13 and 242]. An element $e \in R$ is said to be idempotent if $e^2 = e$. The idempotents e and f are called orthogonal if $ef = fe = 0$. An idempotent e is called local if eRe is a local ring.

Definition 1.1.10 [17, p. 246]. Let M be a right R-module. An element $m \in M$ is called singular element of M if $r_R(m) \subseteq^{ess} R_R$. The set of all singular elements of M is denoted by $Z(M)$. We say M is singular (resp. nonsingular) if $Z(M) = M$ (resp. $Z(M) = 0$).

Definition 1.1.11 [4, 36]. The second singular submodule of a right R-module M is denoted by $Z_2(M)$ and defined by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$. The second singular right ideal of R is defined by $Z_2^r = Z_2(R_R)$.

Definition 1.1.12 [5, p. 15]. A functor ρ: Mod-R ⟶Mod-R is said to be preradical if the following hold:

(1) $\rho(M) \hookrightarrow M$ for all $M \in$ Mod-R.
(2) For any R-homomorphism $\alpha: M \longrightarrow N$ we have that $\alpha\big(\rho(M)\big) \subseteq \rho(N)$.

Definition 1.1.13 [3, p. 175]. A ring R is called Von Neumann regular if $a \in aRa$, for every $a \in R$.

Definition 1.1.14 [22]. A right ideal I of R is said to be lie over a summand of R_R, if there exists a direct decomposition $R_R = A_R \oplus B_R$ with $A \hookrightarrow I$ and $B \cap I \ll R_R$ (it is clear that $I = A \oplus (B \cap I)$).

Definition 1.1.15 [22]. A ring R is said to be semiregular if every finitely generated right ideal of R lies over a direct summand of R_R.

Definition 1.1.16 [18]. A right R-module M is called semilocal if $M/J(M)$ is semisimple. A ring R is said to be semilocal if R_R (or ${}_R R$) is a semilocal R-module.

Definition 1.1.17 [29, p. 187]. A ring R is said to be a semiperfect ring if it satisfies the following conditions:

(1) R is semilocal ring.
(2) Idempotents can be lifted modulo J (i.e.,if for every idempotent $f \in R/J$, there exists idempotent $e \in R$ such that $e + J = f$).

Definition 1.1.18 [24, p. 152]. A ring R is called right semiartinian if every nonzero right R-module has a nonzero socle.

Recall that a subset K of a ring R is said to be left t-nilpotent if for each sequence $a_1, a_2, a_3, \ldots$ of elements of K, $a_1 a_2 \ldots a_n = 0$ for some $n \in \mathbb{N}$ (see [6, p. 239]).

Proposition 1.1.19 [29, Proposition VIII.5.1,p.189]. The following statements are equivalent for a ring R:

(1) R is right semiartinian and semilocal.
(2) R is right semiartinian and semiperfect.
(3) J is left t-nilpotent and R is semilocal.

Definition 1.1.20 [29, p. 189]. A ring R is said to be left perfect if it satisfies the conditions of Proposition 1.1.19.

Definition 1.1.21 [14, p. 68]. A ring R is called right max if every nonzero right R-module has a maximal submodule.

Theorem 1.1.22 [14, Theorem 4.4, p. 69]. A ring R is right max if and only if $J(M) \ll M$ for every nonzero right R-module M.

Definition 1.1.23 [29, p. 12]. A right R-module M is called noetherian if every submodule of M is finitely generated; equivalently, if every strictly ascending chain of submodules (ascending chain condition; briefly, ACC) is finite. A ring R is said to be right noetherian if it is noetherian as right R-module.

Definition 1.1.24 [29, p. 13]. A right R-module M is called artinian if every strictly descending chain of submodules (descending chain condition; briefly, DCC) is finite. A ring R is said to be right artinian if it is artinian as right R-module.

Remark 1.1.25 [16, p. 274]. Every right (or left) artinian ring is right perfect.

Corollary 1.1.26 [16, Corollary 9.3.12, p. 225]. If R is a right artinian ring, then R is a right noetherian.

Definition 1.1.27 [17, p. 189]. A ring R is called right Kasch if every simple right R-module is isomorphic to a simple right ideal of R.

Definition 1.1.28 [24, p. 49]. The dual of right R-module M is $M^d = \mathrm{Hom}_R(M, R_R)$ which is a left R-module via $(r\alpha)(m) = r \cdot \alpha(m)$ for all $r \in R$, $\alpha \in M^d$, and $m \in M$.

Definition 1.1.29 [6, p. 155]. The character module of a right R-module M is $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ which is a left R-module via $(r\alpha)(m) = \alpha(mr)$ for all $r \in R$, $\alpha \in M^+$, and $m \in M$.

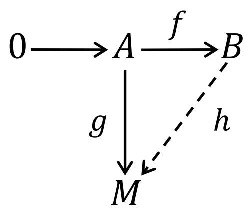
Definition 1.1.30 [23]. A ring R is called right minannihilator if $rl(K) = K$ for every simple right ideal K.

Section Two

1.2 Injective Modules And Some Related Concepts

In this section, we recall the definitions of injective module, flat module, concepts in homological algebra, and some special rings and we list some of their characterizations and properties which are relevant to our work.

Definition 1.2.1 [6, p. 135]. A right R-module M is said to be injective, if every diagram with exact row:



of right R-modules and right R-homomorphisms can be completed commutatively by an R-homomorphism $h: B \longrightarrow M$. Equivalently, we can assume that A is a submodule of B and f can be replaced by the inclusion map $i: A \longrightarrow B$. A ring R is called right self-injective, if R_R is injective.

Example 1.2.2 [6, p. 136].

(1) The $\mathbb{Z}$-module $< 2 > \subseteq \mathbb{Z}$ is not injective.
(2) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Proposition 1.2.3 [6] and [16]. Let M and N be a right R-modules, then:

(1) $\prod_{i \in I} M_i$ is an injective right R-module if and only if each M_i is injective.
(2) A finite direct sum of injective right R-modules is injective.
(3) If N is injective and $N \hookrightarrow M$, then $N \subseteq^{\oplus} M$.
(4) If M injective and $N \subseteq^{\oplus} M$, then N is injective.
(5) If M is injective and $M \cong N$, then N is injective.

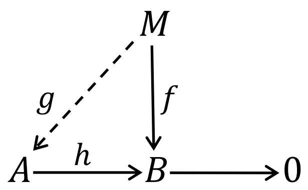
Definition 1.2.4 [21, p. 1]. Let M and N be R-modules. M is said to be N-injective if for every submodule K of N, any R-homomorphism $f: K \longrightarrow M$ can be extended to an R-homomorphism $g: N \longrightarrow M$.

Definition 1.2.5 [24, p. 11]. A right R-module M is said to be quasi-continuous if it satisfies the following conditions:

(1) ($C1$-condition) If every submodule of M is essential in a direct summand of M.
(2) ($C3$-condition) If $N \subseteq^{\oplus} M$, $K \subseteq^{\oplus} M$, and $N \cap K = 0$, then $N \oplus K \subseteq^{\oplus} M$.

Definition 1.2.6 [24, p. 96]. A right R-module M is called right principally injective, if every R-homomorphism $f: aR \longrightarrow M$, $a \in R$, extends to R.

Definition 1.2.7 [6, p. 144]. A right R-module M is said to be projective if each diagram with exact row:



of right R-modules and R-homomorphisms can be completed commutatively by an R-homomorphism $g: M \longrightarrow A$.

Theorem 1.2.8 [16, Theorem 5.3.1, p. 115]. A right R-module M is injective (resp. projective) if every R-monomorphism (resp. R-epimorphism) $\alpha: M \longrightarrow N$ (resp. $\beta: N \longrightarrow M$) is split.

Definition 1.2.9 [24, p.33]. A ring R is called right pseudo-Frobenius (or right PF) if R is right self-injective and semiperfect with $S_r \subseteq^{ess} R_R$.

Definition 1.2.10 [24, p. 20]. A ring R is called quasi-Frobenius (briefly, QF ring) if it is left and right self-injective artinian ring.

Proposition 1.2.11 [6, Proposition 12.5.13, p. 427]. A ring R is QF if and only if every injective right R-module is projective.

Definition 1.2.12 [34]. A ring R is said to be right PS-ring (resp. FS-ring) if S_r is projective (resp. flat).

Proposition 1.2.13 [34, Proposition 8]. Let R be a commutative ring. Then R is an FS-ring if and only if it is PS-ring.

Definition 1.2.14 [2]. Let M and N be right R-modules. M is called soc-N-injective if every R-homomorphism $f: K \longrightarrow M$ extends to N, where K is a semisimple submodule of N; equivalently, for any R-homomorphism $f: \text{soc}(N) \longrightarrow M$ extends to N. M is called soc-injective, if M is soc-R-injective. M is called strongly soc-injective, if M is soc-N-injective for all $N \in$ Mod-R. A ring R is called right soc-injective (resp. strongly right soc-injective), if R_R is soc-injective (resp. strongly soc-injective).

Definition 1.2.15 [27]. A right R-module M is called small injective if every R-homomorphism from a small right ideal of R into M can be extended from R_R to M. A ring R is called right small injective, if R_R is small injective.

Definition 1.2.16 [3, p. 169]. A ring R is called semiprimitive if $J = 0$.

Theorem 1.2.17 [30, Theorem 2.8]. Let R be a ring. Then the following statements are equivalent:

(1) R is semiprimitive.
(2) Every right (or left) R-module is small injective.
(3) Every simple right (or left) R-module is small injective.

Definition 1.2.18 [33]. A right R-module M is called principally small injective if every R-homomorphism from aR to M can be extended from R_R to M, for all $a \in J$. A ring R is called right principally small injective, if R_R is principally small injective.

Definition 1.2.19 [23]. A right R-module M is called mininjective if, for each simple right ideal K of R, every R-homomorphism $f: K \longrightarrow M$ extends to R. A ring R is called right mininjective, if R_R is mininjective.

Definition 1.2.20 [23]. A ring R is called right universally mininjective if $S_r \cap J = 0$.

Definition 1.2.21 [24, p. 68]. A ring R is called right min-PF, if it is a semiperfect, right mininjective, $S_r \subseteq^{ess} R_R$, and $lr(K) = K$ for every simple left ideal $K \subseteq Re$ for some local idempotent $e \in R$.

Definition 1.2.22 [24, p. 62]. A ring R is said to be right minfull if it is semiperfect, right mininjective and $\mathrm{soc}(eR) \neq 0$ for each local idempotent $e \in R$.

Definition 1.2.23 [35]. A ring R is called right simple J-injective if, for any small right ideal I and any R-homomorphism $f: I_R \longrightarrow R_R$ with simple image can be extended to R.

Definition 1.2.24 [36]. A right R-module M is called strongly s-injective if every R-homomorphism from K to M extends to N, for every right R-module N, where $K \subseteq Z(N)$.

Definition 1.2.25 [12, p. 129]. Let R be a ring and $\mathcal{F}$ be a class of right R-modules. An R-homomorphism $f: M \longrightarrow N$ is said to be $\mathcal{F}$-preenvelope of M where $N \in \mathcal{F}$ if, for every R-homomorphism $g: M \longrightarrow F$ with $F \in \mathcal{F}$, there is $h: N \longrightarrow F$ such that $hf = g$. If every $h \in \mathrm{End}(N)$ such that $hf = f$ is an isomorphism, then f is called an $\mathcal{F}$-envelope of M.

Definition 1.2.26 [12, p. 105]. Let R be a ring and $\mathcal{F}$ be a class of right R-modules. An R-homomorphism $f: N \longrightarrow M$ is said to be $\mathcal{F}$-precover of M where $N \in \mathcal{F}$ if, for every R-homomorphism $g: L \longrightarrow M$ with $L \in \mathcal{F}$, there is $h: L \longrightarrow N$ such that $fh = g$. If every $h \in \mathrm{End}(N)$ such that $fh = f$ is an isomorphism, then f is called an $\mathcal{F}$-cover of M.

Definition 1.2.27 [20, Definition 4.2.1, p. 66]. Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a class of left (resp. right) R-modules. The pair ($\mathcal{F}$,$\mathcal{G}$) is said to be almost dual pair if for any left R-module M, $M \in \mathcal{F}$ if and only if $M^+ \in \mathcal{G}$; and $\mathcal{G}$ is closed under direct summands and direct products.

Remark 1.2.28 [12, Definition 2.1.1 and Remark 3.1.8, p. 40 and 70]. Let M be a right R-module, then:

(1) There is an exact sequence $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with each P_i projective. This sequence is called projective resolution of M.
(2) There is an exact sequence $0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$with each E_i injective. This sequence is called injective resolution of M.

Definition 1.2.29 [12, p. 25 and 26]. A sequence $C: \ \cdots \longrightarrow C_2 \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\alpha_0} C_{-1} \xrightarrow{\alpha_{-1}} C_{-2} \longrightarrow \cdots$ is called chain complex if $\alpha_{n-1}\alpha_n = 0$ for all $n \in \mathbb{Z}$. $\ker(\alpha_n)/\mathrm{im}(\alpha_{n+1})$ is called the nth homology module and is denoted by

$H_n(C)$. A chain complex of the form $D: \cdots \longrightarrow D_{-2} \longrightarrow D_{-1} \xrightarrow{\beta_{-1}} D_0 \xrightarrow{\beta_0} D_1 \xrightarrow{\beta_1} D_2 \longrightarrow \cdots$ is called a cochain complex. $\ker(\beta_n)/\mathrm{im}(\beta_{n-1})$ is called the nth cohomology module and is denoted by $H^n(D)$.

Definition 1.2.30 [12, p. 41 and 70]. Let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of right R-module M and consider the deleted projective resolution $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$. If N is a right R-module and L is a left R-module, then:

(1) The ith cohomology module of the complex sequence $0 \longrightarrow \mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \cdots$ is denoted by $\mathrm{Ext}_R^i(M, N)$ (briefly, $\mathrm{Ext}^i(M, N)$).

(2) The ith homology module of complex $\cdots \longrightarrow P_1 \otimes_R L \longrightarrow P_0 \otimes_R L \longrightarrow 0$ is denoted by $\mathrm{Tor}_i^R(M, L)$ (briefly, $\mathrm{Tor}_i(M, L)$).

Theorem 1.2.31[12, Theorem 3.2.1, p. 75]. Let R and S be rings and consider the situation $(A_R,\ {}_RB_S,\ C_S)$. If C is injective, then $\mathrm{Ext}^i\big(A, \mathrm{Hom}_S(B, C)\big) \cong \mathrm{Hom}_S(\mathrm{Tor}_i(A, B), C)$, for all $i \geq 0$.

Theorem 1.2.32 [12, Theorem 3.1.9, p. 70]. The following statements are equivalent for a right R-module E:

(1) E is injective.

(2) $\mathrm{Ext}^i(M, E) = 0$ for all right R-module M.

(3) $\mathrm{Ext}^1(M, E) = 0$ for all right R-module M.

Theorem 1.2.33 [26, Corollary 7.25, p. 421] and [13, Theorem XII.4.4, p. 491]. The following statements are equivalent for a right R-module P:

(1) P is projective.

(2) $\mathrm{Ext}^i(P, M) = 0$ for all right R-module M.

(3) $\mathrm{Ext}^1(P, M) = 0$ for all right R-module M.

Definition 1.2.34 [12, p. 40]. A left R-module M is said to be flat if given any exact sequence $0 \longrightarrow A \longrightarrow B$ of right R-modules, the tensored sequence $0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M$ is exact.

Example 1.2.35.

(1) Every projective R-module is flat (see [6, Examples (1), p. 155]).

(2) The $\mathbb{Z}$-module $\mathbb{Z}^{\mathbb{N}}$ is flat (since $\mathbb{Z}$ is coherent ring), but not projective (see [6, Examples (3), p. 145]).

Proposition 1.2.36 [26, Proposition 3.54, p. 136]. A left R-module M is flat if and only if M^+ is injective.

Theorem 1.2.37 [12, Theorem 2.1.8 and 3.2.10, p. 41 and 78]. The following statements are equivalent for a left R-module F:

(1) F is flat.
(2) $\text{Tor}_i(M, F) = 0$ for all right R-module M.
(3) $\text{Tor}_1(M, F) = 0$ for all right R-module M.
(4) $\text{Tor}_i(R/I, F) = 0$ for all right ideal I.
(5) $\text{Tor}_1(R/I, F) = 0$ for all right ideal I.

Definition 1.2.38 [6, p. 159]. A right R-module M is said to be finitely presented if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ of right R-modules, where F is finitely generated free and K is finitely generated; equivalently, if there is an exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, where F_1 and F_0 are finitely generated free right R-modules.

Definition 1.2.39 [17, p.138]. A ring R is said to be right coherent if every finitely generated right ideal of R is finitely presented.

It is clear that every right noetherian ring is right coherent.

Definition 1.2.40 [9]. A right R-module M is called n-presented if there is an exact sequence $F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ such that each F_i is a finitely generated free right R-modules.

Lemma 1.2.41[9, Lemma 2.7]. Let R and S be rings and consider the situation $(A_R, {}_SB_R, {}_SC)$ with A_R is n-presented and ${}_S\text{C}$ is injective, then $\text{Tor}_{n-1}(\text{A}, \text{Hom}_S(B, C)) \cong \text{Hom}_S(\text{Ext}^{n-1}(A, B), C)$.

Definition 1.2.42 [32, p. 197]. A direct system of R-modules $(M_i, f_{ij})_\Lambda$ consists of a family of right R-modules $\{M_i\}_\Lambda$ and a family of R-homomorphisms $f_{ij}: M_i \longrightarrow M_j$ with $i \leq j$ satisfying $f_{ii} = I_{M_i}$ and $f_{jk}f_{ij} = f_{ik}$ for $i \leq j \leq k$. A direct system of R-homomorphisms from $(M_i, f_{ij})_\Lambda$ into a right R-module M is a family of an R-homomorphisms $\{f_i: M_i \longrightarrow M\}_\Lambda$ with $f_j f_{ij} = f_i$ whenever $i \leq j$. A direct system of R-homomorphisms $\{f_i: M_i \longrightarrow M\}_\Lambda$ is said to be a direct limit of $(M_i, f_{ij})_\Lambda$ if, for every direct system of R-

homomorphisms $\{u_i: M_i \longrightarrow L\}_\Lambda$, $L \in$ Mod-R, there is a unique R-homomorphism $u: M \longrightarrow L$ such that $uf_i = u_i$ for all $i \in \Lambda$. The direct limit is unique and denoted by $M = \varinjlim M_i$.

Definition 1.2.43 [32, p. 274 and 278]. An exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of right R-modules is called pure if every finitely presented right R-module P is projective with respect to this sequence; equivalently, if the sequence $0 \longrightarrow \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C) \longrightarrow 0$ is exact. In this case we call $f(A)$ is a pure submodule of B. A right R-module M is called pure injective if M is injective with respect to every pure exact sequence.

Theorem 1.2.44 [32, 34.5, p. 286]. The exact sequence of right R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is pure if and only if the sequence $0 \longrightarrow C^+ \longrightarrow B^+ \longrightarrow A^+ \longrightarrow 0$ is split.

Theorem 1.2.45 [32, 33.7, p. 279]. A right R-module N is pure injective if and only if every pure sequence $0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$ is split.

Theorem 1.2.46 [15, Theorem 2.5]. Let $\mathcal{F}$ be a class of right R-modules. If $\mathcal{F}$ is closed under pure quotient, then the following statements are equivalent:

(1) $\mathcal{F}$ is closed under direct sums.
(2) $\mathcal{F}$ is precovering.
(3) $\mathcal{F}$ is covering.

Theorem 1.2.47 [32, 34.6, p. 289]. For every left R-module M, then:
(1) M is pure submodule of M^{++}.
(2) M^+ is pure injective.

Definition 1.2.48 [20, Definition 2.4.1, p. 29]. A subclass $\mathcal{F}$ of Mod-R is said to be definable if it is closed under direct products, direct limits and pure submodules.

Definition 1.2.49 [12]. A right R-module M is said to be FP-injective (or absolutely pure) if $\mathrm{Ext}^1(N, M) = 0$ for every finitely presented right R-module N.

Proposition 1.2.50 [10]. Let N be a finitely generated right R-module and $K \hookrightarrow N$. If every R-homomorphism $f: K \longrightarrow M$ extends to N for every FP-injective right R-module M, then K is finitely generated.

Remark 1.2.51 [10]. A right R-module P is finitely presented if and only if $\text{Ext}^1(P, M) = 0$ for all FP-injective right R-module M.

Definition 1.2.52 [19]. A left R-module M is said to be min-flat if $\text{Tor}_1(R/I\,, M) = 0$ for any simple right ideal I of R, equivalently, the sequence $0 \longrightarrow I \otimes_R M \longrightarrow R \otimes_R M$ is exact for any simple right ideal I of R.

Definition 1.2.53 [19]. A ring R is called right min-coherent if every simple right ideal of R is finitely presented.

It is clear that a ring R is right min-coherent if and only if every finitely generated semisimple small right ideal of R is finitely presented.

Lemma 1.2.54 [19, Lemma 3.2]. A left R-module M is min-flat if and only if M^+ is mininjective.

Chapter Two

Section One

2.1 SS-Injective Modules

As a generalization of both soc-injective modules and small injective modules, we will introduce in this section the concept of ss-injective modules and we will give some characterizations and properties of it.

Definition 2.1.1. Let N be a right R-module. A right R-module M is said to be ss-N-injective, if for any semisimple small submodule K of N, any right R-homomorphism $f: K \longrightarrow M$ extends to N. A module M is said to be ss-quasi-injective if M is ss-M-injective. M is said to be ss-injective if M is ss-R-injective. A ring R is said to be right ss-injective if the right R-module R_R is ss-injective.

Example 2.1.2.

(1) Every soc-injective module is ss-injective, but not conversely (see Example 3.2.9).
(2) Every small injective module is ss-injective, but not conversely (see Example 3.2.7).
(3) Every $\mathbb{Z}$-module is ss-injective. In fact, if M is a $\mathbb{Z}$-module, then M is small injective (by Theorem 1.2.17) and hence it is ss-injective.
(4) The two classes of principally small injective rings and ss-injective rings are different (see Example 2.2.4 and Example 3.2.7).

In the following theorem, we will give som basic properties of ss-N-injective modules.

Theorem 2.1.3. The following statements hold:

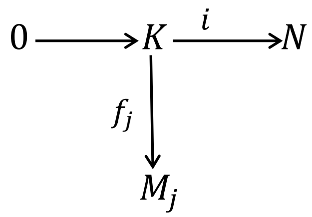
(1) Let N be a right R-module and let $\{M_i: i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i\in I} M_i$ is ss-N-injective if and only if each M_i is ss-N-injective, $i \in I$.
(2) Let M, N and K be right R-modules with $K \subseteq N$. If M is ss-N-injective, then M is ss-K-injective.

(3) Let M, N and K be right R-modules with $M \cong N$. If M is ss-K-injective, then N is ss-K-injective.

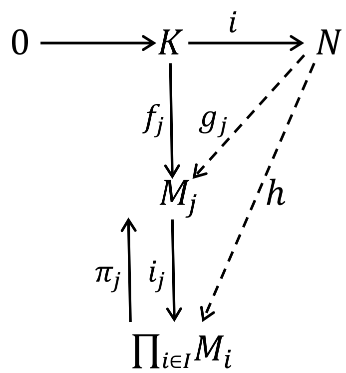
(4) Let M, N and K be right R-modules with $K \cong N$ and M is ss-K-injective. Then M is ss-N-injective.

(5) Let M, N and K be right R-modules with N is a direct summand of M. If M is ss-K-injective, then N is ss-K-injective.

Proof. (1) (⇒) Suppose that $\prod_{i\in I} M_i$ is ss-N-injective. Let $j \in I$ and consider the following diagram:

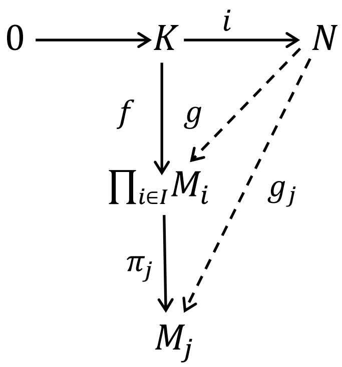


where K is a semisimple small submodule of N. Thus we have the following diagram:



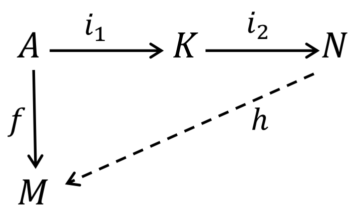
where $i_j: M_j \longrightarrow \prod_{i\in I} M_i$ and $\pi_j: \prod_{i\in I} M_i \longrightarrow M_j$ are the injection and projection right R-homomorphisms, respectively. Since $\prod_{i\in I} M_i$ is ss-N-injective by hypothesis, thus there exists a right R-homomorphism $h: N \longrightarrow \prod_{i\in I} M_i$ such that $hi = i_j f_j$. Put $g_j = \pi_j h: N \longrightarrow M_j$. Thus, we have $g_j i = (\pi_j h)i = \pi_j(i_j f_j) = f_j$. Hence M_j is an ss-N-injective R-module, for all $j \in I$.

(⇐) Suppose that for each $j \in I$, the right R-module M_j is an ss-N-injective. Consider the following diagram:



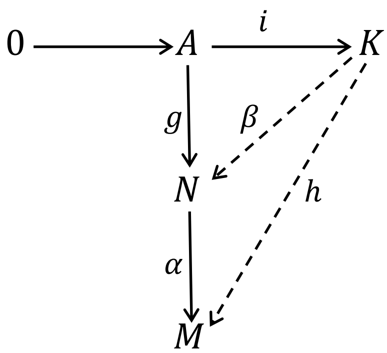
where K is a semisimple small submodule of N. For each $j \in I$, let $\pi_j: \prod_{i\in I} M_i \longrightarrow M_j$ be the projection R-homomorphism. Since each M_j is ss-N-injective, thus there is a right R-homomorphism $g_j: N \longrightarrow M_j$ such that $g_j i = \pi_j f$. Define $g: N \longrightarrow M$ by $g(b) = (g_j(b))_{j\in I}$ for every $b \in N$. It is clear that g is an R-homomorphism. For every $k \in K$, we have that $(gi)(k) = (g_j(i(k)))_{j\in I} = (\pi_j\big(f(k)\big)\,)_{j\in I} = f(k)$, so $\prod_{i\in I} M_i$ is an ss-N-injective.

(2) Suppose that M is an ss-N-injective R-module and let $K \hookrightarrow N$. Consider the following diagram:



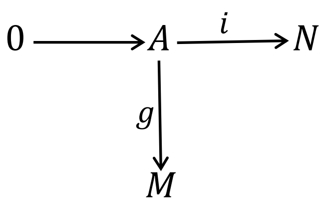
where A is a semisimple small submodule of K and i_1 and i_2 are the inclusion maps. Clearly, A is a semisimple small submodule of N and by hypothesis there exists right R-homomorphism $h: N \longrightarrow M$ is an extension of f and hence $hi_2: K \longrightarrow M$ is an extension of f . Therefore, M is an ss-K-injective.

(3) Suppose that $\alpha: N \longrightarrow M$ is an isomorphism with M is an ss-K-injective and let A be a semisimple small submodule of K, thus we have the following diagram:

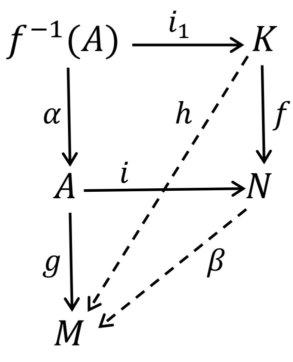


where g any R-homomorphism from A to N and i is the inclusion map. By hypothesis there is an R-homomorphism $h: K \longrightarrow M$ such that $hi = \alpha g$. Put $\beta = \alpha^{-1}h: K \longrightarrow N$. Thus we have $\beta i = \alpha^{-1}hi = \alpha^{-1}\alpha g = g$. Hence N is an ss-K-injective.

(4) Suppose that M is ss-K-injective right R-module and let $f: K \longrightarrow N$ be an isomorphism. Consider the following diagram, where $g: A \longrightarrow M$ is an R-homomorphism, A is a semisimple small submodule of N and i is the inclusion mapping:



The restriction of f to $f^{-1}(A)$ induces an isomorphism $\alpha = f_{|f^{-1}(A)}: f^{-1}(A) \longrightarrow A$, and so we have the following diagram:



where i_1 is the inclusion map. We note that $f^{-1}i = i_1\alpha^{-1}$ because let $a \in A$ and $x = f^{-1}(a)$, then $x \in f^{-1}(A)$. Hence $\alpha(x) = f(x) = f\big(f^{-1}(a)\big) = a$. Thus for all $a \in A$ we have $\alpha^{-1}(a) = x = f^{-1}(a)$ and hence $(f^{-1}i)(a) = f^{-1}(a) = i_1\big(f^{-1}(a)\big) = i_1\big(\alpha^{-1}(a)\big) = (i_1 \circ \alpha^{-1})(a)$. Now, since $f^{-1}(A)$ is semisimple small submodule of K and M is ss-K-injective, then there is an R-homomorphism $h: K \longrightarrow M$ such that $hi_1 = g\alpha$. Put $\beta = hf^{-1}$, so we have $\beta i = hf^{-1}i = hi_1\alpha^{-1} = g\alpha\alpha^{-1} = g$. Hence M is ss-N-injective.

(5) This follows from (1). □

Corollary 2.1.4. The following statements hold:

(1) If N is a right R-module, then a finite direct sum of ss-N-injective modules is again ss-N-injective. Moreover, a finite direct sum of ss-injective modules is again ss-injective.
(2) A direct summand of an ss-quasi-injective (resp., ss-injective) module is again ss-quasi-injective (resp., ss-injective).

Proof. (1) By taking I to be a finite set and applying Theorem 2.1.3 (1).

(2) This follows from Theorem 2.1.3 (5). □

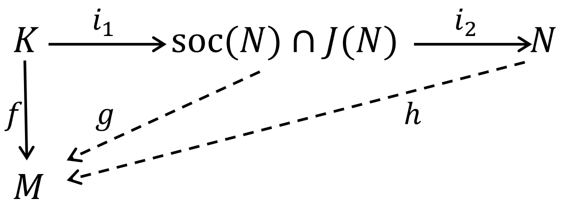
Lemma 2.1.5. Every ss-injective right R-module is a right mininjective.

Proof. Let I be a simple right ideal of R. By [25, Lemma 3.8, p. 29] we have that either I is nilpotent or a direct summand of R. If I is a nilpotent, then $I \subseteq J$ by [6, Proposition 6.2.7, p. 181] and hence I is a simple small right ideal of R. If $I \subseteq^{\oplus} R_R$, then every right R-homomorphism $f: I \longrightarrow M$ (where M is a right R-module) can be extended by fi_I (where i_I is the injection map from R onto I). Thus every ss-injective right R-module is right mininjective. □

Proposition 2.1.6. Let N be a right R-module. If $J(N) \ll N$, then a right R-module M is ss-N-injective if and only if any R-homomorphism $f: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ extends to N.

Proof. (⇒) Since $J(N) \ll N$, then $\mathrm{soc}(N) \cap J(N)$ is semisimple small submodule of N and hence any R-homomorphism $f: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ extends to N.

(⇐) Let K be any semisimple small submodule of N. Since $J(N)$ is the largest small submodule in N and $\mathrm{soc}(N)$ is the largest semisimple submodule in N, then $K \subseteq \mathrm{soc}(N) \cap J(N)$ and we have the following diagram:



where i_1 and i_2 are the inclusion maps and $f: K \longrightarrow M$ is any R-homomorphism. Since K is a direct summand of $\mathrm{soc}(N) \cap J(N)$, then $\mathrm{soc}(N) \cap J(N) = K \oplus L$ for some $L \hookrightarrow N$. Define $g: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ by $g(a) = f(k)$ where $a = k + l, k \in K, l \in L$, so we have $f = gi_1$. Thus there is an R-homomorphism $h: N \longrightarrow M$ such that $hi_2 = g$ and hence $f = gi_1 = hi_2i_1$, so M is ss-N-injective. □

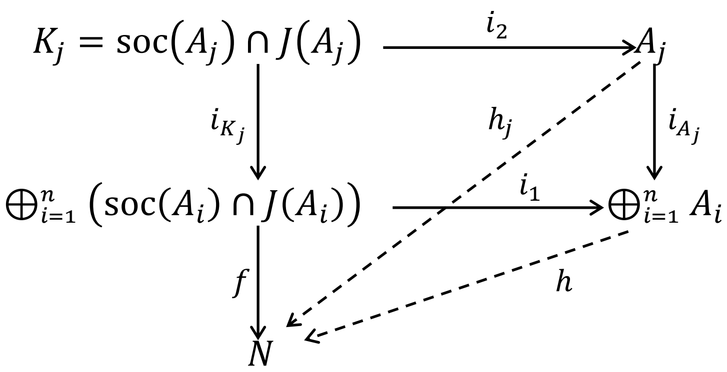
Corollary 2.1.7. If N is a finitely generated right R-module, then a right R-module M is ss-N-injective if and only if any R-homomorphism $f: \mathrm{soc}(N) \cap J(N) \longrightarrow M$ extends to N.

Proof. By Remark 1.1.7 (3) and Proposition 2.1.6. □

Proposition 2.1.8. Let N be a right R-module and $\{A_i : i = 1, 2, \ldots, n\}$ be a family of finitely generated right R-modules. Then N is ss-$\bigoplus_{i=1}^{n} A_i$-injective if and only if N is ss-A_i-injective, for all $i = 1, 2, \ldots, n$.

Proof. (⇒) This follows from Theorem 2.1.3 ((2), (4)).

(⇐) By [5, Proposition (I.4.1) and Proposition (I.1.2), p. 28 and 16] we have $\mathrm{soc}(\bigoplus_{i=1}^{n} A_i) \cap J(\bigoplus_{i=1}^{n} A_i) = (\mathrm{soc} \cap J)(\bigoplus_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} (\mathrm{soc} \cap J)(A_i) = \bigoplus_{i=1}^{n} \left(\mathrm{soc}(A_i) \cap J(A_i)\right)$. For $j = 1, 2, \ldots, n$ consider the following diagram:



where i_1 and i_2 are inclusion maps, and i_{K_j} and i_{A_j} are injection maps. By hypothesis, there is an R-homomorphism $h_j: A_j \longrightarrow N$ such that $h_j i_2 = f i_{K_j}$, also there exists exactly one R-homomorphism $h: \bigoplus_{i=1}^{n} A_i \longrightarrow N$ satisfying $h_j = h i_{A_j}$ by [16, Theorem 4.1.6 (2), p. 83]. Thus $f i_{K_j} = h_j i_2 = h i_{A_j} i_2 = h i_1 i_{K_j}$ for all $j = 1, 2, \dots, n$. Let $(a_1, a_2, \dots, a_n) \in \bigoplus_{i=1}^{n} (\mathrm{soc}(A_i) \cap J(A_i))$, thus $a_j \in \mathrm{soc}(A_j) \cap J(A_j)$, for all $j = 1, 2, \dots, n$, and

$$\begin{aligned} f((a_1, a_2, \dots, a_n)) &= f\big((a_1, 0, \dots, 0) + (0, a_2, 0, \dots, 0) + \cdots + (0, 0, \dots, 0, a_n)\big) \\ &= f\big((a_1, 0, \dots, 0)\big) + f\big((0, a_2, 0, \dots, 0)\big) + \cdots + f((0, 0, \dots, 0, a_n)) \\ &= f\left(i_{K_1}(a_1)\right) + f\left(i_{K_2}(a_2)\right) + \cdots + f\left(i_{K_n}(a_n)\right) \\ &= \left(h i_1 i_{K_1}\right)(a_1) + \left(h i_1 i_{K_2}\right)(a_2) + \cdots + (h i_1 i_{K_n})(a_n) \\ &= (h i_1)\big((a_1, 0, \dots, 0) + (0, a_2, 0, \dots, 0) + \cdots + (0, \dots, 0, a_n)\big) \\ &= (h i_1)((a_1, a_2, \dots, a_n)) \end{aligned}$$

Thus $f = h i_1$ and the proof is complete. □

Corollary 2.1.9. The following statements are hold for a right R-module M:

(1) Let $1 = e_1 + e_2 + \cdots + e_n$ in R, where the e_i are orthogonal idempotents. Then M is ss-injective if and only if M is ss-$e_i R$-injective for every $i = 1, 2, \dots, n$.

(2) For idempotents e and f of R. If $eR \cong fR$ and M is ss-eR-injective, then M is ss-fR-injective.

Proof. (1) From [3, Corollary 7.3, p. 96], we have $R = \bigoplus_{i=1}^{n} e_i R$, thus it follows from Proposition 2.1.8 that M is ss-injective if and only if M is ss-e_iR-injective for all $i = 1, 2, \dots, n$.

(2) This follows from Theorem 2.1.3 (4). □

Proposition 2.1.10. A right R-module M is ss-injective if and only if M is ss-P-injective, for every finitely generated projective right R-module P.

Proof. (⇒) Let M be an ss-injective right R-module, thus it follows from Proposition 2.1.8 that M is ss-R^n-injective for any positive integer number n. Let P be a finitely generated projective R-module, thus by [1, Corollary 3.5.5, p. 138], we have that P is a direct summand of a module isomorphic to R^m for some positive integer number m. Since M is ss-R^m-injective, thus M is ss-P-injective by Theorem 2.1.3 ((2),(4)).

(⇐) By the fact that R is projective. □

Proposition 2.1.11. The following statements are equivalent for a right R-module M:

(1) Every right R-module is ss-M-injective.
(2) Every simple submodule of M is ss-M-injective.
(3) $\mathrm{soc}(M) \cap J(M) = 0$.

Proof. (1)⇒(2) and (3)⇒(1) are clear.

(2)⇒(3) Assume that $\mathrm{soc}(M) \cap J(M) \neq 0$, thus $\mathrm{soc}(M) \cap J(M) = \bigoplus_{i \in I} x_i R$ where x_iR is a simple small submodule of M, for each $i \in I$. Therefore x_iR is ss-M-injective for each $i \in I$ by hypothesis. For any $i \in I$, the inclusion map from x_iR to M is split, so we have that $x_iR \subseteq^{\oplus} M$. Since x_iR is small submodule of M, thus $x_iR = 0$ and hence $x_i = 0$ for all $i \in I$ and this a contradiction. □

Corollary 2.1.12. The following statements are equivalent for a ring R:

(1) R is right universally mininjective.
(2) R is right mininjective and every quotient of a soc-injective right R-module is soc-injective.
(3) R is right mininjective and every quotient of an injective right R-module is soc-injective.

(4) R is right mininjective and every semisimple submodule of a projective right R-module is projective.
(5) Every right R-module is ss-injective.
(6) Every simple right ideal is ss-injective.

Proof. (1)⇔(2)⇔(3)⇔(4) By [23, Lemma 5.1] and [2, Corollary 2.9].

(1)⇔(5)⇔(6) By Proposition 2.1.11. □

Lemma 2.1.13. Let M be an ss-quasi-injective right R-module and $S = \mathrm{End}(M_R)$, then the following statements hold:

(1) $l_M r_R(m) = Sm$ for all $m \in \mathrm{soc}(M) \cap J(M)$.
(2) $r_R(m) \subseteq r_R(n)$, where $m \in \mathrm{soc}(M) \cap J(M)$, $n \in M$ implies $Sn \subseteq Sm$.
(3) $l_S\big(mR \cap r_M(\alpha)\big) = l_S(m) + S\alpha$, where $m \in \mathrm{soc}(M) \cap J(M)$, $\alpha \in S$.
(4) If kR is a simple submodule of M, then Sk is a simple left S-module. For all $k \in J(M)$. Moreover, $\mathrm{soc}(M) \cap J(M) \subseteq \mathrm{soc}({}_S M)$.
(5) $\mathrm{soc}(M) \cap J(M) \subseteq r_M(J({}_S S))$.
(6) $l_S(A \cap B) = l_S(A) + l_S(B)$, for every semisimple small right submodules A and B of M.

Proof. (1) Let $n \in l_M r_R(m)$, then $r_R(m) \subseteq r_R(n)$. Now, let $\gamma: mR \longrightarrow M$ is given by $\gamma(mr) = nr$, thus γ is well define R-homomorphism. By hypothesis, there exists an endomorphism β of M such that $\beta_{|mR} = \gamma$. Therefore, $n = \gamma(m) = \beta(m) \in Sm$, that is $l_M r_R(m) \subseteq Sm$. Conversely, let $sm \in SM$. Thus $smr = 0$ for all $r \in r_R(m)$ and hence $sm \in l_M r_R(m)$. Therefore, $l_M r_R(m) = Sm$.

(2) Let $n \in M$ and $m \in \mathrm{soc}(M) \cap J(M)$. Since $r_R(m) \subseteq r_R(n)$, then $n \in l_M r_R(m)$. By (1), we have $n \in Sm$ as desired.

(3) If $f \in l_S(m) + S\alpha$, then $f = f_1 + f_2$ such that $f_1(m) = 0$ and $f_2 = g\alpha$ for some $g \in S$. For all $n \in mR \cap r_M(\alpha)$, we have $n = mr$ and $\alpha(n) = 0$ for some $r \in R$. Since $f_1(n) = f_1(mr) = f_1(m)r = 0$ and $f_2(n) = g\big(\alpha(n)\big) = g(0) = 0$, thus $f \in l_S\big(mR \cap r_M(\alpha)\big)$ and this implies that $l_S(m) + S\alpha \subseteq l_S\big(mR \cap r_M(\alpha)\big)$. Now, we will prove the other inclusion. Let $g \in l_S\big(mR \cap r_M(\alpha)\big)$. If $r \in r_R\big(\alpha(m)\big)$, then $\alpha(mr) = 0$. So, $mr \in mR \cap r_M(\alpha)$ which yields $r_R\big(\alpha(m)\big) \subseteq r_R\big(g(m)\big)$. Since $m \in \mathrm{soc}(M) \cap J(M)$, thus $\alpha(m) \in \mathrm{soc}(M) \cap J(M)$. By (2), we have that $g(m) = \gamma\alpha(m)$ for some $\gamma \in S$. Therefore,

$g-\gamma\alpha \in l_S(m)$ which leads to $g \in l_S(m)+S\alpha$. Thus $l_S\big(mR \cap r_M(\alpha)\big) = l_S(m)+S\alpha$.

(4) To prove Sk is simple left S-module, we need only show that Sk is cyclic for any nonzero element in it. If $0 \neq \alpha(k) \in Sk$, then $\alpha: kR \longrightarrow \alpha(kR)$ is an isomorphism. Since $\alpha \in S$, then $\alpha(kR) \ll M$. Since M is ss-quasi-injective, thus $\alpha^{-1}: \alpha(kR) \longrightarrow kR$ has an extension $\beta \in S$ and hence $\beta\big(\alpha(k)\big) = \alpha^{-1}\big(\alpha(k)\big) = k$, so $k \in S\alpha k$ which leads to $Sk = S\alpha k$. Therefore, Sk is simple left S-module and this leads to $\mathrm{soc}(M) \cap J(M) \subseteq \mathrm{soc}({}_SM)$.

(5) If mR is simple and small submodule of M, then $m \neq 0$. We claim that $\alpha(m)=0$ for all $\alpha \in J({}_SS)$, thus $mR \subseteq r_M(J({}_SS))$. Otherwise, $\alpha(m) \neq 0$ for some $\alpha \in J({}_SS)$. Thus $\alpha: mR \longrightarrow \alpha(mR)$ is an isomorphism. Now, we need to prove that $r_R\big(\alpha(m)\big) = r_R(m)$. Let $r \in r_R(m)$, so $\alpha(m)r = \alpha(mr) = \alpha(0) = 0$ which leads to $r_R(m) \subseteq r_R\big(\alpha(m)\big)$. The other inclusion, if $r \in r_R\big(\alpha(m)\big)$, then $\alpha(mr) = 0$, that is $mr \in \ker(\alpha) = 0$, so $r \in r_R(m)$. Hence $r_R\big(\alpha(m)\big) = r_R(m)$. Since $m, \alpha(m) \in \mathrm{soc}(M) \cap J(M)$, thus $S\alpha m = Sm$ by (2), and this implies that $m = \beta\alpha(m)$ for some $\beta \in S$, so $(1-\beta\alpha)(m) = 0$. Since $\alpha \in J({}_SS)$, then the element $\beta\alpha$ is quasi-regular by [3, Theorem 15.3, p.166]. Thus $1-\beta\alpha$ is invertible and hence $m=0$ which is a contradiction. This shows that $\mathrm{soc}(M) \cap J(M) \subseteq r_M(J({}_SS))$.

(6) Let $\alpha \in l_S(A \cap B)$ and consider $f: A+B \longrightarrow M$ is given by $f(a+b) = \alpha(a)$, for all $a \in A$ and $b \in B$. Since M is ss-quasi-injective, then there exists $\beta \in S$ such that $f(a+b) = \beta(a+b)$. Thus $\beta(a+b) = \alpha(a)$, so $(\alpha-\beta)(a) = \beta(b)$ which yields $\alpha-\beta \in l_S(A)$. Therefore, $\alpha = \alpha-\beta+\beta \in l_S(A)+l_S(B)$ and this implies that $l_S(A \cap B) \subseteq l_S(A)+l_S(B)$. The other inclusion is obtained by [3,Proposition 2.16,p.38], then the proof is complete. □

Remark 2.1.14. Let M be a right R-module, then $D(S) = \{\alpha \in S = \mathrm{End}(M) \mid r_M(\alpha) \cap mR \neq 0$ for each $0 \neq m \in \mathrm{soc}(M) \cap J(M)\}$ is a left ideal in S.

Proof. If $\alpha \in D(S)$ and $0 \neq m \in \mathrm{soc}(M) \cap J(M)$, thus $0 \neq ma \in r_M(\alpha) \cap mR$, for some $a \in R$ and so $\alpha(ma) = 0$. Since $(-\alpha)(ma) = \alpha(ma)(-1) = 0$, then $ma \in r_M(-\alpha)$ and hence $r_M(-\alpha) \cap mR \neq 0$. Thus $-\alpha \in D(S)$. Now, let $\alpha_1, \alpha_2 \in D(S)$ and $0 \neq m \in \mathrm{soc}(M) \cap J(M)$. We have that $0 \neq ma \in r_M(\alpha_1) \cap mR$ for some $a \in R$. Since $\alpha_2 \in D(S)$, then $-\alpha_2 \in D(S)$ and hence

$0 \neq mab \in r_M(-\alpha_2) \cap mR$ for some $b \in R$. Therefore, $0 \neq mab \in r_M(\alpha_1) \cap r_M(-\alpha_2) \cap mR$. Since $r_M(\alpha_1) \cap r_M(-\alpha_2) = r_M(\alpha_1 + (-\alpha_2)) = r_M(\alpha_1 - \alpha_2)$ by [3, Proposition 2.16, p.38], thus $r_M(\alpha_1 - \alpha_2) \cap mR \neq 0$ for all $0 \neq m \in \operatorname{soc}(M) \cap J(M)$ and hence $\alpha_1 - \alpha_2 \in D(S)$. Also, let $\gamma \in S$ and $\alpha \in D(S)$. Since $r_M(\alpha) \subseteq r_M(\gamma\alpha)$, thus $r_M(\gamma\alpha) \cap mR \neq 0$ for all $0 \neq m \in \operatorname{soc}(M) \cap J(M)$, that is $\gamma\alpha \in D(S)$. Thus $D(S)$ is a left ideal of S. □

Proposition 2.1.15. Let M be an ss-quasi-injective right R-module. Then $r_M(\alpha) \subsetneq r_M(\alpha - \alpha\gamma\alpha)$, for all $\alpha \notin D(S)$ and for some $\gamma \in S = \operatorname{End}(M)$.

Proof. For all $\alpha \notin D(S)$. By hypothesis, we can find $0 \neq m \in \operatorname{soc}(M) \cap J(M)$ such that $r_M(\alpha) \cap mR = 0$. Clearly, $r_R(\alpha(m)) = r_R(m)$, so $Sm = S\alpha m$ by Lemma 2.1.13 (2). Thus $m = \gamma\alpha(m)$ for some $\gamma \in S$ and this implies that $(\alpha - \alpha\gamma\alpha)(m) = 0$. Therefore, $m \in r_M(\alpha - \alpha\gamma\alpha)$, but $m \notin r_M(\alpha)$ and hence the inclusion is strictly. □

Proposition 2.1.16. Let M is an ss-quasi-injective right R-module, then the set $\{\alpha \in S = \operatorname{End}(M) | 1 - \beta\alpha$ is R-monomorphism for all $\beta \in S\}$ is contained in $D(S)$. Moreover, $J({}_S S) \subseteq D(S)$.

Proof. Let $\alpha \notin D(S)$, then there exists $0 \neq m \in \operatorname{soc}(M) \cap J(M)$ such that $r_M(\alpha) \cap mR = 0$. If $r \in r_R(\alpha(m))$, then $\alpha(mr) = 0$ and so $mr \in r_M(\alpha)$. Since $r_M(\alpha) \cap mR = 0$, thus $r \in r_R(m)$ and hence $r_R(\alpha(m)) \subseteq r_R(m)$, so we get $Sm \subseteq S\alpha m$ by Lemma 2.1.13 (2). Therefore, $m \in \ker(1 - \gamma\alpha)$ for some $\gamma \in S$. Since $m \neq 0$, thus $1 - \gamma\alpha$ is not R-monomorphism and hence the inclusion holds. Now, let $\alpha \in J({}_S S)$, then we have $\beta\alpha$ is quasi-regular element by [3,Theorem 15.3, p. 166] and hence $1 - \beta\alpha$ is isomorphism for all $\beta \in S$, which completes the proof. □

Theorem 2.1.17. (ss-Baer's condition) For a right R-module M, the following statements are equivalent:

(1) M is an ss-injective right R-module.

(2) If $S_r \cap J = A \oplus B$, where A and B are riht ideals of R, and $\alpha: A \longrightarrow M$ is an R-homomorphism, then there exists $m \in M$ such that $\alpha(a) = ma$ for all $a \in A$ and $mB = 0$.

Proof. (1)⇒(2) Define $\gamma: S_r \cap J \longrightarrow M$ by $\gamma(a + b) = \alpha(a)$ for all $a \in A, b \in B$. By hypothesis, there is a right R-homomorphism $\beta: R \longrightarrow M$ is an extension of γ, so if $m = \beta(1)$, then $\alpha(a) = \gamma(a) = \beta(a) = \beta(1)a = ma$, for all $a \in A$. Moreover, $mb = \beta(b) = \gamma(b) = \alpha(0) = 0$ for all $b \in B$, so $mB = 0$.

(2)⇒(1) Let $\alpha: I \longrightarrow M$ be any right R-homomorphism, where I is any semisimple small right ideal. By hypothesis, there exists $m \in M$ such that $\alpha(a) = ma$ for all $a \in I$. Define $\beta: R_R \longrightarrow M$ by $\beta(r) = mr$ for all $r \in R$, thus β extends α. □

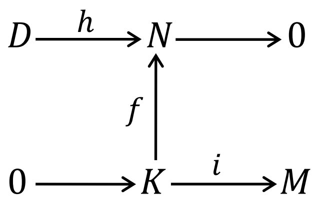
Theorem 2.1.18. If M is a projective right R-module, then the following statements are equivalent:

(1) Every quotient of an ss-M-injective right R-module is ss-M-injective.
(2) Every quotient of a soc-M-injective right R-module is ss-M-injective.
(3) Every quotient of an injective right R-module is ss-M-injective.
(4) Every sum of two ss-M-injective submodules of a right R-module is ss-M-injective.
(5) Every sum of two soc-M-injective submodules of a right R-module is ss-M-injective.
(6) Every sum of two injective submodules of a right R-module is ss-M-injective.
(7) Every semisimple small submodule of M is projective.
(8) Every simple small submodule of M is projective.
(9) $\mathrm{soc}(M) \cap J(M)$ is projective.

Proof. (1)⇒(2)⇒(3), (4)⇒(5)⇒(6) and (9)⇒(7)⇒(8) are obvious.

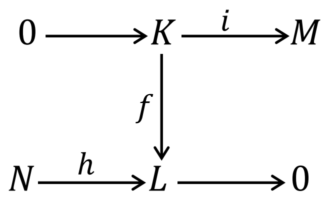
(8)⇒(9) Since $\mathrm{soc}(M) \cap J(M)$ is a direct sum of simple submodules of M and since every simple in $J(M)$ is small in M, thus $\mathrm{soc}(M) \cap J(M)$ is projective.

(3)⇒(7) Let D and N be right R-modules and consider the diagram:



where K is a semisimple small submodule of M, h is a right R-epimorphism, f is a right R-homomorphism, and i is the inclusion map. We can take D to be injective R-module (by [6, Proposition 5.2.10, p. 148]). Since N is ss-M-injective, then we can extend f to an R-homomorphism $\alpha: M \longrightarrow N$. By projectivity of M, thus α can be lifted to an R-homomorphism $\tilde{\alpha}: M \longrightarrow D$ such that $h\tilde{\alpha} = \alpha$. Let $\tilde{f}: K \longrightarrow D$ be the restriction of $\tilde{\alpha}$ over K. Obviously, $h\tilde{f} = f$ and this implies that K is projective.

(7)⇒(1) Let $h: N \longrightarrow L$ be an R-epimorphism, where N and L are right R-modules, and N is ss-M-injective. Consider the following diagram:



where K is a semisimple small submodule of M, $f: K \longrightarrow L$ is an R-homomorphism, and i is the inclusion map. By hypothesis, K is projective, thus there is an R-homomorphism $g: K \longrightarrow N$ such that $hg = f$. Since N is ss-M-injective, then there exists R-homomorphism $\tilde{g}: M \longrightarrow N$ such that $\tilde{g}i = g$. Put $\beta = h\tilde{g}: M \longrightarrow L$. Thus $\beta i = h\tilde{g}i = hg = f$. Hence L is an ss-M-injective right R-module.

(1)⇒(4) Let N_1 and N_2 be two ss-M-injective submodules of a right R-module N. Then $N_1 + N_2$ is a homomorphic image of the direct sum $N_1 \oplus N_2$. Since $N_1 \oplus N_2$ is ss-M-injective, thus $N_1 + N_2$ is ss-M-injective by hypothesis.

(6)⇒(3) Let E be an injective right R-module and $N \hookrightarrow E$. Let $Q = E \oplus E, K = \{(n,n) \mid n \in N\}, \bar{Q} = Q/K, H_1 = \{y + K \in \bar{Q} \mid y \in E \oplus 0\}$ and $H_2 = \{y + K \in \bar{Q} \mid y \in 0 \oplus E\}$. Then $\bar{Q} = H_1 + H_2$. Since $(E \oplus 0) \cap K = 0$ and $(0 \oplus E) \cap K = 0$, thus $E \cong H_i$, $i = 1, 2$. Since $H_1 \cap H_2 = \{y + K \in \bar{Q} \mid y \in N \oplus 0\} = \{y + K \in \bar{Q} \mid y \in 0 \oplus N\}$, thus $H_1 \cap H_2 \cong N$ under $y \longmapsto y + K$ for all $y \in N \oplus 0$. By hypothesis, $\bar{Q}$ is ss-M-injective. Since H_1 is injective, thus $\bar{Q} = H_1 \oplus A$ for some $A \hookrightarrow \bar{Q}$, so $A \cong (H_1 + H_2)/H_1 \cong H_2/(H_1 \cap H_2) \cong E/N$. By Theorem 2.1.3 ((3),(5)), E/N is ss-M-injective. □

The following corollary gives a new characterizations of PS-rings.

Corollary 2.1.19. The following statements are equivalent for a ring R:

(1) Every quotient of an ss-injective right R-module is ss-injective.
(2) Every quotient of a soc-injective right R-module is ss-injective.
(3) Every quotient of a small injective right R-module is ss-injective.

(4) Every quotient of an injective right R-module is ss-injective.

(5) Every sum of two ss-injective submodules of any right R-module is ss-injective.

(6) Every sum of two soc-injective submodules of any right R-module is ss-injective.

(7) Every sum of two small injective submodules of any right R-module is ss-injective.

(8) Every sum of two injective submodules of any right R-module is ss-injective.

(9) Every semisimple small submodule of any projective right R-module is projective.

(10) Every semisimple small submodule of any finitely generated projective right R-module is projective.

(11) Every semisimple small submodule of R_R is projective.

(12) Every simple small submodule of R_R is projective.

(13) $S_r \cap J$ is projective.

(14) S_r is projective (R is a right PS-ring).

Proof. The equivalence between (1), (2), (4), (5), (6), (8), (11), (12) and (13) is from Theorem 2.1.18.

(1)⇒(3)⇒(4), (5)⇒(7)⇒(8) and (9)⇒(10)⇒(13) are clear.

(14)⇒(9) By [2, Corollary 2.9].

(13)⇒(14) Let $S_r = (S_r \cap J) \oplus A$, where $A = \bigoplus_{i \in I} S_i$ and S_i is a simple right ideal and direct summand of R_R, for all $i \in I$. Thus A is projective, but $S_r \cap J$ is also projective, so it follows that S_r is projective. □

Theorem 2.1.20. If every singular simple right R-module is ss-injective, then $r(a) \subseteq^{\oplus} R_R$ for every $a \in S_r \cap J$ and S_r is projective.

Proof. Let $a \in S_r \cap J$ and let $A = RaR + r(a)$. Thus there exists $B \hookrightarrow R_R$ such that $A \oplus B \subseteq^{ess} R_R$. Assert that $A \oplus B \neq R_R$, then we find $I \subseteq^{max} R_R$ such that $A \oplus B \subseteq I$, and so $I \subseteq^{ess} R_R$.Since R/I is singular right R-module by [17, Example 7.6 (3), p. 247], then R/I is ss-injective. Consider the map $\alpha: aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ which is well define R-homomorphism. Thus, there exists $c \in R$ with $1 + I = ca + I$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$ which leads to $1 \in I$, a contradiction. Thus $A \oplus B = R_R$ and hence $RaR + (r(a) \oplus B) = R$. Since $RaR \ll R_R$, then $r(a) \subseteq^{\oplus} R_R$. Put $r(a) = (1 - e)R$, for some $e^2 = e \in R$, so it follows that $ax = aex$ (because $(1-e)x \in r(a)$, and so $a(1-e)x = 0$) for all $x \in R$ and this leads to $aR = aeR$. Let $\gamma: eR \longrightarrow aeR$ be defined by $\gamma(er) = aer$ for all $r \in R$. Then γ is a well defined R-epimorphism. Clearly, $\ker(\gamma) = \{er: \ aer = 0\} = \{er: \ er \in r(a)\} = eR \cap r(a) = 0$. Hence γ is an isomorphism and so aR is projective. Since $S_r \cap J$ is a direct sum of simple small right ideals, thus $S_r \cap J$ is projective and it follows from Corollary 2.1.19 that S_r is projective. □

Corollary 2.1.21. A ring R is right mininjective and every singular simple right R-module is ss-injective if and only if R is a right universally mininjective.

Proof. By Theorem 2.1.20 and [23, Lemma 5.1]. □

Recall that a ring R is called zero insertive if $aRb = 0$ for all $a, b \in R$ with $ab = 0$ (see [30]).

Lemma 2.1.22 [30, Lemma 2.11]. Let R be a zero insertive ring, then $RaR + r(a) \subseteq^{ess} R_R$ for every $a \in R$.

Proposition 2.1.23. Let R be a zero insertive ring. If every singular simple right R-module is ss-injective, then R is right universally mininjective.

Proof. Let $a \in S_r \cap J$. We claim that $RaR + r(a) = R$, thus $r(a) = R$ (since $RaR \ll R$), so $a = 0$ and this means that $S_r \cap J = 0$. Otherwise, if $RaR + r(a) \subsetneq R$, then there exists a maximal right ideal I of R such that $RaR + r(a) \subseteq I$. Since $I \subseteq^{ess} R_R$ by Lemma 2.1.22, then R/I is ss-injective by hypothesis. Consider $\alpha: aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ for all $r \in R$ which is well defined R-homomorphism. Thus $1 + I = ca + I$ for some $c \in R$. Since $ca \in RaR \subseteq I$, then $1 \in I$ and this contradicts the maximality of I, so we must have $RaR + r(a) = R$ and this ends the proof. □

Theorem 2.1.24. If M is a finitely generated right R-module, then the following statements are equivalent:

(1) $\mathrm{soc}(M) \cap J(M)$ is a noetherian R-module.
(2) $\mathrm{soc}(M) \cap J(M)$ is finitely generated.
(3) Any direct sum of ss-M-injective right R-modules is ss-M-injective.
(4) Any direct sum of soc-M-injective right R-modules is ss-M-injective.
(5) Any direct sum of injective right R-modules is ss-M-injective.

(6) $K^{(S)}$ is ss-M-injective for every injective right R-module K and for any index set S.

(7) $K^{(\mathbb{N})}$ is ss-M-injective for every injective right R-module K.

Proof. (1)⇒(2) and (3)⇒(4)⇒(5)⇒(6)⇒(7) Clear.

(2)⇒(3) Let $E = \bigoplus_{i\in I} M_i$ be a direct sum of ss-M-injective right R-modules and $f: N \longrightarrow E$ be a right R-homomorphism where N is a semisimple small submodule of M. Since $\mathrm{soc}(M) \cap J(M)$ is finitely generated, thus N is finitely generated and hence $f(N) \subseteq \bigoplus_{i\in I_1} M_i$, for a finite subset I_1 of I. Since a finite direct sums of ss-M-injective right R-modules is ss-M-injective, thus $\bigoplus_{i\in I_1} M_i$ is ss-M-injective and hence f can be extended to an R-homomorphism $g: M \longrightarrow E$. Thus E is ss-M-injective.

(7)⇒(1) Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of submodules of $\mathrm{soc}(M) \cap J(M)$. For each $i \geq 1$, let $E_i = E(M/N_i)$ and $E = \bigoplus_{i=1}^{\infty} E_i$. For every $i \geq 1$, we put $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus \left(\prod_{\substack{j=1 \\ j\neq i}}^{\infty} E_j\right)$, then M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus \left(\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{\infty} E_j\right)$ is ss-M-injective, so it follows from Theorem 2.1.3 (5) that E is ss-M-injective. Define $f: U = \bigcup_{i=1}^{\infty} N_i \longrightarrow E$ by $f(m) = (m + N_i)_i$. It is clear that f is a well defined R-homomorphism. Since M is finitely generated, thus $\mathrm{soc}(M) \cap J(M)$ is a semisimple small submodule of M and hence $\bigcup_{i=1}^{\infty} N_i$ is a semisimple small submodule of M, so f can be extended to a right R-homomorphism $g: M \longrightarrow E$. Since M is finitely generated, then we have $g(M) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$ for some n and hence $f(U) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$. Since $\pi_i f(x) = \pi_i\left(\left(x + N_j\right)_{j\geq 1}\right) = x + N_i$ for all $x \in U$ and $i \geq 1$, where $\pi_i: \bigoplus_{j\geq 1} E(M/N_i) \longrightarrow E(M/N_i)$ be the projection map. Thus $\pi_i f(U) = U/N_i$ for all $i \geq 1$. Since $f(U) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$. Thus $U/N_i = \pi_i f(U) = 0$, for all $i \geq n+1$, so $U = N_i$ for all $i \geq n+1$ and hence the chain $N_1 \subseteq N_2 \subseteq \cdots$ terminates at N_{n+1}. Thus $\mathrm{soc}(M) \cap J(M)$ is a noetherian R-module. □

Corollary 2.1.25. If N is a finitely generated right R-module, then the following statements are equivalent:

(1) $\mathrm{soc}(N) \cap J(N)$ is finitely generated.

(2) $M^{(S)}$ is ss-N-injective for every soc-N-injective right R-module M and for any index set S.
(3) $M^{(S)}$ is ss-N-injective for every ss-N-injective right R-module M and for any index set S.
(4) $M^{(\mathbb{N})}$ is ss-N-injective for every soc-N-injective right R-module M.
(5) $M^{(\mathbb{N})}$ is ss-N-injective for every ss-N-injective right R-module M.

Proof. By Theorem 2.1.24. □

Corollary 2.1.26. The following statements are equivalent for a ring R:

(1) $S_r \cap J$ is finitely generated.
(2) Any direct sum of ss-injective right R-modules is ss-injective.
(3) Any direct sum of soc-injective right R-modules is ss-injective.
(4) Any direct sum of small injective right R-modules is ss-injective.
(5) Any direct sum of injective right R-modules is ss-injective.
(6) $M^{(S)}$ is ss-injective for every injective right R-module M and for any index set S.
(7) $M^{(S)}$ is ss-injective for every soc-injective right R-module M and for any index set S.
(8) $M^{(S)}$ is ss-injective for every small injective right R-module M and for any index set S.
(9) $M^{(S)}$ is ss-injective for every ss-injective right R-module M and for any index set S.
(10) $M^{(\mathbb{N})}$ is ss-injective for every injective right R-module M.
(11) $M^{(\mathbb{N})}$ is ss-injective for every soc-injective right R-module M.
(12) $M^{(\mathbb{N})}$ is ss-injective for every small injective right R-module M.
(13) $M^{(\mathbb{N})}$ is ss-injective for every ss-injective right R-module M.

Proof. By applying Theorem 2.1.24 and Corollary 2.1.25. □

Remark 2.1.27. Let M be a right R-module. We denote that $r_u(N) = \{a \in S_r \cap J| \quad Na = 0\}$ and $l_M(K) = \{m \in M| \quad mK = 0\}$ where $N \subseteq M$ and $K \subseteq S_r \cap J$. Clearly, $r_u(N) \hookrightarrow (S_r \cap J)_R$ and $l_M(K) \hookrightarrow_S M$, where $S = \mathrm{End}(M_R)$, and we have the following:

(1) $N \subseteq l_M r_u(N)$ for all $N \subseteq M$.
(2) $K \subseteq r_u l_M(K)$ for all $K \subseteq S_r \cap J$.
(3) $r_u l_M r_u(N) = r_u(N)$ for all $N \subseteq M$.

(4) $l_M r_u l_M(K) = l_M(K)$ for all $K \subseteq S_r \cap J$.

Proof. (1) Let $x \in N$, then $xr = 0$ for all $r \in r_u(N)$, so $x \in l_M r_u(N)$.

(2) Similarly of (1).

(3) Let $r \in r_u(N)$, then $xr = 0$ for all $x \in l_M r_u(N)$, that is $r \in r_u l_M r_u(N)$, and so $r_u(N) \subseteq r_u l_M r_u(N)$. The second inclusion is obtained by (1).

(4) Similarly of (3). □

Lemma 2.1.28. For a right R-module M, the following statements are equivalent:

(1) For right ideals of the form $r_u(N)$, the ring R satisfies ACC, where $N \subseteq M$.
(2) The ring R satisfies the DCC for left S-modules of the form $l_M(K)$, where $K \subseteq S_r \cap J$.
(3) For each semisimple small right ideal I of R, there exists a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$.

Proof. (1)⇔(2) Clear.

(2)⇒(3) Consider $\Omega = \{l_M(A) | \quad A$ is finitely generated right ideal and $A \subseteq I\}$ which is nonempty set because $M \in \Omega$. Now, let K be a finitely generated right ideal of R contained in I such that $l_M(K)$ is the minimal in Ω. Put $B = K + xR$, where $x \in I$. Thus B is a finitely generated right ideal contained in I and $l_M(B) \subseteq l_M(K)$. But since $l_M(K)$ is minimal in Ω, then $l_M(B) = l_M(K)$ which yields $l_M(K)x = 0$, for all $x \in I$. Therefore, $l_M(K)I = 0$ and hence $l_M(K) \subseteq l_M(I)$. But $l_M(I) \subseteq l_M(K)$, so $l_M(I) = l_M(K)$.

(3)⇒(1) Suppose that $r_u(M_1) \subseteq r_u(M_2) \subseteq \cdots \subseteq r_u(M_n) \subseteq \cdots$ where $M_i \subseteq M$ for each i. Put $D_i = l_M r_u(M_i)$ for each i, and $I = \bigcup_{i=1}^{\infty} r_u(M_i)$, then $I \subseteq S_r \cap J$. By hypothesis, there exists a finitely generated right ideal K of R contained in I such that $l_M(I) = l_M(K)$. Since K is a finitely generated, thus there exists $t \in \mathbb{N}$ such that $K \subseteq r_u(M_n)$ for all $n \geq t$, that is $l_M(K) \supseteq l_M r_u(M_n) = D_n$ for all $n \geq t$. Since $l_M(K) = l_M(I) = l_M(\bigcup_{i=1}^{\infty} r_u(M_i)) = \bigcap_{i=1}^{\infty} l_M r_u(M_i) = \bigcap_{i=1}^{\infty} D_i \subseteq D_n$, then $l_M(K) = D_n$ for all $n \geq t$. Since $D_n = l_M r_u(M_n)$, then $r_u(M_n) = r_u l_M r_u(M_n) = r_u(D_n) = r_u l_M(K)$ for all $n \geq t$. Thus $r_u(M_n) = r_u(M_t)$ for all $n \geq t$. Hence (3) implies (1), which ends the proof. □

The first part in the following proposition is obtained directly by Corollary 2.1.26, but we will prove it by different way.

Proposition 2.1.29. Let E be an ss-injective right R-module. Then $E^{(\mathbb{N})}$ is ss-injective if and only if the ring R satisfies the ACC for right ideals of form $r_u(N)$, where $N \subseteq E$.

Proof. ($\Rightarrow$) Suppose that $r_u(N_1) \subsetneq r_u(N_2) \subsetneq \cdots \subsetneq r_u(N_m) \subsetneq \cdots$ be a strictly chain, where $N_i \subseteq E$. Then we get, $l_E r_u(N_1) \supsetneq l_E r_u(N_2) \supsetneq \cdots \supsetneq l_E r_u(N_m) \supsetneq \cdots$. For each $i \geq 1$, we can find $t_i \in l_E r_u(N_i) - l_E r_u(N_{i+1})$ and $a_{i+1} \in r_u(N_{i+1})$ such that $t_i a_{i+1} \neq 0$. Let $L = \bigcup_{i=1}^{\infty} r_u(N_i)$, then for all $\ell \in L$ there exists $m_\ell \geq 1$ such that $\ell \in r_u(N_i)$ for all $i \geq m_\ell$ and this implies that $t_i \ell = 0$ for all $i \geq m_\ell$. Put $\bar{t} = (t_i)_i$, we have $\bar{t}\ell \in E^{(\mathbb{N})}$ for every $\ell \in L$. Consider $\alpha_{\bar{t}}: L \longrightarrow E^{(\mathbb{N})}$ is given by $\alpha_{\bar{t}}(\ell) = \bar{t}\ell$, then $\alpha_{\bar{t}}$ is a well define R-homomorphism. Since L is semisimple small right ideal, thus $\alpha_{\bar{t}}$ extends to $\gamma: R \longrightarrow E^{(\mathbb{N})}$ (by hypothesis). Hence $\alpha_{\bar{t}}(\ell) = \bar{t}\ell = \gamma(\ell) = \gamma(1)\ell$. Thus there exists $k \geq 1$ such that $t_i \ell = 0$ for all $i \geq k$ and all $\ell \in L$ (since $\gamma(1) \in E^{(\mathbb{N})}$), but this contradicts with $t_k a_{k+1} \neq 0$.

($\Leftarrow$) Let $\alpha: I \longrightarrow E^{(\mathbb{N})}$ be an R-homomorphism, where I is a semisimple small right ideal, then it follows from Lemma 2.1.28 that there is a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$. Since $E^{\mathbb{N}}$ is ss-injective, thus $\alpha = a\,\cdot$ for some $a \in E^{\mathbb{N}}$. Write $K = \bigoplus_{i=1}^{m} r_i R$, so we have $\alpha(r_i) = a r_i \in E^{(\mathbb{N})}$, $i = 1, 2, \ldots, m$. Thus, there exists $\tilde{a} \in E^{(\mathbb{N})}$ such that $a_n r_i = \tilde{a}_n r_i$ for all $n \in \mathbb{N}$, $i = 1, 2, \ldots, m$, where a_n is the nth coordinate of a. Since K is generated by $\{r_1, r_2, \ldots, r_m\}$, thus $ar = \tilde{a}r$ for all $r \in K$. Therefore, $a_n - \tilde{a}_n \in l_M(K) = l_M(I)$ for all $n \in \mathbb{N}$ which leads to $a_n r = \tilde{a}_n r$ for all $r \in I$ and $n \in \mathbb{N}$, so $ar = \tilde{a}r$ for all $r \in I$. Thus there exists $\tilde{a} \in E^{(\mathbb{N})}$ such that $\alpha(r) = \tilde{a}r$ for all $r \in I$ and this means that $E^{(\mathbb{N})}$ is ss-injective. □

Theorem 2.1.30. Let R be a ring, then the following statements are equivalent:

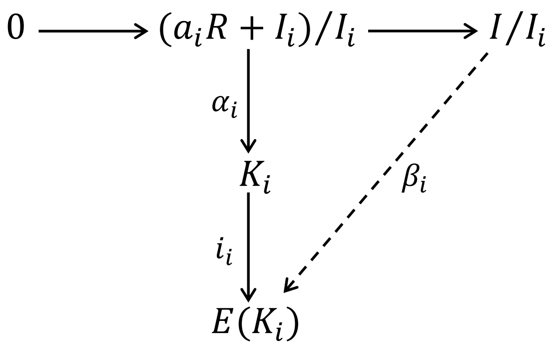
(1) $S_r \cap J$ is finitely generated.

(2) $\bigoplus_{i=1}^{\infty} E(M_i)$ is ss-injective right R-module for every family of simple right R-modules $\{M_i\}_{i \in \mathbb{N}}$.

Proof. (1)⇒(2) By Corollary 2.1.26.

(2)⇒(1) Let $I_1 \subsetneq I_2 \subsetneq \cdots$ be a properly ascending chain of semisimple small right ideals of R. It is clear that $I = \bigcup_{i=1}^{\infty} I_i$ is a semisimple small right ideal of R. For every $i \geq 1$, there exists $a_i \in I$, $a_i \notin I_i$ and consider $N_i/I_i \subseteq^{max} (a_i R + I_i)/I_i$, so $K_i = (a_i R + I_i)/N_i$ is a simple right R-module.

Define $\alpha_i: (a_iR + I_i)/I_i \longrightarrow (a_iR + I_i)/N_i$ by $\alpha_i(x + I_i) = x + N_i$ which is right R-epimorphism. Consider the following diagram:



where i_i is the inclusion map. Thus there exists $\beta_i: I/I_i \longrightarrow E(K_i)$ such that $\beta_i = i_i\alpha_i$. Since $a_i \notin N_i$, then $\beta_i(a_i + I_i) = i_i(\alpha_i(a_i + I_i)) = a_i + N_i \neq 0$ for each $i \geq 1$. If $b \in I$, then there exists $n_b \geq 1$ such that $b \in I_i$ for all $i \geq n_b$ and hence $\beta_i(b + I_i) = 0$ for all $i \geq n_b$. Thus we can define $\gamma: I \longrightarrow \bigoplus_{i=1}^{\infty} E(K_i)$ by $\gamma(b) = (\beta_i(b + I_i))_i$. Then there exists $\tilde{\gamma}: R \longrightarrow \bigoplus_{i=1}^{\infty} E(K_i)$ such that $\tilde{\gamma}_{|I} = \gamma$ by hypothesis. Put $\tilde{\gamma}(1) = (c_i)_i$, thus there exists $n \geq 1$ with $c_i = 0$ for all $i \geq n$. Since $(\beta_i(b + I_i))_i = \gamma(b) = \tilde{\gamma}(b) = \tilde{\gamma}(1)b = (c_ib)_i$ for all $b \in I$, thus $\beta_i(b + I_i) = c_ib$ for all $i \geq 1$, so it follows that $\beta_i(b + I_i) = 0$ for all $i \geq n$ and all $b \in I$ and this contradicts with $\beta_n(a_n + I_n) \neq 0$. Thus (2) implies (1). □

In the next results, we will give some relations between ss-injectivity and other injectivities.

Lemma 2.1.31. Let $M$and C be right R-modules and $N \hookrightarrow M$ with M/N is a semisimple. Then every R-homomorphism from a submodule (resp. semisimple submodule) A of M to C can be extended to an R-homomorphism from M to C if and only if every R-homomorphism from a submodule (resp. semisimple submodule) B of N to C can be extended to an R-homomorphism from M to C.

Proof. (⇒) is obtained directly.

(⇐) Let A be a submodule of a right R-module M and let f be an R-homomorphism from A to C. Since M/N is a semisimple, thus there exists $L \hookrightarrow M$ such that $A + L = M$ and $A \cap L \subseteq N$ (see [18, Proposition 2.1]). Thus there exists an R-homomorphism $g: M \longrightarrow C$ such that $g(x) = f(x)$ for all $x \in A \cap L$. Define $h: M \longrightarrow C$ such that for any $x = a + \ell$, $a \in A$, $\ell \in L$, $h(x) = f(a) + g(\ell)$. Thus h is a well define R-homomorphism, because if $a_1 + \ell_1 = a_2 + \ell_2$, $a_i \in A$, $\ell_i \in L$, $i = 1,2$, then $a_1 - a_2 = \ell_2 - \ell_1 \in A \cap L$, that is $f(a_1 - a_2) = g(\ell_2 - \ell_1)$ which leads to $h(a_1 + \ell_1) = h(a_2 + \ell_2)$. Therefore h is a well define R-homomorphism and extension of f. □

Corollary 2.1.32. For right R-modules M and N, the following hold:

(1) If M is finitely generated and $M/J(M)$ is semisimple right R-module, then N is soc-M-injective if and only if N is ss-M-injective.
(2) If $M/\mathrm{soc}(M)$ is semisimple right R-module, then N is soc-M-injective if and only if N is M-injective.
(3) If R/S_r is semisimple as right R-module, then N is soc-injective if and only if N is injective.
(4) If R/S_r is semisimple as right R-module, then N is ss-injective if and only if N is small injective.

Proof. (1) (⇒) Clear.

(⇐) Since N is a right ss-M-injective, thus every R-homomorphism from a semisimple small submodule of M to N extends to M. Since M is finitely generated, thus $J(M) \ll M$ and hence every R-homomorphism from any semisimple submodule of $J(M)$ to N extends to M. Since $M/J(M)$ is semisimple, thus every R-homomorphism from any semisimple submodule of M to N extends to M by Lemma 2.1.31. Therefore, N is soc-M-injective right R-module.

(2) (⇒) Since N is soc-M-injective. Thus every R-homomorphism from any submodule of $\mathrm{soc}(M)$ to N extends to M. Since $M/\mathrm{soc}(M)$ is semisimple, thus Lemma 2.1.31 implies that every R-homomorphism from any submodule of M to N extends to M. Hence N is M-injective.

(⇐) Clear.

(3) By (2).

(4) Since R/S_r is semisimple as right R-module, thus $J(R/S_r) = 0$. By Remark 1.1.7 (5), we have $J \subseteq S_r$ and hence $J = J \cap S_r$. Thus N is ss-injective if and only is N is small injective. □

Corollary 2.1.33. Let R be a semilocal ring, then $S_r \cap J$ is finitely generated if and only if S_r is finitely generated.

Proof. Suppose that $S_r \cap J$ is finitely generated. By Corollary 2.1.26, every direct sum of soc-injective right R-modules is ss-injective. Thus it follows from Corollary 2.1.32 (1) and [2, Corollary 2.11] that S_r is finitely generated. □

Proposition 2.1.34. The following statements are equivalent for a right R-module M:

(1) M is ss-injective.

(2) The sequence $0 \longrightarrow \mathrm{Hom}_R(R/A, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(A, M) \longrightarrow 0$ is exact for all $A \hookrightarrow S_r \cap J$, where i and π are the inclusion and canonical maps, respectively.

(3) The sequence $0 \longrightarrow \mathrm{Hom}_R(R/S_r \cap J, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(S_r \cap J, M) \longrightarrow 0$ is exact, where i and π are the inclusion and canonical maps, respectively.

(4) $\mathrm{Ext}^1(R/A, M) = 0$ for all $A \hookrightarrow S_r \cap J$.

(5) $\mathrm{Ext}^1(R/S_r \cap J, M) = 0$.

Proof. (1)⇒(2) Let $f \in \mathrm{Hom}_R(A, M)$, then there is $g \in \mathrm{Hom}_R(R, M)$ such that $f = gi$ and hence $f = i^*(g)$, so i^* is an R-epimorphism.

(2)⇒(1) Clear.

(1)⇔(3) is similar to (1)⇔(2).

(2)⇔(4) From the exactness of the sequence $0 \longrightarrow \mathrm{Hom}_R(R/A, M) \xrightarrow{\pi^*} \mathrm{Hom}_R(R, M) \xrightarrow{i^*} \mathrm{Hom}_R(A, M) \longrightarrow \mathrm{Ext}^1(R/A, M) \longrightarrow 0$ (see [13, Theorem XII.4.4, p. 491]).

(3)⇔(5) is similar to (2)⇔(4). □

Section Two

2.2 SS-Injective Rings

In this section, we will study ss-injective rings with some characterizations and properties of them.

Proposition 2.2.1. The following statements are equivalent for a ring R:

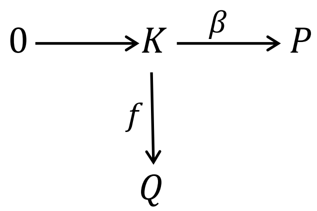
(1) R is a right ss-injective ring.

(2) If K is a semisimple right R-module, P and Q are finitely generated projective right R-modules, $\beta: K \longrightarrow P$ is an R-monomorphism with $\beta(K) \ll P$ and $f: K \longrightarrow Q$ is an R-homomorphism, then f can be extended to an R-homomorphism $h: P \longrightarrow Q$.

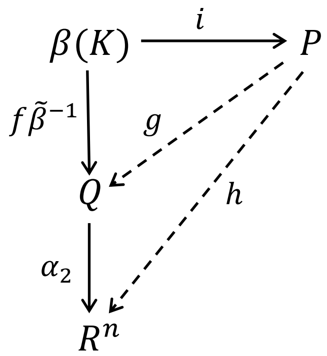
(3) If M be a right semisimple R-module and f is a nonzero R-monomorphism from M to R_R with $f(M) \ll R_R$, then $M^d = Rf$.

Proof. (2)⇒(1) Clear.

(1)⇒(2) Consider the following diagram:



where P and Q are finitely generated projective right R-modules, β is an R-monomorphism and K is a semisimple R-module with $\beta(K) \ll P$. Since Q is finitely generated, then there is an R-epimorphism $\alpha_1: R^n \longrightarrow Q$ for some positive integer number n. Since Q is a projective, then there is an R-homomorphism $\alpha_2: Q \longrightarrow R^n$ such that $\alpha_1 \alpha_2 = I_Q$. Thus we have the following diagram:



where $\tilde{\beta}: K \longrightarrow \beta(K)$ is defined by $\tilde{\beta}(a) = \beta(a)$ for all $a \in K$ and i is the inclusion map. Since R is a right ss-injective ring, it follows from Proposition 2.1.10 and Corollary 2.1.4 (1) that R^n is a right ss-P-injective R-module. So there exists an R-homomorphism $h: P \longrightarrow R^n$ such that $hi = \alpha_2 f\tilde{\beta}^{-1}$. Put $g = \alpha_1 h: P \longrightarrow Q$. Thus $gi = (\alpha_1 h)i = \alpha_1\left(\alpha_2 f\tilde{\beta}^{-1}\right) = f\tilde{\beta}^{-1}$ and hence $(g\beta)(a) = g\left(i\big(\beta(a)\big)\right) = \left(f\tilde{\beta}^{-1}\right)\big(\beta(a)\big) = f(a)$ for all $a \in K$. Therefore, there is an R-homomorphism $g: P \longrightarrow Q$ such that $g\beta = f$.

(1)⇒(3) Let $g \in M^d$, we have $gf^{-1}: f(M) \longrightarrow R_R$, since $f(M)$ is a semisimple small right ideal of R and R is a right ss-injective ring (by hypothesis), thus there exists $a \in R$ such that $(gf^{-1})(k) = ak$, for all $k \in f(M)$.Now, let $m \in M$, then $f(m) \in f(M)$ and hence $g\left(f^{-1}\big(f(m)\big)\right) = af(m)$. Therefore, $g(m) = af(m)$, for all $m \in M$. Hence $M^d = Rf$.

(3)⇒(1) Let $f: K \longrightarrow R$ be a right R-homomorphism, where K is a semisimple small right ideal of R and $i: K \longrightarrow R$ be the inclusion map, thus by (3) we have $K^d = Ri$ and hence $f = ci$ in K^d for some $c \in R$. Thus there is $c \in R$ such that $f(a) = ca$ for all $a \in K$ and this implies that R is a right ss-injective ring. □

Example 2.2.2.

(1) Every universally mininjective ring is ss-injective, but not conversely (see Example 3.2.8).
(2) The two classes of universally mininjective rings and soc-injective rings are different (see Example 3.2.8 and Example 3.2.9).

Corollary 2.2.3. Let R be a right ss-injective ring. Then:
(1) R is a right mininjective ring.
(2) $lr(a) = Ra$ for all $a \in S_r \cap J$.

(3) $r(a) \subseteq r(b)$, $a \in S_r \cap J$, $b \in R$ implies $Rb \subseteq Ra$.
(4) $l\big(bR \cap r(a)\big) = l(b) + Ra$, for all $a \in S_r \cap J$, $b \in R$.
(5) $l(K_1 \cap K_2) = l(K_1) + l(K_2)$, for all semisimple small right ideals K_1 and K_2 of R.

Proof. (1) By Lemma 2.1.5.

(2), (3), (4) and (5) are obtained by Lemma 2.1.13 and [32, 11.11, p. 88]. □

The following is an example of a right mininjective ring which is not right ss-injective.

Example 2.2.4. (The Björk Example [24, Example 2.5, p. 38]). Let F be a field and let $a \longmapsto \bar{a}$ be an isomorphism $F \longrightarrow \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. By [24, Example 2.5 and 5.2, p. 38 and 97] we have R is a right principally injective and local ring. It is mentioned in [2, Example 4.15], that R is not right soc-injective. Since R is local, thus by Corollary 2.1.32 (1), R is not right ss-injective ring.

Theorem 2.2.5. Let R be a right ss-injective ring. Then:

(1) $S_r \cap J \subseteq Z_r$.
(2) If the ascending chain $r(a_1) \subseteq r(a_2a_1) \subseteq \cdots \subseteq r(a_n \ldots a_2a_1) \subseteq \cdots$ terminates for any sequence $a_1, a_2, \ldots$ in $Z_r \cap S_r$, then $S_r \cap J$ is right t-nilpotent and $S_r \cap J = Z_r \cap S_r$.

Proof. (1) Let $a \in S_r \cap J$ and $bR \cap r(a) = 0$ for any $b \in R$. By Corollary 2.2.3 (4), $l(b) + Ra = l\big(bR \cap r(a)\big) = l(0) = R$, so $l(b) = R$ because $a \in J$, implies that $b = 0$. Thus $r(a) \subseteq^{ess} R_R$ and hence $S_r \cap J \subseteq Z_r$.

(2) For any sequence $x_1, x_2, \ldots$ in $Z_r \cap S_r$, we have $r(x_1) \subseteq r(x_2x_1) \subseteq \cdots$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r(x_m \ldots x_2x_1) = r(x_{m+1}x_m \ldots x_2x_1)$. If $x_m \ldots x_2x_1 \neq 0$, then $(x_m \ldots x_2x_1)R \cap r(x_{m+1}) \neq 0$(because $r(x_{m+1}) \subseteq^{ess} R_R$) and hence $0 \neq x_m \ldots x_2x_1r \in r(x_{m+1})$ for some $r \in R$. Thus $x_{m+1}x_m \ldots x_2x_1r = 0$ and this implies that $x_m \ldots x_2x_1r = 0$, a contradiction. Thus $Z_r \cap S_r$ is right t-nilpotent, so $Z_r \cap S_r \subseteq J$. Therefore, $S_r \cap J = Z_r \cap S_r$ by (1). □

Proposition 2.2.6. Let R be a right ss-injective ring. Then :

(1) If Ra is a simple left ideal of R, then $\mathrm{soc}(aR) \cap J(aR)$ is zero or simple.

(2) $rl(S_r \cap J) = S_r \cap J$ if and only if $rl(N) = N$ for all semisimple small right ideals N of R.

Proof. (1) Suppose that $\mathrm{soc}(aR) \cap J(aR)$ is a nonzero. Let x_1R and x_2R be any simple small right ideals of R with $x_i \in aR$, $i = 1, 2$. If $x_1R \cap x_2R = 0$, then by Corollary 2.2.3 (5), $l(x_1) + l(x_2) = R$. Since $x_i \in aR$, thus $x_i = ar_i$ for some $r_i \in R$, $i = 1, 2$, that is $l(a) \subseteq l(ar_i) = l(x_i)$, $i = 1, 2$. Since Ra is a simple, then $l(a) \subseteq^{max} R$, that is $l(x_1) = l(x_2) = l(a)$. Therefore, $l(a) = R$ and hence $a = 0$ and this contradicts the minimality of Ra. Thus $\mathrm{soc}(aR) \cap J(aR)$ is simple.

(2) Suppose that $rl(S_r \cap J) = S_r \cap J$ and let N be a semisimple small right ideal of R, trivially we have $N \subseteq rl(N)$. If $N \cap xR = 0$ for some $x \in rl(N)$, then by Corollary 2.2.3 (5), $l(N \cap xR) = l(N) + l(xR) = R$, since $x \in rl(N) \subseteq rl(S_r \cap J) = S_r \cap J$. If $y \in l(N)$, then $yx = 0$, that is $y(xr) = 0$ for all $r \in R$ and hence $l(N) \subseteq l(xR)$. Thus $l(xR) = R$, so $x = 0$ and this means that $N \subseteq^{ess} rl(N)$. Since $N \subseteq^{ess} rl(N) \subseteq rl(S_r \cap J) = S_r \cap J$, it follows that $N = rl(N)$. The converse is trivial. □

Lemma 2.2.7. Let R be a ring then $rl(N) = N$, for all semisimple small right ideals N of R if and only if $r(l(N) \cap Ra) = N + r(a)$, for all semisimple small right ideals N of R and all $a \in R$.

Proof. By the same argument of [2, Lemma 4.7]. □

Lemma 2.2.8. Let K be an m-generated semisimple right ideal lies over summand of R_R. If R is a right ss-injective ring, then every R-homomorphism from K to R_R can be extended to an endomorphism of R_R.

Proof. Let $\alpha: K \longrightarrow R$ be a right R-homomorphism. By hypothesis, $K = eR \oplus B$, for some $e^2 = e \in R$, where B is an m-generated semisimple small right ideal of R. Now, we need to prove that $K = eR \oplus (1-e)B$. Clearly, $eR + (1-e)B$ is a direct sum. Let $x \in K$, then $x = a + b$, for some $a \in eR, b \in B$, so we can write $x = a + eb + (1-e)b$ and this implies that $x \in eR \oplus (1-e)B$. Conversely, let $x \in eR \oplus (1-e)B$. Thus $x = a + (1-e)b$, for some $a \in eR, b \in B$. We obtain $x = a + (1-e)b = (a - eb) + b \in eR \oplus B$. It is obvious that $(1-e)B$ is an m-generated semisimple small right ideal. Since R is a right ss-injective, then there exists $\gamma \in \mathrm{End}(R_R)$ such that $\gamma_{|(1-e)B} = \alpha_{|(1-e)B}$. Define $\beta: R_R \longrightarrow R_R$ by $\beta(x) = \alpha(ex) + \gamma\big((1-e)x\big)$, for all $x \in R$ which is well defined R-homomorphism. If $x \in K$, then $x = a + b$ where $a \in eR$ and $b \in (1 - e)B$, so $\beta(x) = \alpha(ex) + \gamma\big((1 - e)x\big) = \alpha(a) + \gamma(b) = \alpha(a) + \alpha(b) = \alpha(x)$ which yields β is an extension of α. □

Corollary 2.2.9. Let R be a ring such that every finitely generated semisimple right ideal lies over a summand of R_R (in particular, R is a semiregular ring) . If R is a right ss-injective ring, then every R-homomorphism from a finitely generated semisimple right ideal to R extends to R.

Proof. By Lemma 2.2.8. □

Corollary 2.2.10. Let S_r be a finitely generated and lies over summand of R_R. Then R is a right ss-injective ring if and only if R is a right soc-injective ring.

Proof. By Lemma 2.2.8. □

Lemma 2.2.11. A ring R is a right minannihilator if and only if $rl(K) = K$ for any simple small right ideal K of R.

Proof. (⇒) This is clear.

(⇐) Let K be any simple right ideal of R. Thus either $K = eR$ for some $e^2 = e \in R$ or $K \subseteq J$ by [25, Lemma 3.8, p. 29]. If $K = eR$ with $e^2 = e$, then $rl(K) = rl(eR) = rl(e)$. Let $x \in rl(e)$. Since $e^2 = e$, thus $(e - 1)e = 0$ and hence $e - 1 \in l(e)$ and this implies that $(e - 1)x = 0$. Thus $x = ex$ and hence $x \in eR$. Therefore, $rl(e) \subseteq eR$. Since $eR \subseteq rl(e)$, thus $rl(e) = eR$ and hence R is a right minannihilator. □

Similarly, we can prove the following lemma.

Lemma 2.2.12. A ring R is a left minannihilator if and only if $lr(K) = K$ for any simple small left ideal K of R.

Corollary 2.2.13. For a right ss-injective ring R, the following hold:

(1) If $rl(S_r \cap J) = S_r \cap J$, then R is right minannihilator.
(2) If $S_\ell \subseteq S_r$, then:
(a) $S_\ell = S_r$.
(b) R is a left minannihilator ring.

Proof. (1) Let aR be a simple small right ideal of R, thus $rl(a) = aR$ by Proposition 2.2.6 (2). Therefore, R is a right minannihilator ring.

(2) (a) Since R is a right ss-injective, then it is right mininjective and it follows from [23, Proposition 1.14 (4)] that $S_\ell = S_r$.

(b) If Ra is a simple small left ideal of R, then $lr(a) = Ra$ by Corollary 2.2.3 (2) and hence R is a left minannihilator ring. □

The following two results extend the results [2, Proposition 4.6 and Theorem 4.12] from the soc-injective rings to the ss-injective rings.

Proposition 2.2.14. The following statements are equivalent for a right ss-injective ring R:

(1) $S_\ell \subseteq S_r$.
(2) $S_\ell = S_r$.
(3) R is a left mininjective ring.

Proof. (1)⇒(2) By Corollary 2.2.13 (2) (a).

(2)⇒(3) By Corollary 2.2.13 (2) and [24, Corollary 2.34, p. 53], we must show that R is right minannihilator ring. Let aR be a simple small right ideal, then Ra is a simple small left ideal by [23, Theorem 1.14]. Let $0 \neq x \in rl(aR)$, then $l(a) \subseteq l(x)$. Since $l(a) \subseteq^{max} R$, thus $l(a) = l(x)$ and hence Rx is simple left ideal, that is $x \in S_r$. Now, if $Rx = Re$ for some $e^2 = e \in R$, then $e = rx$ for some $0 \neq r \in R$. Since $(e-1)e = 0$, then $(e-1)rx = 0$, that is $(e-1)ra = 0$ and this implies that $ra \in eR$. Thus $raR \subseteq eR$, but eR is semisimple right ideal, so $raR \subseteq^{\oplus} R$ and hence $ra = 0$. Therefore, $rx = 0$, that is $e = 0$, a contradiction. Thus $x \in J$ and hence $x \in S_r \cap J$. Therefore, $aR \subseteq rl(aR) \subseteq S_r \cap J$. Now, let $aR \cap yR = 0$ for some $y \in rl(aR)$, thus $l(aR) + l(yR) = l(aR \cap yR) = R$. Since $y \in rl(aR)$, thus $l(aR) \subseteq l(yR)$ and hence $l(yR) = R$, that is $y = 0$. Therefore, $aR \subseteq^{ess} rl(aR)$, so $aR = rl(aR)$ as desired.

(3)⇒(1) Follows from [24, Corollary 2.34, p. 53]. □

Corollary 2.2.15. Let R be a right ss-injective ring, semiperfect with $S_r \subseteq^{ess} R_R$. Then R is a right minfull ring and the following statements hold:

(1) Every simple right ideal of R is essential in a summand.
(2) $\mathrm{soc}(eR)$ is simple and essential in eR for every local idempotent $e \in R$. Moreover, R is right finitely cogenerated.
(3) For every semisimple right ideal I of R, there exists $e^2 = e \in R$ such that $I \subseteq^{ess} rl(I) \subseteq^{ess} eR$.

(4) $S_r \subseteq S_\ell \subseteq rl(S_r)$.

(5) If I is a semisimple right ideal of R and aR is a simple right ideal of R with $I \cap aR = 0$, then $rl(I \oplus aR) = rl(I) \oplus rl(aR)$.

(6) $rl(\bigoplus_{i=1}^{n} a_iR) = \bigoplus_{i=1}^{n} rl(a_iR)$, where $\bigoplus_{i=1}^{n} a_iR$ is a direct sum of simple right ideals.

(7) The following statements are equivalent:

(a) $S_r = rl(S_r)$.

(b) $K = rl(K)$, for every semisimple right ideals K of R.

(c) $kR = rl(kR)$, for every simple right ideals kR of R.

(d) $S_r = S_\ell$.

(e) $\mathrm{soc}(Re)$ is a simple for all local idempotent $e \in R$.

(f) $\mathrm{soc}(Re) = S_re$, for all local idempotent $e \in R$.

(g) R is a left mininjective.

(h) $L = lr(L)$, for every semisimple left ideals L of R.

(i) R is a left minfull ring.

(j) $S_r \cap J = rl(S_r \cap J)$.

(k) $K = rl(K)$, for every semisimple small right ideals K of R.

(l) $L = lr(L)$, for every semisimple small left ideals L of R.

(8) If R satisfies any condition of (7), then $r(S_\ell \cap J) \subseteq^{ess} R_R$.

Proof. (1), (2), (3), (4), (5) and (6) are obtained by Corollary 2.1.32 (1) and [2, Theorem 4.12].

(7) The equivalence of (a), (b), (c), (d), (e), (f), (g), (h) and (i) follows from Corollary 2.1.32 (1) and [2, Theorem 4.12].

(b)⇒(j) Clear.

(j)⇔(k) By Proposition 2.2.6 (2).

(k)⇒(c) By Corollary 2.2.13 (1).

(h)⇒(l) Clear.

(l)⇒(d) Let Ra be a simple left ideal of R. By hypothesis, $lr(A) = A$ for any simple small left ideal A of R. By Lemma 2.2.12, $lr(A) = A$, for any simple left ideal A of R and hence $lr(Ra) = Ra$. Thus R is a right min-PF ring and it follows from [23, Theorem 3.14] that $S_r = S_\ell$.

(8) Let K be a right ideal of R such that $r(S_\ell \cap J) \cap K = 0$. Then $K\,r(S_\ell \cap J) = 0$ and we have $K \subseteq lr(S_\ell \cap J) = S_\ell \cap J = S_r \cap J$. Now, $r\big((S_\ell \cap J) +$

$l(K)\big) = r(S_\ell \cap J) \cap K = 0$. Since R is left Kasch, then $(S_\ell \cap J) + l(K) = R$ by [17, Corollary 8.28, p. 281]. Thus $l(K) = R$ and hence $K = 0$, so $r(S_\ell \cap J) \subseteq^{ess} R_R$. □

Recall that a ring R is said to be right V-ring if every simple right R-module is injective; equivalently, if $J(N) = 0$ for all $N \in$ Mod-R (see [17, p. 97 and 99].

N. Zeyada, S. Hussein and A. Amin [38] introduced the notion almost-injective, a right R-module M is called almost-injective if $M = E \oplus K$, where E is injective and K has zero radical. They proved that, every almost-injective right R-module is an injective if and only if every almost-injective is a quasi-continuous if and only if R is a semilocal ring (see [38, Theorem 2.12]). After reflect of [38, Theorem 2.12] we found it is not true always, so most of the other results in [38] are not necessary to be correct, because they are based on [38, Theorem2.12]. The following example shows that the contradiction in [38, Theorem 2.12] is exist.

Example 2.2.16.

(1) Let R be an artinian ring. Assume that R is not semisimple ring, then R is not right V-ring. Thus there is simple right R-module is not injective. Therefore, there is almost-injective right R-module is not injective. So it follows from [38, Theorem 2.12] that R is not semilocal. Hence, R is not right artinian and this a contradiction. Thus every right artinian ring is semisimple, but this is not true in general (see below example).

(2) The ring $\mathbb{Z}_8$ is semilocal. Since $< \bar{4} >= \{\bar{0}, \bar{4}\}$ is almost-injective as $\mathbb{Z}_8$-module, then $< \bar{4} >$ is injective $\mathbb{Z}_8$-module by [38, Theorem 2.12]. Thus $< \bar{4} > \subseteq^{\oplus} \mathbb{Z}_8$ and this a contradiction.

The following Theorem is a new version of [38, Theorem 2.12] in terms of ss-injectivity.

Theorem 2.2.17. The following statements are equivalent for a ring R:

(1) R is a semiprimitive and every almost-injective right R-module is quasi-continuous.

(2) R is a right ss-injective and right minannihilator ring, J is a right artinian, and every almost-injective right R-module is quasi-continuous.

(3) R is a semisimple ring.

Proof. (1)⇒(2) and (3)⇒(1) are clear.

(2)⇒(3) Let M be a right R-module with zero Jacobson radical and let K be a nonzero submodule of M. Thus $K \oplus M$ is a quasi-continuous module. By [21, Corollary 2.14, p. 23], K is an M-injective. Thus $K \subseteq^{\oplus} M$ and hence M is semisimple. In particular, R/J is a semisimple R-module and hence R/J is artinian by [16, Theorem 9.2.2 (b), p. 219], so R is semilocal ring. Since J is a right artinian, then R is a right artinian. So, it follows from Corollary 2.2.15 (7) that R is right and left mininjective. Thus [23, Corollary 4.8] implies that R is QF ring. By hypothesis $R \oplus (R/J)$ is quasi-continuous (since R is self-injective), so again by [21, Corollary 2.14, p. 23] we have that R/J is an injective. Since R is QF ring, then R/J is a projective (see Proposition 1.2.11). Thus the canonical map $\pi: R \longrightarrow R/J$ is a splits and hence $J \subseteq^{\oplus} R$, that is $J = 0$. Therefore, R is semisimple. □

Note. It is mentioned in [37], that the result [38, Theorem 2.12] is not true but they didn't give a counterexample.

Section Three

2.3 SS-Flat Modules

In this section, we will introduce the dual concept to ss-injective module namely, ss-flat module. We will give some results in terms of ss-injectivity and ss-flatness.

Definition 2.3.1. A left R-module M is said to be ss-flat if $\mathrm{Tor}_1(R/(S_r \cap J), M) = 0$.

Example 2.3.2.

(1) Any flat module is ss-flat, but the converse is not true. For example the $\mathbb{Z}$-module $\mathbb{Z}_n$ is not flat for all $n \geq 2$ (see [6,Example , p.155]), but it is clear that $\mathbb{Z}_n$ as $\mathbb{Z}$-module is ss-flat for all $n \geq 2$.

(2) Every ss-flat module is min-flat, since if M is an ss-flat left R-module, then M^+ is an ss-injective right R-module (by Lemma 2.3.3) and hence from Lemma 2.1.5 we have that M^+ is right mininjective. By Lemma 1.2.54, M is min-flat.

(3) In the Björk Example (Example 2.2.4) we have that the ring R is right mininjective ring but not right ss-injective ring. If $\dim({}_{\bar{F}}F)$ is finite, then R right artinian by [2, Example 4.15]. Therefore, R is a right coherent ring. Thus R^+ is a left min-flat R-module by [19, Theorem 4.5], but the left R-module R^+ is not ss-flat by Theorem 2.3.10 below.

Lemma 2.3.3. The following statements are equivalent :

(1) M is an ss-flat left R-module.

(2) M^+ is an ss-injective right R-module.

(3) $\mathrm{Tor}_1(R/A, M) = 0$, for every semisimple small right ideal A of R.

(4) $\mathrm{Tor}_1(R/B, M) = 0$, for every finitely generated semisimple small right ideal B of R.

(5) The sequence $0 \longrightarrow (S_r \cap J)\otimes_R M \longrightarrow R_R \otimes_R M$ is exact.

(6) The sequence $0 \longrightarrow A\otimes_R M \longrightarrow R_R \otimes_R M$ is exact for every finitely generated semisimple small right ideal A of R.

Proof. (1)⇔(2) This follows from $\mathrm{Ext}^1(R/(S_r \cap J)\ , M^+) \cong \mathrm{Tor}_1(R/(S_r \cap J)\ , M)^+$ (see Theorem 1.2.31).

(2)⇒(3) Let A be a semisimple small right ideal of R. By Theorem 1.2.31 and Corollary 2.1.7 $\mathrm{Tor}_1(R/A\ , M)^+ \cong \mathrm{Ext}^1(R/A\ , M^+) = 0$. Thus $\mathrm{Tor}_1(R/A\ , M) = 0$, since $\mathbb{Q}/\mathbb{Z}$ is injective cogenerator.

(3)⇒(1) This is clear.

(4)⇒(3) Let I be a semisimple small right ideal of R, so $I = \varinjlim I_i$, where I_i is a finitely generated semisimple small right ideal of R, $f_{ij}: I_i \longrightarrow I_j$ is the inclusion map, and (I_i, f_{ij}) is a direct system (see [12, Example 1.5.5 (2), p. 32]). Clearly, $(R/I_i\ , h_{ij})$ is a direct system of R-modules where $h_{ij}: R/I_i \longrightarrow R/I_j$ is defined by $h_{ij}(a + I_i) = a + I_j$ with direct limit $(h_i\ , \varinjlim R/I_i)$. Since the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_i & \xrightarrow{i_i} & R & \xrightarrow{\pi_i} & R/I_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{ij}} & & \| & & \downarrow{\scriptstyle h_{ij}} & & \\ 0 & \longrightarrow & I_j & \xrightarrow{i_j} & R & \xrightarrow{\pi_j} & R/I_j & \longrightarrow & 0 \end{array}$$

where i_i and π_i are the inclusion and canonical maps, respectively. By [32, 24.6, p. 200], we have the exact sequence $0 \longrightarrow I \xrightarrow{i} R \xrightarrow{u} \varinjlim R/I_i \longrightarrow 0$. It follows from [32, 24.4, p. 199] that the following diagram is commutative:

$$\begin{array}{ccccc} R & \xrightarrow{\pi_i} & R/I_i & \longrightarrow & 0 \\ \| & & \downarrow{\scriptstyle h_i} & & \\ R & \xrightarrow{u} & \varinjlim R/I_i & \longrightarrow & 0 \end{array}$$

Thus the family of mappings $\left\{g_i \mid \quad g_i: R/I_i \longrightarrow R/\varinjlim I_i\ , \text{where } g_i(a + I_i) = a + \varinjlim I_i \right\}$ forms a direct system of homomorphisms, since for $i \leq j$, we get $g_j h_{ij}(a + I_i) = g_j\big(a + I_j\big) = a + \varinjlim I_i = g_i(a + I_i)$ for all $a + I_i \in R/I_i$.

Thus there is an R-homomorphism α such that the following diagram is commutative with short exact rows (see Definition 1.2.42):

$$\begin{array}{ccccccccc} & & & & \multicolumn{3}{c}{\overset{u}{\longrightarrow}} & & \\ 0 \longrightarrow & I & \overset{i}{\longrightarrow} & R & \overset{\pi_i}{\longrightarrow} & R/I_i & \overset{h_i}{\longrightarrow} & \varinjlim R/I_i & \longrightarrow 0 \\ & \| & & \| & & \| & & \downarrow \alpha & \\ 0 \longrightarrow & I & \overset{i}{\longrightarrow} & R & \overset{\pi_i}{\longrightarrow} & R/I_i & \overset{g_i}{\longrightarrow} & R/\varinjlim I_i & \longrightarrow 0 \\ & & & & \multicolumn{3}{c}{\underset{\pi}{\longrightarrow}} & & \end{array}$$

where π is the canonical map, so it follows from [3, Exercise 11 (1), p. 52] that $\varinjlim R/I_i \cong R/\varinjlim I_i$. Therefore,

$$\begin{aligned} \mathrm{Tor}_1(R/I\,, M) &= \mathrm{Tor}_1\left(R/\varinjlim I_i\,, M\right) \\ &\cong \mathrm{Tor}_1\left(\varinjlim R/I_i\,, M\right) \quad \text{(by [13, Theorem XII.5.4 (4), p. 494])} \\ &\cong \varinjlim \mathrm{Tor}_1(R/I_i\,, M) = 0 \quad \text{(by [26, Proposition 7.8, p. 410]).} \end{aligned}$$

(3)⇒(4) Clear.

(1)⇔(5) By [13, Theorem XII.5.4 (3), p. 494], we have the exact sequence $0 \longrightarrow \mathrm{Tor}_1(R/(S_r \cap J)\,, M) \longrightarrow (S_r \cap J) \otimes_R M \longrightarrow R_R \otimes_R M.$ Thus the equivalence between (1) and (5) is true.

(4)⇔(6) is similar to ((1)⇔(5)). □

In the following, we will use the symbol SSI (resp. SSF) to denote the classes of ss-injective right (resp. ss-flat left) R-modules.

Corollary 2.3.4. The pair (SSF, SSI) is an almost dual pair.

Proof. By Lemma 2.3.3 and Theorem 2.1.3 (1) and (5).

Lemma 2.3.5. For a ring R, the following statements hold:

(1) If $S_r \cap J$ is finitely generated, then every pure submodule of ss-injective right R-module is ss-injective.
(2) Every pure submodule of ss-flat left R-module is ss-flat.

(3) Every direct limts (direct sums) of ss-flat left R-modules is ss-flat.
(4) If M, N are left R-modules, $M \cong N$, and M is ss-flat, then N is ss-flat.

Proof. (1) Let M be an ss-injective right R-module and N be a pure submodule of M. Since $R/(S_r \cap J)$ is a finitely presented, thus the sequence $\mathrm{Hom}_R(R/(S_r \cap J)\ , M) \longrightarrow \mathrm{Hom}_R(R/(S_r \cap J)\ , M/N) \longrightarrow 0$ is exact. By [13, Theorem XII.4.4 (4), p. 491], we have the exact sequence $\mathrm{Hom}_R(R/(S_r \cap J)\ , M) \longrightarrow \mathrm{Hom}_R(R/(S_r \cap J), M/N) \longrightarrow \mathrm{Ext}^1(R/(S_r \cap J), N) \longrightarrow \mathrm{Ext}^1(R/(S_r \cap J), M)$ which leads to $\mathrm{Ext}^1(R/(S_r \cap J)\ , N) = 0$. Hence N is an ss-injective right R-module.

(2), (3) and (4) By Corollary 2.3.4 and [20, Proposition 4.2.8, p. 70]. □

In the following definition, we will introduce the concept of ss-coherent ring as a generalization of coherent ring

Definition 2.3.6. A ring R is said to be right ss-coherent ring, if R is a right min-coherent and $S_r \cap J$ is finitely generated; equivalently, if $S_r \cap J$ is finitely presented.

Example 2.3.7.

(1) Every coherent ring is ss-coherent.
(2) Every ss-coherent ring is min-coherent.
(3) Let R be a commutative ring, then the polynomial ring $R[x]$ is not coherent ring with zero socle by [19, Remark 4.2 (3)]. Hence $R[x]$ is an ss-coherent ring but not coherent.

Corollary 2.3.8. A right ideal $S_r \cap J$ of a ring R is finitely generated if and only if every FP-injective right R-module is an ss-injective.

Proof. By Proposition 1.2.50. □

In the next theorem we give a new characterizations of min-coherent rings in terms of ss-injective and ss-flat modules.

Theorem 2.3.9. The following statements are equivalent for a ring R:

(1) R is a right min-coherent ring.
(2) If M is an ss-injective right R-module, then M^+ is ss-flat.
(3) If M is an ss-injective right R-module, then M^{++} is ss-injective.
(4) A left R-module N is ss-flat if and only if N^{++} is ss-flat.

(5) SSF is closed under direct products.

(6) ${}_R R^S$ is ss-flat for any index set S.

(7) $\mathrm{Ext}^2(R/I, M) = 0$ for every FP-injective right R-module M and every finitely generated semisimple small right ideal I.

(8) If $0 \longrightarrow N \longrightarrow M \longrightarrow H \longrightarrow 0$ is an exact sequence of right R-modules with N is FP-injective and M is ss-injective, then $\mathrm{Ext}^1(R/I, H) = 0$ for every finitely generated semisimple small right ideal I.

(9) Every left R-module has an (SSF)-preenvelope.

(10) If $\alpha: M \longrightarrow N$ is an (SSI)-preenvelope of a right R-module M, then $\alpha^+: N^+ \longrightarrow M^+$ is an (SSF)-precover of M^+.

(11) For any positive integer n and any $b_1, \dots, b_n \in S_r \cap J$, then the right ideal $\{r \in R: b_1 r + b_2 r_2 + \cdots + b_n r_n = 0$ for some $r_2, \cdots, r_n \in R\}$ is finitely generated.

(12) For any finitely generated semisimple small right ideal A of R and any $x \in S_r \cap J$, then $\{r \in R | xr \in A\}$ is finitely generated.

(13) $r(x)$ is finitely generated for any simple right ideal xR.

(14) Every simple submodule of a projective right R-module is finitely presented.

Proof. (1)⇒(2) Let I be a finitely generated semisimple small right ideal of R, thus there is an exact sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} I \longrightarrow 0$ in which F_i is a finitely generated free right R-module, $i = 1,2$ by hypothesis. Therefore, the sequence $F_2 \xrightarrow{\alpha_2} F_1 \xrightarrow{\beta} R \xrightarrow{\pi} R/I \longrightarrow 0$ is exact, where $i: I \longrightarrow R$ and $\pi: R \longrightarrow R/I$ are the inclusion and the canonical maps, respectively and $\beta = i\alpha_1$. Thus R/I is 2-presented and hence Lemma 1.2.41 implies that $\mathrm{Tor}_1(R/I, M^+) \cong \mathrm{Ext}^1(R/I, M)^+ = 0$. Therefore, M^+ is an ss-flat left R-module.

(2)⇒(3) By Lemma 2.3.3.

(3)⇒(4) Assume that N is an ss-flat left R-module, thus N^+ is an ss-injective by Lemma 2.3.3 and this implies that N^{+++} is an ss-injective by (3). So N^{++} is an ss-flat by Lemma 2.3.3 again. The converse is obtained by Theorem 1.2.47 (1) and Lemma 2.3.5 (2).

(4)⇒(5) By (4), $(SSF)^{++} \subseteq SSF$. Since (SSF, SSI) is an almost dual pair (by Corollary 2.3.4), thus [20, Proposition 4.3.1 and Proposition 4.2.8 (3), p. 85 and 70] implies that SSF is closed under direct products.

(5)⇒(6) Obvious.

(6)⇒(1) Since every ss-flat left R-module is min-flat, thus the result follows from [19, Theorem 4.5].

(1)⇒(7) Let I be a finitely generated semisimple small right ideal of R and let M be a FP-injective right R-module. By [13, Theorem XII.4.4 (3), p. 491], we get the exact sequence $\mathrm{Ext}^1(I, M) \longrightarrow \mathrm{Ext}^2(R/I\,, M) \longrightarrow \mathrm{Ext}^2(R, M)$. But $\mathrm{Ext}^1(I, M) = 0$ (since M is FP-injective and I is a finitely presented) and $\mathrm{Ext}^2(R, M) = 0$ (since R is projective). Thus $\mathrm{Ext}^2(R/I\,, M) = 0$.

(7)⇒(8) If $0 \longrightarrow N \longrightarrow M \longrightarrow H \longrightarrow 0$ is an exact sequence of right R-modules, where N is FP-injective and M is ss-injective and let I be a finitely generated semisimple small right ideal of R. By [13, Theorem XII.4.4 (4)), p. 491], we get an exact sequence $0 = \mathrm{Ext}^1(R/I\,, M) \longrightarrow \mathrm{Ext}^1(R/I\,, H) \longrightarrow \mathrm{Ext}^2(R/I\,, N) = 0$. Thus $\mathrm{Ext}^1(R/I\,, H) = 0$ for every finitely generated semisimple small right ideal I of R.

(8)⇒(1) Let N be a FP-injective right R-module, thus we have the exact sequence $0 \longrightarrow N \longrightarrow E(N) \longrightarrow E(N)/N \longrightarrow 0$. Let I be a finitely generated semisimple small right ideal of R, thus $\mathrm{Ext}^1(R/I\,, E(N)/N) = 0$ by hypothesis. So it follows from [13, Theorem XII.4.4 (4), p. 491] that the sequence $0 = \mathrm{Ext}^1(R/I\,, E(N)/N) \longrightarrow \mathrm{Ext}^2(R/I\,, N) \longrightarrow \mathrm{Ext}^2(R/I\,, E(N)) = 0$ is exact, and so $\mathrm{Ext}^2(R/I\,, N) = 0$. Hence we have the exact sequence $0 = \mathrm{Ext}^1(R, N) \longrightarrow \mathrm{Ext}^1(I, N) \longrightarrow \mathrm{Ext}^2(R/I\,, N) = 0$ (see [13, Theorem XII.4.4 (3)), p. 491]). Thus $\mathrm{Ext}^1(I, N) = 0$ and this implies that I is a finitely presented (see Remark 1.2.51). Therefore, R is a right min-coherent.

(5)⇔(9) By Corollary 2.3.4 and [20, Proposition 4.2.8 (3), p. 70].

(2)⇒(10) Since $(SSI)^+ \subseteq SSF$ (by hypothesis) and $(SSF)^+ \subseteq SSI$ (by Lemma 2.3.3), thus the result follows from [11, Corollary 3.2, p. 1137].

(10)⇒(2) By taking M is an ss-injective right R-module in (10).

(1)⇒(11) Let $b_1\,, b_2\,, \dots, b_n \in S_r \cap J$. Put $K_1 = b_1R + b_2R + \cdots + b_nR$ and $K_2 = b_2R + \cdots + b_nR$. Thus $K_1 = b_1R + K_2$. Define $f: R \longrightarrow K_1/K_2$ by $f(r) = b_1r + K_2$ which is a well-define R-epimorphism, because if $r_1 = r_2 \in R$, then $b_1r_1 - b_1r_2 = 0 \in K_2$, that is $b_1r_1 + K_2 = b_1r_2 + K_2$. Now, we have that $\ker(f) = \{r \in R | b_1r + K_2 = K_2\} = \{r \in R|\, b_1r \in K_2\} = \{r \in R|\, b_1\, r + b_2\, r_2\, + \cdots + b_n\, r_n = 0 \text{ for some } r_2, \cdots, r_n \in R\}$. By (1) and using [17, Lemma

4.54, p. 141], we have that K_1/K_2 is a finitely presented. But $R/\ker(f) \cong K_1/K_2$, so $\ker(f)$ is finitely generated.

(11)⇒(12) Let $x \in S_r \cap J$ and A be any finitely generated semisimple small right ideal of R, then $A = \bigoplus_{i=1}^{n} a_i R$, so we have that $\{r \in R | xr \in A\} = \{r \in R |\ x\, r + a_1\, r_1 + \cdots + a_n\, r_n = 0 \text{ for some } r_1, \cdots, r_n \in R\}$ is finitely generated by hypothesis.

(12)⇒(13) By taking $A = 0$.

(13)⇒(1) Let xR be a simple right ideal. Since $r(x)$ is finitely generated and $xR \cong R/r(x)$, thus xR is finitely presented.

(1)⇒(14) Let $S_r = \bigoplus_{i \in I} a_i R$, where $a_i R$ is a simple right ideal for each $i \in I$. If P is a projective right R-module, then P is isomorphic to a direct summand of $R^{(S)}$ for some index set S. Let A be any simple submodule of P, then $A \cong B \leq^{\oplus} \bigoplus_S S_r = \bigoplus_S \bigoplus_{i \in I} a_i R$. Since A is finitely generated, then there are finite index sets $S_0 \subseteq S$ and $I_0 \subseteq I$ such that $A \cong B \leq^{\oplus} \bigoplus_{S_0} \bigoplus_{i \in I_0} a_i R$, so it follows from [17, Lemma 4.54, p. 141] that A is finitely presented.

(14)⇒(1) Clear. □

Theorem 2.3.10. The following statements are equivalent for a ring R:

(1) R is a right ss-coherent ring.
(2) A right R-module M is ss-injective if and only if M^+ is ss-flat.
(3) A right R-module M is ss-injective if and only if M^{++} is ss-injective.
(4) SSI is closed under direct limits.
(5) $S_r \cap J$ is finitely generated and every pure quotient of ss-injective right R-module is ss-injective.
(6) The following two conditions hold:
    (a) Every right R-module has an (SSI)-cover.
    (b) Every pure quotient of ss-injective right R-module is ss-injective.

Proof. (1)⇒(2) Let M^+ be an ss-flat, then M^{++} is an ss-injective by Lemma 2.3.3, so it follows from Theorem 1.2.47 (1) and Lemma 2.3.5 (1) that M is ss-injective. The converse is obtained by Theorem 2.3.9.

(2)⇒(3) Let M^{++} be an ss-injective, thus M^+ is an ss-flat by Lemma 2.3.3 and hence M is ss-injective by hypothesis. The converse is true by Theorem 2.3.9.

(3)⇒(1) Let M be an FP-injective right R-module, then the exact sequence $0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0$ is pure by [28, Proposition 2.6], so it follows from Theorem 1.2.44 that the sequence $0 \longrightarrow M^{++} \longrightarrow E(M)^{++} \longrightarrow (E(M)/M)^{++} \longrightarrow 0$ is split. Since $E(M)^{++}$ is an ss-injective by hypothesis, thus M^{++} is ss-injective and hence M is an ss-injective by hypothesis again. Therefore, $S_r \cap J$ is finitely generated by Corollary 2.3.8, and so $S_r \cap J$ is finitely presented by Theorem 2.3.9. Thus R is a right ss-coherent ring.

(1)⇒(4) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a direct system of ss-injective right R-modules. Since $S_r \cap J$ is finitely presented, then $R/S_r \cap J$ is 2-presented, so it follows from [9, Lemma 2.9 (2)] that $\mathrm{Ext}^1\left(R/(S_r \cap J), \varinjlim M_\lambda\right) \cong \varinjlim \mathrm{Ext}^1(R/(S_r \cap J), M_\lambda)$ $= 0$. Hence $\varinjlim M_\lambda$ is ss-injective.

(4)⇒(2) Let $\{E_i : i \in I\}$ be a family of injective right R-modules. Since $\bigoplus_{i\in I} E_i = \varinjlim \{\bigoplus_{i\in I_0} E_i : I_0 \subseteq I, I_0 \text{ finite} \}$ (see [32, p. 206]), then $\bigoplus_{i\in I} E_i$ is ss-injective and hence $S_r \cap J$ is a finitely generated by Corollary 2.1.26. By Lemma 2.3.5, SSI is closed under pure submodules. Since SSI is closed under direct products (by Theorem 2.1.3 (1)) and since SSI is closed under direct limlts (by hypothesis), thus SSI is a definable class. By [20, Proposition 4.3.8, p. 89], (SSI, SSF) is an almost dual pair and hence a right R-module M is an ss-injective if and only if M^+ is an ss-flat

(2)⇒(5) By the equivalence between (1) and (2), we have that $S_r \cap J$ is a finitely generated. Now, let $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ be a pure exact sequence of right R-modules with M is ss-injective, so it follows from Theorem 1.2.44 that the sequence $0 \longrightarrow (M/N)^+ \longrightarrow M^+ \longrightarrow N^+ \longrightarrow 0$ is split. By hypothesis, M^+ is ss-flat, so $(M/N)^+$ is ss-flat. Thus M/N is ss-injective by hypothesis again.

(5)⇒(4) Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a direct system of ss-injective right R-modules. By [32, 33.9 (2), p. 279], there is a pure exact sequence $\bigoplus_{\lambda\in\Lambda} M_\lambda \longrightarrow \varinjlim M_\lambda \longrightarrow 0$. Since $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is ss-injective by Corollary 2.1.26, thus $\varinjlim M_\lambda$ is ss-injective by hypothesis.

(5)⇔(6) By Corollary 2.1.26 and Theorem 1.2.46. □

Corollary 2.3.11. A ring R is ss-coherent if and only if it is min-coherent and the class SSI is closed under pure submodules.

Proof. (⇒) Suppose that R is an ss-coherent ring, thus R is a min-coherent and $S_r \cap J$ is a finitely generated right ideal of R. By Lemma 2.3.5 (1), SSI is closed under pure submodules.

(⇐) Let M be any ss-injective right R-module. Since R is a min-coherent, thus Theorem 2.3.9 implies that M^+ is an ss-flat. Conversely, let M be any right R-module such that M^+ is ss-flat. By Lemma 2.3.3, M^{++} is an ss-injective. Since M is a pure submodule of M^{++} (by Theorem 1.2.47 (1)) and since SSI is a closed under pure submodule (by hypothesis) it follows that M is an ss-injective. Hence for any right R-module M, we have that M is an ss-injective if and only if M^+ is an ss-flat. Thus Theorem 2.3.10 implies that R is an ss-coherent. □

Corollary 2.3.12. For a right min-coherent ring R, the following statements are equivalent:

(1) Every ss-flat left R-module is flat.
(2) Every ss-injective right R-module is FP-injective.
(3) Every ss-injective pure injective right R-module is injective.

Proof. (1)⇒(2) For any ss-injective right R-module M, then M^+ is ss-flat by Theorem 2.3.9, and so M^+ is flat by hypothesis. Thus M^{++} is an injective by Proposition 1.2.36. Since M is a pure submodule of M^{++}, then M is an FP-injective by [32, 35.8, p. 301].

(2)⇒(3) By [28, Proposition 2.6] and Theorem 1.2.45.

(3)⇒(1) Assume that N is an ss-flat left R-module, thus N^+ is an ss-injective pure injective by Lemma 2.3.3 and Theorem 1.2.47 (2). Thus N^+ is an injective, and so N is a flat by Proposition 1.2.36. □

Proposition 2.3.13. For a right ss-coherent ring R, the following statements are equivalent:

(1) R is a right ss-injective ring.
(2) Every left R-module has a monic ss-flat preenvelope.
(3) Every right R-module has epic ss-injective cover.
(4) Every injective left R-module is ss-flat.

(5) Every flat right R-module is ss-injective.

Proof. (1)⇒(2) Let N be a left R-module, then there is an epimorphism $\alpha: R_R^{(S)} \longrightarrow N^+$ for some index set S by [26, Theorem 2.35, p. 58], and so there is an R-monomorphism $g: N \longrightarrow (R_R^+)^S$ by applying [13, Proposition XI.2.3, p. 420], [32, 11.10 (2) (ii), p. 87] and Theorem 1.2.47 (1), respectively. In the other hand, N has ss-flat preenvelope $f: N \longrightarrow F$ by Theorem 2.3.9. Since $(R_R^+)^S$ is ss-flat by Theorem 2.3.9 again, thus there is an R-homomorphism $h: F \longrightarrow (R_R^+)^S$ such that $hf = g$, so this means that f is an R-monomorphism.

(2)⇒(4) Let N be an injective left R-module, then there is an R-monomorphism $f: N \longrightarrow F$ with F is ss-flat. But $N \cong f(N) \subseteq^{\oplus} F$, so we have that N is ss-flat by Lemma 2.3.5 (4).

(4)⇒(5) Let M be a flat right R-module, then M^+ is an injective and hence ss-flat. Thus M is ss-injective by Theorem 2.3.10.

(5)⇒(1) Obvious, since R_R is flat.

(1)⇒(3) Let M be any right R-module, then M has ss-injective cover, say, $g: N \longrightarrow M$ by Theorem 2.3.10. By [26, Theorem 2.35, p. 58], there is an R-epimorphism $f: R_R^{(S)} \longrightarrow M$ for some index set S. Since $R_R^{(S)}$ is ss-injective by Corollary 2.1.26, then there is a R-homomorphism $h: R_R^{(S)} \longrightarrow N$ such that $gh = f$, so g is an R-epimorphism.

(3)⇒(1) Let $f: N \longrightarrow R_R$ be an epic ss-injective cover. Since R_R is a projective, then there is an R-homomorphism $g: R_R \longrightarrow N$ such that $fg = I_R$, thus f is split, and so $N = \ker(f) \oplus B$ for some ss-injective submodule B of N. Therefore $R_R \cong N/\ker(f) \cong B$ is ss-injective. □

Proposition 2.3.14. The class SSI is closed under cokernels of homomorphisms if and only if $\operatorname{coker}(\alpha)$ is an ss-injective for every ss-injective right R-module M and $\alpha \in \operatorname{End}(M)$.

Proof. (⇒) Clear.

(⇐) Let A and B be any ss-injective right R-modules and f be any R-homomorphism from A to B. Define $\alpha: A \oplus B \longrightarrow A \oplus B$ by $\alpha\big((x,y)\big) = \big(0, f(x)\big)$. Therefore, $(A \oplus B)/\operatorname{im}(\alpha) \cong (A \oplus B)/(0 \oplus \operatorname{im}(f)) \cong A \oplus (B/\operatorname{im}(f))$ is an ss-injective. Thus $B/\operatorname{im}(f)$ is an ss-injective. □

Proposition 2.3.15. The class SSF is closed under kernels of homomorphisms if and only if $\ker(\alpha)$ is ss-flat, for every ss-flat left R-module M and $\alpha \in \mathrm{End}(M)$.

Proof. (⇒) Clear.

(⇐) Let $g: N \longrightarrow M$ be any R-homomorphism with N and M are ss-flat left R-modules. Define $\alpha: N \oplus M \longrightarrow N \oplus M$ by $\alpha\big((a,b)\big) = \big(0, g(a)\big)$. Thus $\ker(\alpha)$ $= \ker(g) \oplus M$ is ss-flat by hypothesis and hence $\ker(g)$ is an ss-flat. □

Theorem 2.3.16. If R is a commutative ring, then the following statements are equivalent:

(1) R is a min-coherent ring.
(2) $\mathrm{Hom}_R(M, N)$ is an ss-flat for all ss-injective R-modules M and all injective R-modules N.
(3) $\mathrm{Hom}_R(M, N)$ is an ss-flat for all injective R-modules M and N.
(4) $\mathrm{Hom}_R(M, N)$ is an ss-flat for all projective R-modules M and N.
(5) $\mathrm{Hom}_R(M, N)$ is an ss-flat for all projective R-modules M and all ss-flat R-modules N.

Proof. (1)⇒(2) If I is a finitely generated semisimple small ideal of R, then I is finitely presented. By [13, Theorem XII.4.4 (3)), p. 491], we have the exact sequence $0 \longrightarrow \mathrm{Hom}_R(R/I\,, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow 0$. Thus the sequence $0 \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(I, M), N) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(R, M), N) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(R/I\,, M), N) \longrightarrow 0$ is exact by [13, Theorem XII.4.4 (3)), p. 491] again. Thus we have the exact sequence $0 \longrightarrow \mathrm{Hom}_R(M, N)\otimes_R I \longrightarrow \mathrm{Hom}_R(M, N)\otimes_R R \longrightarrow \mathrm{Hom}_R(M, N)\otimes_R (R/I) \longrightarrow 0$ by [12, Theorem 3.2.11, p. 78] and this implies that $\mathrm{Hom}_R(M, N)$ is an ss-flat.

(2)⇒(3) Clear.

(3)⇒(1) By [6, Proposition 2.3.4, p. 66] and [26,Theorem 2.75, p. 92], we have that $(R^{++})^S \cong \big(\mathrm{Hom}_{\mathbb{Z}}(R^+\otimes_R R, \mathbb{Q}/\mathbb{Z})\big)^S \cong \big(\mathrm{Hom}_R(R^+, R^+)\big)^S$ for any index set S. Thus $(R^{++})^S \cong \mathrm{Hom}_R(R^+, (R^+)^S)$ is an ss-flat for any index set S by [32, 11.10 (2), p. 87] and since R^+ and $(R^+)^S$ are injective. Since R^S is a pure submodule of $(R^{++})^S$ by Theorem 1.2.47 (1) and [7, Lemma 1 (2)], so it follows from Lemma 2.3.5 (2) that R^S is an ss-flat for any index set S. Thus (1) follows from Theorem 2.3.9.

(1)⇒(5) Since M is a projective R-module, thus there is a projective R-module P such that $M \oplus P \cong R^{(S)}$ for some index set S. Therefore, $\mathrm{Hom}_R(M,N) \oplus \mathrm{Hom}_R(P,N) \cong \mathrm{Hom}_R(R^{(S)},N) \cong \left(\mathrm{Hom}_R(R,N)\right)^S \cong N^S$ by [32, 11.10 and 11.11, p. 87 and 88]. But N^S is an ss-flat by Theorem 2.3.9, thus $\mathrm{Hom}_R(M,N)$ is an ss-flat.

(5)⇒(4) Clear.

(4)⇒(1) For any index set S, by [32, 11.10 and 11.11, p. 87 and 88], we have that $R^S \cong \mathrm{Hom}_R(R^{(S)},R)$. Thus R^S is ss-flat by (4), so it follows from Theorem 2.3.9 that (1) holds. □

Corollary 2.3.17. The following are equivalent for a commutative ss-coherent ring R:

(1) M is an ss-injective R-module.
(2) $\mathrm{Hom}_R(M,N)$ is an ss-flat for any injective R-module N.
(3) $M \otimes_R N$ is an ss-injective for any flat R-module N.

Proof. (1)⇒(2) By Theorem 2.3.16.

(2)⇒(3) By [26, Theorem 2.75, p. 92], we have that $(M \otimes_R N)^+ \cong \mathrm{Hom}_R(M,N^+)$ for any R-module N. If N is flat, then N^+ is an injective by Proposition 1.2.36, so $(M \otimes_R N)^+$ is an ss-flat by hypothesis. Therefore $M \otimes_R N$ is an ss-injective by Theorem 2.3.10.

(3)⇒(1) This follows from [6, Proposition 2.3.4, p. 66], since R is a flat. □

Corollary 2.3.18. Let R be a commutative ss-coherent ring and SSF is closed under kernels of homomorphisms. Then the following statements hold for any R-module N:

(1) $\mathrm{Hom}_R(M,N)$ is an ss-flat for any ss-injective R-module M.
(2) $\mathrm{Hom}_R(N,M)$ is an ss-flat for any ss-flat R-module M.
(3) $M \otimes_R N$ is an ss-injective for any ss-injective R-module M.

Proof. (1) Let M be an ss-injective R-module. It is clear that the exact sequence $0 \longrightarrow N \longrightarrow E_0 \longrightarrow E_1$ induces the exact sequence $0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,E_0) \longrightarrow \mathrm{Hom}_R(M,E_1)$ where E_0 and E_1 are injective R-modules. By Theorem 2.3.16, we have that $\mathrm{Hom}_R(M,E_0)$ and $\mathrm{Hom}_R(M,E_1)$ are ss-flat, thus $\mathrm{Hom}_R(M,N)$ is an ss-flat by hypothesis.

(2) Let M be an ss-flat R-module, so we have the exact sequence $0 \longrightarrow \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Hom}_R(F_0, M) \longrightarrow \mathrm{Hom}_R(F_1, M)$ where F_0 and F_1 are free R-modules. By Theorem 2.3.16, the modules $\mathrm{Hom}_R(F_0, M)$ and $\mathrm{Hom}_R(F_1, M)$ are ss-flat. Therefore, $\mathrm{Hom}_R(N, M)$ is an ss-flat by hypothesis.

(3) Let M be any ss-injective R-module, then $(M \otimes_R N)^+ \cong \mathrm{Hom}_R(M, N^+)$ is an ss-flat by [26, Theorem 2.75, p. 92] and applying (1), and hence $M \otimes_R N$ is ss-injective by Theorem 2.3.10. □

Theorem 2.3.19. Let R be a commutative ss-coherent ring. Then the following statements are equivalent:

(1) R is an ss-injective ring.
(2) $\mathrm{Hom}_R(M, N)$ is an ss-injective for any projective R-module M and any flat R-module N.
(3) $\mathrm{Hom}_R(M, N)$ is an ss-injective for any projective R-modules M and N.
(4) $\mathrm{Hom}_R(M, N)$ is an ss-injective for any injective R-modules M and N.
(5) $\mathrm{Hom}_R(M, N)$ is an ss-flat for any flat R-module M and any injective R-module N.
(6) $M \otimes_R N$ is an ss-flat for any flat R-module M and any injective R-module N.

Proof. (1)⇒(2) Since R is an ss-injective, thus every flat R-module is an ss-injective by Proposition 2.3.13. Let M be a projective R-module, then $M \oplus P \cong R^{(S)}$ for some projective R-module P and for some index set S. Thus for all flat R-module N, we have $\mathrm{Hom}_R(M, N) \oplus \mathrm{Hom}_R(P, N) \cong \mathrm{Hom}_R(R^{(S)}, N) \cong N^S$ by [32, 11.10 and 11.11]. Since N^S is an ss-injective, thus $\mathrm{Hom}_R(M, N)$ is an ss-injective.

(2)⇒(3) Clear.

(3)⇒(1) Since $R \cong \mathrm{Hom}_R(R, R)$ by [32, 11.11, p. 88], thus R is an ss-injective ring.

(1)⇒(4) By Theorem 1.2.31, $\mathrm{Ext}^1\big(R/(S_r \cap J), \mathrm{Hom}_R(M, N)\big) \cong \mathrm{Hom}_R(\mathrm{Tor}_1(R/(S_r \cap J), M), N)$ for all injective R-modules M and N. By Proposition 2.3.13, M is an ss-flat. Thus $\mathrm{Tor}_1(R/(S_r \cap J), M) = 0$ and hence $\mathrm{Hom}_R(M, N)$ is an ss-injective.

(4)⇒(1) To prove R is an ss-injective ring, we need prove that every injective R-module is ss-flat (see Proposition 2.3.13). Now, let M be any injective R-

module, then $\text{Hom}_R(M, R^+)$ is an ss-injective, so $0 = \text{Ext}^1\big(R/(S_r \cap J)\ , \text{Hom}_R(M, R^+)\big) \cong \text{Hom}_R(\text{Tor}_1(R/(S_r \cap J), M), R^+) \cong (\text{Tor}_1(R/(S_r \cap J)\, , M) \otimes_R R)^+ \cong \text{Tor}_1(R/(S_r \cap J)\, , M)^+$ by applying Theorem 1.2.31, [26, Theorem 2.75, p. 92] and [6, Proposition 2.3.4, p. 66]. Therefore, $Tor_1(R/(S_r \cap J)\, , M) = 0$, since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator. Thus M is an ss-flat.

(5)⇒(1) and (6)⇒(1) By taking $M = R$ and using [32, 11.11, p. 88] and [6, Proposition 2.3.4, p. 66].

(1)⇒(5) Let M be a flat R-module and N be an injective R-module, then $\text{Hom}_R(M, N)$ is injective. Therefore, $\text{Hom}_R(M, N)$ is an ss-flat by Proposition 2.3.13.

(1)⇒(6) Let M be a flat R-module and let N be an injective R-module. Then N is ss-flat by Proposition 2.3.13, so the sequence $0 \longrightarrow N \otimes_R (S_r \cap J) \longrightarrow N$ is an exact. Since M is flat, then the sequence $0 \longrightarrow M \otimes_R N \otimes_R (S_r \cap J) \longrightarrow M \otimes_R N$ is exact and this implies that $M \otimes_R N$ is an ss-flat. □

Proposition 2.3.20. Let R be a commutative ring. Then the following statements are equivalent:

(1) M is an ss-flat.
(2) $\text{Hom}_R(M, N)$ is an ss-injective for all injective R-module N.
(3) $M \otimes_R N$ is an ss-flat for all flat R-module N.

Proof. (1)⇒(2) Let N be any injective R-module. Since $\text{Ext}^1\big(R/(S_r \cap J)\ , \text{Hom}_R(M, N)\big) \cong \text{Hom}_R(\text{Tor}_1(R/(S_r \cap J)\ , M), N) = 0$ by Theorem 1.2.31, then $\text{Hom}_R(M, N)$ is an ss-injective.

(2)⇒(3) Let N be a flat R-module. Then N^+ is an injective by Proposition 1.2.36. So it follows from [26, Theorem 2.75, p. 92] that $(M \otimes_R N)^+ \cong \text{Hom}_R(M, N^+)$ is ss-injective. Thus $M \otimes_R N$ is an ss-flat by Lemma 2.3.3.

(3)⇒(1) Follows from [6, Proposition 2.3.4, p. 66]. □

Proposition 2.3.21. Let R be a commutative ring and M be a semisimple R-module. If M is an ss-flat, then $\text{End}(M)$ is an ss-injective as R-module.

Proof. By [6, p. 157], there is a group epimorphism $\varphi: (S_r \cap J) \otimes_R M \longrightarrow (S_r \cap J)M$ given by $a \otimes x \longmapsto ax$ for each generator $a \otimes x \in (S_r \cap J) \otimes_R M$. Thus we have the commutative diagram:

$$\begin{array}{ccc} 0 \longrightarrow (S_r \cap J) \otimes_R M & \xrightarrow{i_1 \otimes I_M} & R \otimes_R M \\ \varphi \big\downarrow & & \big\downarrow f \\ 0 \longrightarrow (S_r \cap J)M & \xrightarrow{\quad i_2 \quad} & M \end{array}$$

where i_1 and i_2 are the inclusion maps, and f is an isomorphism defined by [6, Proposition 2.3.4, p. 66]. Since $f(i_1 \otimes I_M)$ is a $\mathbb{Z}$-monomorphism, then φ is an isomorphism. Therefore $(S_r \cap J) \otimes_R M \cong (S_r \cap J)M \subseteq J(M) = 0$ by Remark 1.1.7 (4). So it follows from [26, Theorem 2.75, p. 92] that $0 = \mathrm{Hom}_R((S_r \cap J) \otimes_R M, M) \cong \mathrm{Hom}_R(S_r \cap J, \mathrm{End}(M))$. But the sequence $0 = \mathrm{Hom}_R(S_r \cap J, \mathrm{End}(M)) \longrightarrow \mathrm{Ext}^1(R/(S_r \cap J), \mathrm{End}(M)) \longrightarrow \mathrm{Ext}^1(R, \mathrm{End}(M)) = 0$ is exact by [13, Theorem XII.4.4 (3)), p. 491]. Thus $\mathrm{Ext}^1(R/(S_r \cap J), \mathrm{End}(M)) = 0$ and hence $\mathrm{End}(M)$ is an ss-injective as R-module. □

Proposition 2.3.22. Let R be a commutative ring and M be a simple R-module. Then M is ss-flat if and only if M is ss-injective.

Proof. (⇒) Let $M = mR$ be a simple R-module. Define $f: \mathrm{Hom}_R(mR, mR) \longrightarrow mR$ by $f(\alpha) = \alpha(m)$. We assert that f is a well define R-homomorphism. Let $\alpha_1 = \alpha_2$, then $\alpha_1(m) = \alpha_2(m)$, so $f(\alpha_1) = f(\alpha_2)$. Now, let $\alpha_1, \alpha_2 \in \mathrm{End}(M)$ and $r_1, r_2 \in R$, then $f(r_1\alpha_1 + r_2\alpha_2) = (r_1\alpha_1 + r_2\alpha_2)(m) = (r_1\alpha_1)(m) + (r_2\alpha_2)(m) = r_1\alpha_1(m) + r_2\alpha_2(m) = r_1 f(\alpha_1) + r_2 f(\alpha_2)$ proving the assertion. Since $f(\mathrm{End}(M)) = M$ and $\ker(f) = \{\alpha \in \mathrm{End}(M) | \, f(\alpha) = 0\} = \{\alpha \in \mathrm{End}(M) | \alpha(m) = 0\} = \{\alpha \in \mathrm{End}(M) | m \in \ker(\alpha)\} = 0$, then $\mathrm{End}(M) \cong M$ and hence M is an ss-injective by Proposition 2.3.21.

(⇐) Let $\{S_\lambda\}_{\lambda \in \Lambda}$ be a family of all simple R-modules and $E = E(\bigoplus_{\lambda \in \Lambda} S_\lambda)$. Then $\mathrm{Hom}_R(M, E) \cong M$ by the proof of [31, Lemma 2.6], so it follows from Theorem 1.2.31 that $\mathrm{Ext}^1(R/(S_r \cap J), M) \cong \mathrm{Hom}_R(\mathrm{Tor}_1(R/(S_r \cap J), M), E)$. Since M is an ss-injective, then $\mathrm{Hom}_R(\mathrm{Tor}_1(R/(S_r \cap J), M), E) = 0$. But E is an injective cogenerator (by using [3, Corollary 18.19, p. 212]), thus we get $\mathrm{Tor}_1(R/(S_r \cap J), M) = 0$ (see [12, definition 3.2.7, p. 77]) and hence M is an ss-flat. □

The following corollary extends Proposition 1.2.13.

Corollary 2.3.23. The following statements are equivalent for a commutative ring R:

(1) R is a universally mininjective.

(2) R is a PS-ring.

(3) R is an FS-ring.

(4) S_r is an ss-flat.

Proof. By Proposition 2.3.22 and Corollary 2.1.12. □

Chapter Three

Section One

3.1 Strongly SS-Injective Modules

In this section, we will introduce and study the concept of strongly ss-injective modules and we will characterize semiprimitive rings, artinian rings and QF rings in terms of this concept.

Definition 3.1.1. A right R-module M is said to be strongly ss-injective if M is ss-N-injective, for all right R-module N. A ring R is said to be strongly right ss-injective if the right R-module R_R is strongly ss-injective.

Example 3.1.2.

(5) Every strongly soc-injective module is strongly ss-injective, but not conversely (see Example 3.2.9).
(6) Every strongly ss-injective module is ss-injective, but not conversely (see Example 3.2.8).

Proposition 3.1.3. A right R-module M is a strongly ss-injective if and only if every R-homomorphism $\alpha: A \longrightarrow M$ extends to N, for all right R-module N, where $A \ll N$ and $\alpha(A)$ is a semisimple submodule in M.

Proof. (⇐) Clear.

(⇒) Let A be a small submodule of N, and $\alpha: A \longrightarrow M$ be an R-homomorphism with $\alpha(A)$ is a semisimple submodule of M. If $B = \ker(\alpha)$, then α induces an R-homomorphism $\tilde{\alpha}: A/B \longrightarrow M$ defined by $\tilde{\alpha}(a + B) = \alpha(a)$, for all $a \in A$. Clearly, $\tilde{\alpha}$ is well define because if $a_1 + B = a_2 + B$ we have $a_1 - a_2 \in B$, so $\alpha(a_1) = \alpha(a_2)$, that is $\tilde{\alpha}(a_1 + B) = \tilde{\alpha}(a_2 + B)$. Since M is strongly ss-injective and A/B is semisimple and small in N/B, thus $\tilde{\alpha}$ extends to an R-homomorphism $\gamma: N/B \longrightarrow M$. If $\pi: N \longrightarrow N/B$ is the canonical map, then the R-homomorphism $\beta = \gamma\pi: N \longrightarrow M$ is an extension of α such that if $a \in A$, then $\beta(a) = (\gamma\pi)(a) = \gamma(a + B) = \tilde{\alpha}(a + B) = \alpha(a)$ as desired. □

Corollary 3.1.4. The following statements hold:

(3) A finite direct sum of strongly ss-injective modules is again strongly ss-injective.
(4) A direct summand of strongly ss-injective module is again strongly ss-injective.

Corollary 3.1.5. Let R be a ring. Then:

(1) If M is a semisimple strongly ss-injective right R-module, then M is a small injective.
(2) If every simple right R-module is strongly ss-injective, then R is a semiprimitive ring.

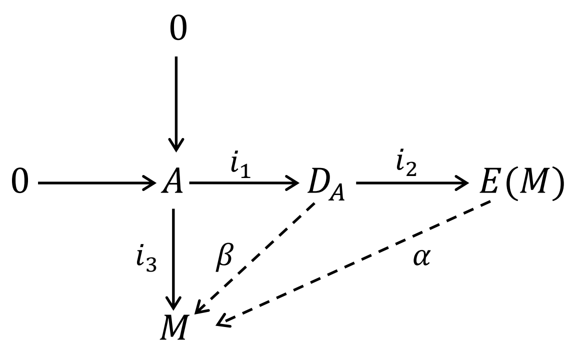
Proof. (1) By Proposition 3.1.3.

(2) By (1) and applying Theorem 1.2.17. □

Remark 3.1.6. The converse of Corollary 3.1.5 is not true (see Example 3.1.11).

Theorem 3.1.7. If M is a strongly ss-injective (or just ss-$E(M)$-injective) right R-module, then for every semisimple small submodule A of M, there is an injective R-module E_A such that $M = E_A \oplus T_A$ where $T_A \hookrightarrow M$ with $T_A \cap A = 0$. Moreover, if $A \neq 0$, then E_A can be taken $A \subseteq^{ess} E_A$.

Proof. Let A be a semisimple small submodule of M. If $A = 0$, we end the proof by taking $E_A = 0$ and $T_A = M$. Suppose that $A \neq 0$ and consider the following diagram:



where i_1, i_2 and i_3 are inclusion maps and $D_A = E(A)$ is the injective hull of A in $E(M)$. Since M is strongly ss-injective, thus M is ss-$E(M)$-injective. Since A is a semisimple small submodule of M, so it follows from Lemma 1.1.2 (1) that A is a semisimple small submodule in $E(M)$ and hence there exists an R-homomorphism $\alpha: E(M) \longrightarrow M$ such that $\alpha i_2 i_1 = i_3$. Put $\beta = \alpha i_2: D_A \longrightarrow M$, thus β is an extension of i_3. Let $x \in \ker(\beta) \cap A$, then $x \in \ker(\beta)$ and $x =$

$i_1(x) \in A$ and hence $\beta\big(i_1(x)\big) = \beta(x) = 0$. Therefore, $x = i_3(x) = \beta\big(i_1(x)\big) = 0$, and so $\ker(\beta) \cap A = 0$. Since $A \subseteq^{ess} D_A$, thus β is an R-monomorphism. Put $E_A = \beta(D_A)$. Since E_A is an injective submodule of M, thus $M = E_A \oplus T_A$ for some $T_A \hookrightarrow M$. Since $\beta(A) = A$, thus $A \subseteq \beta(D_A) = E_A$ and this means that $T_A \cap A = 0$. Moreover, define $\tilde{\beta} = \beta: D_A \longrightarrow E_A$, thus $\tilde{\beta}$ is an isomorphism. Since $A \subseteq^{ess} D_A$, thus $\tilde{\beta}(A) \subseteq^{ess} E_A$. But $\tilde{\beta}(A) = \beta(A) = A$, so $A \subseteq^{ess} E_A$. □

Corollary 3.1.8. If M is a right R-module has a semisimple small submodule A such that $A \subseteq^{ess} M$, then the following statements are equivalent:

(1) M is injective.
(2) M is strongly ss-injective.
(3) M is ss-$E(M)$-injective.

Proof. (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(1) By Theorem 3.1.7, we can write $M = E_A \oplus T_A$ where E_A injective and $T_A \cap A = 0$. Since $A \subseteq^{ess} M$, thus $T_A = 0$ and hence $M = E_A$. Therefore M is an injective right R-module. □

Example 3.1.9. $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective.

Proof. Assume that $\mathbb{Z}_4$ is strongly ss-injective $\mathbb{Z}$-module. Let $A =< \bar{2} >= \{\bar{0}, \bar{2}\}$. It is clear that A is a semisimple small and essential submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module. By Corollary 3.1.8, $\mathbb{Z}_4$ is injective $\mathbb{Z}$-module and this a contradiction. Thus $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective. Moreover, Since $E(\mathbb{Z}_{2^2}) = \mathbb{Z}_{2^\infty}$ as $\mathbb{Z}$-module (see [24, p. 6]), thus $\mathbb{Z}_4$ is not ss-$\mathbb{Z}_{2^\infty}$-injective, by Corollary 3.1.8. □

Corollary 3.1.10. Let M be a right R-module such that $\mathrm{soc}(M) \cap J(M) \ll M$ (in particular, if M is finitely generated). If M is strongly ss-injective, then $M = E \oplus T$, where E is injective and $T \cap \mathrm{soc}(M) \cap J(M) = 0$. Moreover, if $\mathrm{soc}(M) \cap J(M) \neq 0$, then we can take $\mathrm{soc}(M) \cap J(M) \subseteq^{ess} E$.

Proof. By taking $A = \mathrm{soc}(M) \cap J(M)$ and applying Theorem 3.1.7. □

The following example shows that the converse of Theorem 3.1.7 and Corollary 3.1.10 is not true.

Example 3.1.11. Let $M=\mathbb{Z}_6$ as $\mathbb{Z}$-module. Since $J(M)=0$ and $\operatorname{soc}(M)=M$, thus $\operatorname{soc}(M)\cap J(M)=0$. So, we can write $M=0\bigoplus M$ with $M\cap\big(\operatorname{soc}(M)\cap J(M)\big)=0$. Let $N=\mathbb{Z}_8$ as $\mathbb{Z}$-module. Since $J(N)=<\bar{2}>$ and $\operatorname{soc}(N)=<\bar{4}>$. Define $\gamma:\operatorname{soc}(N)\cap J(N)\longrightarrow M$ by $\gamma(\bar{4})=\bar{3}$, thus γ is a $\mathbb{Z}$-homomorphism. Assume that M is strongly ss-injective, thus M is ss-N-injective, so there exists $\mathbb{Z}$-homomorphism $\beta: N\longrightarrow M$ such that $\beta\circ i=\gamma$, where i is the inclusion map from $\operatorname{soc}(N)\cap J(N)$ to N. Since $\beta\big(J(N)\big)\subseteq J(M)$, thus $\bar{3}=\gamma(\bar{4})=\beta(\bar{4})\in\beta\big(J(N)\big)\subseteq J(M)=0$ and this contradiction, so M is not strongly ss-injective $\mathbb{Z}$-module.

We can prove the following corollary by using Proposition 2.1.11.

Corollary 3.1.12. The following statements are equivalent:

(1) $\operatorname{soc}(M)\cap J(M)=0$, for all right R-module M.
(2) Every right R-module is strongly ss-injective.
(3) Every simple right R-module is strongly ss-injective.

In the next results, we will give the connection between strongly ss-injective modules and strongly soc-injective modules and we provide many new equivalences of artinian rings and QF rings.

Theorem 3.1.13. If R is a right perfect ring, then M is a strongly soc-injective right R-module if and only if M is a strongly ss-injective.

Proof. (⇒) Clear.

(⇐) Let R be a right perfect ring and M be a strongly ss-injective right R-module. Since R is a semilocal ring, thus it follows from [18, Theorem 3.5] that every right R-module N is semilocal and hence $N/J(N)$ is semisimple right R-module. Since R is a right perfect ring, thus it is right max (see [14, Theorem 4.3, p. 69]) and hence the Jacobson radical of every right R-module is small by Theorem 1.1.22. Thus $N/J(N)$ is semisimple and $J(N)\ll N$, for any $N\in$ Mod-R. Since M is strongly ss-injective, thus every R-homomorphism from a semisimple small submodule of N to M extends to N, for every $N\in$ Mod-R, and this implies that every R-homomorphism from any semisimple submodule of $J(N)$ to M extends to N, for every $N\in$ Mod-R. Since $N/J(N)$ is semisimple right R-module, for every $N\in$ Mod-R. Thus Lemma 2.1.31 implies that every

R-homomorphism from any semisimple submodule of N to M extends to N, for every $N \in$ Mod-R and hence M is strongly soc-injective. □

The result [2, Proposition 3.7] is improved by below corollary.

Corollary 3.1.14. A ring R is QF if and only if every strongly ss-injective right R-module is projective.

Proof. (⇒) If R is QF ring, then R is a right perfect ring, so by Theorem 3.1.13 and [2, Proposition 3.7] we have that every strongly ss-injective right R-module is projective.

(⇐) By hypothesis we have that every injective right R-module is projective and hence R is QF ring (see Proposition 1.2.11). □

The results of I. Amin, M. Yousif and N. Zeyada [2, Theorem 3.3 and 3.6] gave a equivalent statements to characterize the noetherian rings and semiartinian rings. In the next theorem we obtain characterizations to artinian rings in terms of strongly ss-injective and strongly soc-injective modules.

Theorem 3.1.15. The following statements are equivalent for a ring R:

(1) Every direct sum of strongly ss-injective right R-modules is injective.
(2) Every direct sum of strongly soc-injective right R-modules is injective.
(3) R is right artinian.

Proof. (1)⇒(2) Clear.

(2)⇒(3) Since every direct sum of strongly soc-injective right R-modules is injective. Thus R is right noetherian and right semiartinian by [2, Theorem 3.3 and Theorem 3.6], so it follows from [29, Proposition VIII.5.2, p. 189] that R is right artinian.

(3)⇒(1) By hypothesis, R is right perfect and right noetherian. It follows from Theorem 3.1.13 and [2, Theorem 3.3] that every direct sum of strongly ss-injective right R-modules is strongly soc-injective. Since R is right semiartinian, so [2, Theorem 3.6] implies that every direct sum of strongly ss-injective right R-modules is injective. □

Recall that a submodule K of a right R-module M is called t-essential in M (written $K \subseteq^{tes} M$) if for every submodule L of M, $K \cap L \subseteq Z_2(M)$ implies that $L \subseteq Z_2(M)$ (see [4]). A right R-module M is said to be t-semisimple if every submodule A of M there exists a direct summand B of M such that $B \subseteq^{tes} A$ (see [4]). A ring R is said to be right GV-ring (resp. SI-ring) if every simple singular (resp. singular) right R-module is injective (see [36]). In the next results, we will give the connection between injectivity and strongly s-injectivity and we characterize V-rings, GV-rings, SI-rings and semisimple rings by this connection.

Theorem 3.1.16. If R is a right t-semisimple, then a right R-module M is injective if and only if M is strongly s-injective.

Proof. (⇒) Obvious.

(⇐) Let M be a strongly s-injective, thus $Z_2(M)$ is injective by [36, Proposition 3, p. 27]. Thus every R-homomorphism $f: K \longrightarrow M$, where $K \subseteq Z_2^r$ extends to R by [36, Lemma 1, p. 26]. Since R is a right t-semisimple, thus R/Z_2^r is a right semisimple by [4, Theorem 2.3]. So by applying Lemma 2.1.31, we conclude that M is injective. □

Corollary 3.1.17. A ring R is right SI and right t-semisimple if and only if it is semisimple.

Proof. (⇒) Since R is a right SI-ring, thus every right R-module is strongly s-injective by [36, Theorem 1, p. 29]. By Theorem 3.1.16, we have that every right R-module is injective and hence R is semisimple ring.

(⇐) Clear. □

Corollary 3.1.18. If R is a right t-semisimple ring. Then R is right V-ring if and only if R is right GV-ring.

Proof. By [36, Proposition 5, p. 28] and Theorem 3.1.16. □

Corollary 3.1.19. If R is a right t-semisimple ring, then R/S_r is noetherian right R-module if and only if R is right noetherian.

Proof. If R/S_r is noetherian right R-module, then every direct sum of injective right R-modules is strongly s-injective by [36, Proposition 6]. Since R is right t-semisimple, so it follows from Theorem 3.1.16 that every direct sum of injective right R-modules is injective and hence R is right noetherian. The converse is clear. □

Section Two

3.2 Strongly SS-Injective Rings

In this section, we will give some results on strongly ss-injective rings and we will characterize semisimple and QF rings.

A ring R is strongly right soc-injective iff every finitely generated projective right R-module is strongly soc-injective.

Proposition 3.2.1. A ring R is strongly right ss-injective if and only if every finitely generated projective right R-module is a strongly ss-injective.

Proof. Since a finite direct sum of strongly ss-injective modules is a strongly ss-injective, so every finitely generated free right R-module is strongly ss-injective. But a direct summand of strongly ss-injective is a strongly ss-injective. Therefore, every finitely generated projective is a strongly ss-injective. The converse is clear. □

A ring R is said to be right Ikeda-Nakayama ring if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R (see [24, p. 148]). In the following proposition, the strongly ss-injectivity gives a new version of "Ikeda-Nakayama rings".

Proposition 3.2.2. Let R be a strongly right ss-injective ring, then $l(N \cap K) = l(N) + l(K)$ for all semisimple small right ideals N and all right ideals K of R.

Proof. Suppose that $x \in l(N \cap K)$ and define $\alpha: N + K \longrightarrow R_R$ by $\alpha(a + b) = xa$ for all $a \in N$ and $b \in K$. Clearly, α is well define, because if $a_1 + b_1 = a_2 + b_2$, then $a_1 - a_2 = b_2 - b_1$, that is $x(a_1 - a_2) = 0$, so $\alpha(a_1 + b_1) = \alpha(a_2 + b_2)$. Define the R-homomorphism $\tilde{\alpha}: (N + K)/K \longrightarrow R_R$ by $\tilde{\alpha}(a + K) = xa$ for all $a \in N$ which induced by α. Since $(N + K)/K \subseteq \operatorname{soc}(R/K) \cap J(R/K)$ and R is a strongly right ss-injective, $\tilde{\alpha}$ can be extended to an R-homomorphism $\gamma: R/K \longrightarrow R_R$. If $\gamma(1 + K) = y$, for some $y \in R$, then $y(a + b) = xa$, for all $a \in N$ and $b \in K$. In particular, $ya = xa$ for all $a \in N$ and $yb = 0$ for all $b \in K$. Hence $x = (x - y) + y \in l(N) + l(K)$. Therefore,

$l(N \cap K) \subseteq l(N) + l(K)$. Since the converse is always holds, thus the proof is complete. □

Corollary 3.2.3. Every strongly right ss-injective ring is a right simple J-injective.

Proof. By Proposition 3.1.3. □

Remark 3.2.4. The converse of Corollary 3.2.3 is not true (see Example 3.2.8).

Proposition 3.2.5. Let R be a right Kasch and strongly right ss-injective. Then:

(1) $rl(K) = K$, for every small right ideal K of R. Moreover, R is right minannihilator.

(2) If R is left Kasch, then $r(J) \subseteq^{ess} R_R$.

Proof.(1) By Corollary 3.2.3 and [35, Lemma 2.4].

(2) Let K be a right ideal of R and $r(J) \cap K = 0$. Then $Kr(J) = 0$ and we obtain $K \subseteq lr(J) = J$, because R is left Kasch. By (1), we have $r\big(J + l(K)\big) = r(J) \cap K = 0$ and this means that $J + l(K) = R$ (since R is left Kasch). Thus $K = 0$ and hence $r(J) \subseteq^{ess} R_R$. □

Lemma 3.2.6 [17, Corollary 3.73, p. 97]. A commutative ring R is von Neumann regular if and only if every simple R-module is injective.

The following examples show that the three classes of rings: strongly ss-injective rings, soc-injective rings and small injective rings are different.

Example 3.2.7. Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n}: \ p \text{ does not divide } n\}$, the localization ring of $\mathbb{Z}$ at the prime p. Then R is a commutative local ring and it has zero socle but not principally small injective (see [33, Example 4]). Since $S_r = 0$, thus R is strongly soc-injective ring and hence R is strongly ss-injective ring.

Example 3.2.8. Let $R = \left\{\begin{pmatrix} n & x \\ 0 & n \end{pmatrix}: \ n \in \mathbb{Z}, x \in \mathbb{Z}_2\right\}$. Thus R is a commutative ring, $J = S_r = \left\{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}: \ x \in \mathbb{Z}_2\right\}$ and R is small injective (see [30, Example (i)]). Let $A = J$ and $B = \left\{\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}: \ n \in \mathbb{Z}\right\}$, then $l(A) = \left\{\begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix}: \ n \in \mathbb{Z}, y \in \mathbb{Z}_2\right\}$ and $l(B) = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}: \ y \in \mathbb{Z}_2\right\}$. Thus

$l(A) + l(B) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} : \ n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\}$. Since $A \cap B = 0$, then $l(A \cap B) = R$ and this implies that $l(A) + l(B) \neq l(A \cap B)$. Therefore R is not strongly ss-injective and not strongly soc-injective by Proposition 3.2.2.

Example 3.2.9. Let $F = \mathbb{Z}_2$ be the field of two elements, $F_i = F$ for $i = 1, 2, \dots$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \bigoplus_{i=1}^{\infty} F_i$. If R is the subring of Q generated by 1 and S, then R is a von Neumann regular ring (see [36, Example (1), p. 28]). Since R is commutative, thus every simple R- module is injective by Lemma 3.2.6. Thus R is V-ring and hence and hence $J(N) = 0$ for every right R-module N. It follows from Corollary 3.1.12 that every R-module is a strongly ss-injective. In particular, R is a strongly ss-injective ring. But R is not soc-injective (see [36, Example (1)]).

Example 3.2.10. Let $R = \mathbb{Z}_2[x_1, x_2, \dots]$ where $\mathbb{Z}_2$ is the field of two elements, $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2 \neq 0$ for all i and j. If $m = x_i^2$, then R is a commutative, local, soc-injective ring with $J = \text{span}\{m, x_1, x_2, \dots\}$, and R has simple essential socle $J^2 = \mathbb{Z}_2 m$ (see [2, Example 5.7]). It follows from [2, Example 5.7] that the R-homomorphism $\gamma: J \longrightarrow R$ which is given by $\gamma(a) = a^2$ for all $a \in J$ with simple image can not extend to R, then R is not simple J-injective and not small injective, so it follows from Corollary 3.2.3 that R is not strongly ss-injective.

Recall that a ring R is called right minsymmetric if aR is simple, $a \in R$, implies that Ra is simple.

Theorem 3.2.11. A ring R is QF if and only if R is a strongly right ss-injective and right noetherian ring with $S_r \subseteq^{ess} R_R$.

Proof. (⇒) This is clear.

(⇐) By Corollary 2.2.3 (1), R is a right minsymmetric. It follows from [30, Lemma 2.2] that R is right perfect. Thus, R is strongly right soc-injective, by Theorem 3.1.13. Since $S_r \subseteq^{ess} R_R$, so it follows from [2, Corollary 3.2] that R is a self-injective and hence R is QF. □

Corollary 3.2.12. For a ring R, the following statements are true:

(1) R is a semisimple if and only if $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective.

(2) R is QF if and only if R is a strongly right ss-injective, semiperfect with essential right socle and R/S_r is noetherian as right R-module.

Proof. (1) Suppose that $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective, then R is a right noetherian right V-ring by [2, Proposition 3.12], so it follows from Corollary 3.1.12 that R is a strongly right ss-injective. Thus R is QF by Theorem 3.2.11. But $J = 0$, so R is a semisimple. The converse is clear.

(2) By [23, Theorem 2.9], $J = Z_r$. Since R/Z_2^r is a homomorphic image of R/Z_r and R is a semilocal ring, thus R is a right t-semisimple. By Corollary 3.1.19, R is right noetherian, so it follows from Theorem 3.2.11 that R is QF. The converse is clear. □

Theorem 3.2.13. A ring R is QF if and only if R is strongly right ss-injective, $l(J^2)$ is a countable generated left ideal, $S_r \subseteq^{ess} R_R$ and the chain $r(x_1) \subseteq r(x_2x_1) \subseteq \cdots \subseteq r(x_nx_{n-1} \ldots x_2x_1) \subseteq \cdots$ terminates for every infinite sequence $x_1, x_2, \ldots$ in R.

Proof. (⇒) Since R is QF , then R is right self-injective, right noetherian and right semiartinian. Therefore, R is strongly right ss-injective, $l(J^2)$ is a countable generated left ideal, $S_r \subseteq^{ess} R_R$ and the chain $r(x_1) \subseteq r(x_2x_1) \subseteq \cdots \subseteq r(x_nx_{n-1} \ldots x_2x_1) \subseteq \cdots$ terminates for every infinite sequence $x_1, x_2, \ldots$ in R.

(⇐) By [30, Lemma 2.2], R is right perfect. Since $S_r \subseteq^{ess} R_R$, thus R is right Kasch by [23, Theorem 3.7]. Since R is a strongly right ss-injective, thus R is a right simple J-injective, by Corollary 3.2.3. Now, by Proposition 3.2.5 (1) we have $rl(S_r \cap J) = S_r \cap J$, so Corollary 2.2.15 (7) leads to $S_r = S_\ell$. By [24, Lemma 3.36, p. 73], $S_2^r = l(J^2)$. The result now follows from [35, Theorem 2.18]. □

Remark 3.2.14. The condition $S_r \subseteq^{ess} R_R$ in Theorem 3.2.11 and Theorem 3.2.13 can not be deleted, because $\mathbb{Z}$ is a strongly ss-injective noetherian ring but not QF.

The following two results extend a result [2, Proposition 5.8] that a left perfect ring, strongly left and right soc-injective ring is QF.

Corollary 3.2.15. A ring R is QF ring if and only if it is left perfect, strongly left and right ss-injective ring.

Proof. By Corollary 3.2.3 and [35, Corollary 2.12]. □

Theorem 3.2.16. For a ring R, the following statements are equivalent:

(1) R is a QF ring.
(2) R is a strongly left and right ss-injective, right Kasch and J is left t-nilpotent.
(3) R is a strongly left and right ss-injective, left Kasch and J is left t-nilpotent.

Proof. (1)⇒(2) and (1)⇒(3) are clear.

(3)⇒(1) Suppose that xR is simple right ideal. Thus either $rl(x) = xR \subseteq^{\oplus} R_R$ or $x \in J$. If $x \in J$, then $rl(x) = xR$ (since R is right minannihilator by Proposition 3.2.5), so Theorem 3.1.7 implies that $rl(x) \subseteq^{ess} E \subseteq^{\oplus} R_R$. Therefore, $rl(x)$ is an essential in a direct summand of R_R for every simple right ideal xR. Let K be a left maximal ideal of R. Since R is a left Kasch, thus $r(K) \neq 0$ by [17, Corollary 8.28, p. 281]. Choose $0 \neq y \in r(K)$, so $K \subseteq l(y)$ and we conclude that $K = l(y)$. Since $Ry \cong R/l(y)$, thus Ry is simple left ideal. But R is a left mininjective ring, so yR is a simple right ideal by [23, Theorem 1.14] and this implies that $r(K) \subseteq^{ess} eR$ for some $e^2 = e \in R$ (since $r(K) = rl(y)$). Thus R is semiperfect by [24, Lemma 4.1, p. 79] and hence R is a left perfect (since J is left t-nilpotent), so it follows from Corollary 3.2.15 that R is QF.

(2)⇒(1) is similar to proof of (3)⇒(1). □

Theorem 3.2.17. The ring R is QF if and only if R is a strongly left and right ss-injective, left and right Kasch, and the chain $l(a_1) \subseteq l(a_1a_2) \subseteq \cdots \subseteq l(a_1a_2 \dots a_n) \subseteq \cdots$ terminates for every $a_1, a_2, \dots \in Z_\ell$.

Proof. (⇒) Clear.

(⇐) By Proposition 3.2.5, $l(J)$ is essential in ${}_RR$. Thus $J \subseteq Z_\ell$. Let $a_1, a_2, \dots \in J$, we have $l(a_1) \subseteq l(a_1a_2) \subseteq \cdots \subseteq l(a_1a_2 \dots a_n) \subseteq \cdots$. Thus there exists $k \in \mathbb{N}$ such that $l(a_1 \dots a_k) = l(a_1 \dots a_k a_{k+1})$ (by hypothesis). Suppose that $a_1 \dots a_k \neq 0$, so $R(a_1 \dots a_k) \cap l(a_{k+1}) \neq 0$ (since $l(a_{k+1})$ is essential in ${}_RR$). Thus $ra_1 \dots a_k \neq 0$ and $ra_1 \dots a_k a_{k+1} = 0$ for some $r \in R$, a contradiction.

Therefore, $a_1 \dots a_k = 0$ and hence J is left t-nilpotent, so it follows from Theorem 3.2.16 that R is QF. □

Corollary 3.2.18. The ring R is QF if and only if R is strongly left and right ss-injective with essential right socle, and the chain $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \cdots$ terminates for every infinite sequence $a_1, a_2, \dots$ in R.

Proof. By [30,Lemma 2.2] and Corollary 3.2.15. □

References

[1] W. A. Adkins and S. H. Weintranb, Algebra: an approach via module theory, Springer-Verlag, New York, 1992.

[2] I. Amin, M. Yousif and N. Zeyada, Soc-injective rings and modules, Comm. Algebra 33 (2005), 4229-4250.

[3] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, Berlin-New York, 1974.

[4] Sh. Asgari, A. Haghany and Y. Tolooei, T-semisimple modules and T-semisimple rings, Comm. Algebra 41 (2013), 1882-1902.

[5] L. Bican, T. Kepka and P. Neme, Rings, Modules and Preradicals, Marcel Dekker, Inc, New York, 1982.

[6] P. E. Bland, Rings and Their Modules, Walter de Gruyter & Co., Berlin, 2011.

[7] T. J. Cheatham and D. R. Stone, Flat and projective character modules, Proc. Amer. Math. Soc. 81 (1981), 175-177.

[8] J. L. Chen and N. Q. Ding, A note on existence of envelopes and covers, Bull. Austral. Math. Soc. 54 (1996), 383-390.

[9] J. L. Chen and N. Q. Ding, On n-coherent rings, Comm. Algebra 24 (1996), 3211-3216.

[10] E. E. Enochs, A note on absolutely pure modules, Canad. Math. Bull. 19 (1976), 361-362.

[11] E. E. Enochs and Z. Y. Huang, Injective envelopes and (Gorenstein) flat covers, Algebr. Represent. Theor., 15 (2012), 1131-1145.

[12] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter, 2000.

[13] P. A. Grillet, Abstract Algebra, 2nd edition, GTM 242, Springer, 2007.

[14] Y. Hirano, Rings whose simple modules have some properties, pp.63-76 in J. L. Chen, N. Q. Ding and H. Marubayashi, Advances in Ring Theory, Proceedings of the 4th China-Japan-Korea International Conference, World Scientific Publishing Co. Pte. Ltd., 2005.

[15] H. Holm and P. Jørgensen, Covers, precovers, and purity, Illinois J. Math., 52 (2008), 691-703.

[16] F. Kasch, Modules and Rings, Academic Press, New York, 1982.

[17] T.Y. Lam, Lectures on Modules and Rings, GTM 189, Springer-Verlag, New York, 1999.

[18] C. Lomp, On semilocal modules and rings, Comm. Algebra 27 (1999), 1921-1935.

[19] L. Mao, Min-flat modules and min-coherent rings, Comm. Algebra 35 (2007), 635-650.

[20] A. R. Mehdi, Purity relative to classes of finitely presented modules, PhD Thesis, Manchester University, 2013.

[21] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, Cambridge University Press, Cambridge, 1990.

[22] W. K. Nicholson, Semiregular modules and rings, Canadian J. Math. 28 (1976), 1105-1120.

[23] W. K. Nicholson and M. F. Yousif, Mininjective rings, J. Algebra 187 (1997), 548-578.

[24] W. K. Nicholson and M. F. Yousif, Quasi-Frobenius rings, Cambridge Tracts in Math., 158, Cambridge University Press, Cambridge, 2003.

[25] D. S. Passman, A coruse in ring theory, AMS Chelsea Publishing, 2004.

[26] J. J. Rotman, An Introduction to Homological Algebra, Springer, 2009.

[27] L. Shen and J. Chen, New characterizations of quasi-Frobenius rings, Comm. Algebra 34 (2006), 2157-2165.

[28] B. Stenström, Coherent rings and FP-injective modules, J. London. Math. Soc. 2 (1970), 323-329.

[29] B. Stenström, Rings of Quotients, Sepringer-Verlage, Berlin-Heidelberg-New York, 1975.

[30] L.V. Thuyet and T. C. Quynh, On small injective rings and modules, J. Algebra and Its Applications 8 (2009), 379-387.

[31] R. Ware, Endomorphism rings of projective modules, Trans. Amer. Math. Soc. 155 (1971), 233-256.

[32] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, 1991.

[33] Y. Xiang, Principally small injective rings, Kyungpook Math. J. 51 (2011), 177-185.

[34] Y. Xiao, Rings with flat socles, Proc. Amer. Math. Soc. 123 (1995), 2391-2395.

[35] M. F. Yousif and Y. Q. Zhou, FP-injective, simple-injective and quasi-Frobenius rings, Comm. Algebra 32 (2004), 2273-2285.

[36] N. Zeyada, S-injective modules and rings, Advances in Pure Math. 4 (2014), 25-33.

[37] N. Zeyada and A. Amin, Aspects on weak, s-CS and almost injective rings, arxiv: Math., RA/ 1509.05916v1, 19 Sep 2015.

[38] N. Zeyada, S. Hussein and A. Amin, Rad-injective and almost-injective modules and rings, Algebra Colloquium 18 (2011), 411-418.

الخلاصة

في هذا العمل، قدمنا ودرسنا مفهوم المقاسات الأغمارية من النمط - ss كتعميم لكل من المقاسات الأغمارية من النمط - soc والمقاسات الأغمارية الصغيرة. كذلك قمنا بتقديم ودراسة مفهوم المقاسات المسطحة من النمط - ss كمفهوم رديف للمقاسات الأغمارية من النمط - ss. مفهوم المقاسات الأغمارية القوية من النمط - ss تم تعريفها باستخدام المقاسات الأغمارية من النمط - ss كتعميم للمقاسات الأغمارية القوية من النمط - soc. تشخيصات مختلفة لهذه المقاسات والحلقات قد أعطيت. باستخدام المقاسات الأغمارية من النمط - ss، قدمنا العديد من التشخيصات الجديدة الأخرى للحلقات شبه البسيطة، للحلقات شبه الفروبينوسية، للحلقات الارتينية والحلقات الأغمارية من النمط - min الشاملة. العديد من النتائج في المصادر قد حسنت ووسعت بواسطة نتائج في هذه الرسالة.

جمهورية العراق
وزارة التعليم العالي والبحث العلمي
جامعة القادسية/ كلية علوم الحاسوب والرياضيات
قسم الرياضيات

المقاسات الأغمارية من النمط – SS

ومفاهيم ذات العلاقة

رسالة

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من قبل
عادل سالم تايه البـديري

بأشراف
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