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New Strong Differential Subordinations and Superordinations of Symmetric Analytic Functions

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Abstract

The concept of strong differential subordinations was introduced in [1], [2] by Antonio and Romaguera and developed in [6,8]. The dual concept of strong differential superordination was introduced in [4] and developed in [5,7]. In this paper, we introduce two new classes of symmetric analytic functions defined by strong differential subordination and superordination. Also we study some properties of these classes.

Keywords: Analytic function, Strong differential subordination, Strong differential superordination, Convex function

1 Introduction and Preliminaries

Denote by U the open unit disk of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disk of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}): f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, z \in U, \zeta \in \overline{U}\}$, where $a_j(\zeta)$ are holomorphic functions in $\overline{U}$ for $j \geq n$.

Let $\mathcal{A}^*_\zeta$ denote the class of functions of the form:

$$f(z,\zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta)z^k, \quad (z \in U, \zeta \in \overline{U}), \tag{1.1}$$

which are analytic in $U \times \overline{U}$ and $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$.

Definition 1.1 [4]. We denote by Q_ζ the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where

$$E(f,\zeta) = \left\{\xi \in \partial U: \lim_{z \to \xi} f(z,\zeta) = \infty\right\},$$

and $f'_z(\xi,\zeta) \neq 0$ for $\xi \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q_ζ with $f(0,\zeta) = a$ is denoted by $Q_\zeta(a)$.

Definition 1.2 [4]. Let $f(z,\zeta), F(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $F(z,\zeta)$ if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z,\zeta) = F(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec F(z,\zeta), z \in U, \zeta \in \overline{U}$.

Remark 1.1 [4].

(i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition1.2 is equivalent to $f(0,\zeta) = F(0,\zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset F(U \times \overline{U})$.

(ii) If $f(z,\zeta) = f(z)$ and $F(z,\zeta) = F(z)$, the strong subordination becomes the usual notion of subordination.

If $f(z,\zeta)$ strongly subordinate to $F(z,\zeta)$, then $F(z,\zeta)$ strongly superordinate to $f(z,\zeta)$.

Lemma 1.1 [3]. Let $h(z,\zeta)$ be a univalent with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\mu \in \mathbb{C} \setminus \{0\}$ with $Re(\mu) \geq 0$. If $p \in \mathcal{H}^*[a,1,\zeta]$ and

$$p(z,\zeta) + \frac{1}{\mu} z p'_z(z,\zeta) \prec\prec h(z,\zeta), \quad (z \in U, \zeta \in \overline{U}), \tag{1.2}$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \quad (z \in U, \zeta \in \overline{U}),$$

where $q(z,\zeta) = \mu z^{-\mu} \int_0^z h(t,\zeta)\, t^{\mu-1} dt$ is convex and it is the best dominant of (1.2).

Lemma 1.2 [4]. Let $h(z,\zeta)$ be a convex with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\mu \in \mathbb{C} \setminus \{0\}$ with $Re(\mu) \geq 0$. If $p \in \mathcal{H}^*[a,1,\zeta] \cap Q_\zeta$, $p(z,\zeta) + \frac{1}{\mu} z p_z'(z,\zeta)$ is univalent in $U \times \overline{U}$ and

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\mu} z p_z'(z,\zeta), \quad (z \in U, \zeta \in \overline{U}), \tag{1.3}$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \ (z \in U, \zeta \in \overline{U}),$$

where $q(z,\zeta) = \mu z^{-\mu} \int_0^z h(t,\zeta)\, t^{\mu-1} dt$ is convex and it is the best subordinant of (1.3).

2 Main Results

Definition 2.1. Let $\psi(z,\zeta)$ be an analytic function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $\lambda > 0$. A function $f \in \mathcal{A}_\zeta^*$ is said to be in the class $S(\lambda;\psi)$ if it satisfies the strong differential subordination

$$(1-\lambda)\left(\frac{f(z,\zeta) - f(-z,\zeta)}{2z}\right) + \lambda\left(\frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2}\right) \prec\prec \psi(z,\zeta).$$

A function $f \in \mathcal{A}_\zeta^*$ is said to be in the class $T(\lambda;\psi)$ if it satisfies the strong differential superordination

$$\psi(z,\zeta) \prec\prec (1-\lambda)\left(\frac{f(z,\zeta) - f(-z,\zeta)}{2z}\right) + \lambda\left(\frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2}\right).$$

Theorem 2.1. Let $\psi(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $\lambda > 0$. If $f \in S(\lambda;\psi)$, then there exists a convex function $q(z,\zeta)$ such that $q(z,\zeta) \prec\prec \psi(z,\zeta)$ and $f \in S(0;q)$.

Proof. Suppose that

$$p(z,\zeta) = \frac{f(z,\zeta) - f(-z,\zeta)}{2z} = 1 + \frac{1}{2}\sum_{k=2}^{\infty}(1-(-1)^k)\, a_k(\zeta) z^{k-1}. \tag{2.1}$$

Then, $p \in \mathcal{H}^*[1,1,\zeta]$.
Since $f \in S(\lambda;\psi)$, then we have

$$(1-\lambda)\left(\frac{f(z,\zeta) - f(-z,\zeta)}{2z}\right) + \lambda\left(\frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2}\right) \prec\prec \psi(z,\zeta). \tag{2.2}$$

From (2.1) and (2.2), we get

$$(1-\lambda)\left(\frac{f(z,\zeta) - f(-z,\zeta)}{2z}\right) + \lambda\left(\frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2}\right) = p(z,\zeta) + \lambda z p_z'(z,\zeta) \prec\prec \psi(z,\zeta).$$

By using Lemma 1.1, we obtain

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta).$$

So

$$\frac{f(z,\zeta)-f(-z,\zeta)}{2z} \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta),$$

where

$$q(z,\zeta)=\frac{1}{\lambda}z^{-\frac{1}{\lambda}}\int_0^z \psi(t,\zeta)\, t^{\frac{1}{\lambda}-1}dt$$

is convex and it is the best dominant.

Theorem 2.2. Let $\psi(z,\zeta)$ be a convex function in $U\times\overline{U}$ with $\psi(0,\zeta)=1$ for every $\zeta\in\overline{U}$ and $\lambda>0$. If $f\in T(\lambda;\psi)$, $\frac{f(z,\zeta)-f(-z,\zeta)}{2z}\in\mathcal{H}^*[1,1,\zeta]\cap Q_\zeta$ and

$$(1-\lambda)\left(\frac{f(z,\zeta)-f(-z,\zeta)}{2z}\right)+\lambda\left(\frac{f_z'(z,\zeta)-f_z'(-z,\zeta)}{2}\right)$$

is univalent in $U\times\overline{U}$, then there exists a convex function $q(z,\zeta)$ such that $f\in T(0;q)$.

Proof. Let the function $p(z,\zeta)$ be defined by (2.1). Then $p\in\mathcal{H}^*[1,1,\zeta]\cap Q_\zeta$. After a short calculation and considering $f\in T(\lambda;\psi)$, we can conclude that

$$\psi(z,\zeta)\prec\prec p(z,\zeta)+\lambda z p_z'(z,\zeta).$$

By using Lemma 1.2, we obtain

$$q(z,\zeta)\prec\prec p(z,\zeta).$$

So

$$q(z,\zeta)\prec\prec \frac{f(z,\zeta)-f(-z,\zeta)}{2z},$$

where

$$q(z,\zeta)=\frac{1}{\lambda}z^{-\frac{1}{\lambda}}\int_0^z \psi(t,\zeta)\, t^{\frac{1}{\lambda}-1}dt$$

is convex and it is the best subordinant.

If we combine the results of Theorem 2.1 and Theorem 2.2, we obtain the following strong differential "sandwich theorem".

Theorem 2.3. Let $\psi_1(z,\zeta)$ and $\psi_2(z,\zeta)$ be convex functions in $U\times\overline{U}$ with $\psi_1(0,\zeta)=\psi_2(0,\zeta)=1$ for every $\zeta\in\overline{U}$ and $\lambda>0$. If $f\in S(\lambda;\psi_1)\cap T(\lambda;\psi_2)$, $\frac{f(z,\zeta)-f(-z,\zeta)}{2z}\in\mathcal{H}^*[1,1,\zeta]\cap Q_\zeta$ and

$$(1-\lambda)\left(\frac{f(z,\zeta)-f(-z,\zeta)}{2z}\right)+\lambda\left(\frac{f_z'(z,\zeta)-f_z'(-z,\zeta)}{2}\right)$$

is univalent in $U\times\overline{U}$, then

$$f\in S(0;q_1)\cap T(0;q_2),$$

where

$$q_1(z,\zeta)=\frac{1}{\lambda}z^{-\frac{1}{\lambda}}\int_0^z \psi_1(t,\zeta)\, t^{\frac{1}{\lambda}-1}dt$$

and

$$q_2(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \psi_2(t,\zeta)\, t^{\frac{1}{\lambda}-1} dt.$$

The functions q_1 and q_2 are convex.

Theorem 2.4. Let $\psi(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and

$$G(z,\zeta) = \frac{\epsilon+2}{z^{\epsilon+1}} \int_0^z t^{\epsilon} f(t,\zeta) dt, \quad (z \in U, \zeta \in \overline{U}, Re(\epsilon) > -2\,). \qquad (2.3)$$

If $f \in S(1;\psi)$, then there exists a convex function $q(z,\zeta)$ such that $q(z,\zeta) \prec\prec \psi(z,\zeta)$ and $G \in S(1;q)$.

Proof. Suppose that

$$p(z,\zeta) = \frac{G_z'(z,\zeta) - G_z'(-z,\zeta)}{2}, \quad (z \in U, \zeta \in \overline{U}). \qquad (2.4)$$

Then, $p \in \mathcal{H}^*[1,1,\zeta]$.
From (2.3), we have

$$z^{\epsilon+1} G(z,\zeta) = (\epsilon+2) \int_0^z t^{\epsilon} f(t,\zeta) dt. \qquad (2.5)$$

Differentiating both sides of (2.5) with respect to z, we get

$$f(z,\zeta) = \frac{(\epsilon+1)G(z,\zeta) + zG_z'(z,\zeta)}{\epsilon+2}\,. \qquad (2.6)$$

By using (2.4) and (2.6), we obtain

$$p(z,\zeta) + \frac{1}{\epsilon+2} zp_z'(z,\zeta) = \frac{\epsilon+1}{\epsilon+2} p(z,\zeta) + \frac{1}{\epsilon+2}\big(zp_z'(z,\zeta) + p(z,\zeta)\big)$$
$$= \frac{\big((\epsilon+1)G(z,\zeta) + zG_z'(z,\zeta)\big)_z' - \big((\epsilon+1)G(-z,\zeta) + zG_z'(-z,\zeta)\big)_z'}{2(\epsilon+2)}$$
$$= \frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2}. \qquad (2.7)$$

Since $f \in S(1;\psi)$, then we have

$$\frac{f_z'(z,\zeta) - f_z'(-z,\zeta)}{2} \prec\prec \psi(z,\zeta). \qquad (2.8)$$

From (2.7) and (2.8), we arrive at

$$p(z,\zeta) + \frac{1}{\epsilon+2} zp_z'(z,\zeta) \prec\prec \psi(z,\zeta).$$

By using Lemma 1.1, we obtain

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta).$$

So

$$\frac{G_z'(z,\zeta) - G_z'(-z,\zeta)}{2} \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta),$$

where

$$q(z,\zeta) = (\epsilon+2)z^{-(\epsilon+2)}\int_0^z \psi(t,\zeta)\, t^{\epsilon+1}dt$$

is convex and it is the best dominant.

Theorem 2.5. Let $\psi(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $G(z,\zeta)$ is given by (2.3). If $f \in T(1;\psi)$, $\frac{G_z'(z,\zeta)-G_z'(-z,\zeta)}{2} \in \mathcal{H}^*[1,1,\zeta] \cap Q_\zeta$ and $\frac{f_z'(z,\zeta)-f_z'(-z,\zeta)}{2}$ is univalent in $U \times \overline{U}$, then there exists a convex function $q(z,\zeta)$ such that $G \in T(1;q)$.

Proof. Let the function $p(z,\zeta)$ be defined by (2.4). Then $p \in \mathcal{H}^*[1,1,\zeta] \cap Q_\zeta$. After a short calculation and considering $f \in T(1;\psi)$, we can conclude that

$$\psi(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\epsilon+2} zp_z'(z,\zeta).$$

By using Lemma 1.2, we obtain

$$q(z,\zeta) \prec\prec p(z,\zeta).$$

So

$$q(z,\zeta) \prec\prec \frac{G_z'(z,\zeta)-G_z'(-z,\zeta)}{2},$$

where

$$q(z,\zeta) = (\epsilon+2)z^{-(\epsilon+2)}\int_0^z \psi(t,\zeta)\, t^{\epsilon+1}dt$$

is convex and it is the best subordinant.

If we combine the results of Theorem 2.4 and Theorem 2.5, we obtain the following strong differential "sandwich theorem".

Theorem 2.6. Let $\psi_1(z,\zeta)$ and $\psi_2(z,\zeta)$ be convex functions in $U \times \overline{U}$ with $\psi_1(0,\zeta) = \psi_2(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $G(z,\zeta)$ is given by (2.3). If $f \in S(1;\psi_1) \cap T(1;\psi_2)$, $\frac{G_z'(z,\zeta)-G_z'(-z,\zeta)}{2} \in \mathcal{H}^*[1,1,\zeta] \cap Q_\zeta$ and $\frac{f_z'(z,\zeta)-f_z'(-z,\zeta)}{2}$ is univalent in $U \times \overline{U}$, then

$$G \in S(1;q_1) \cap T(1;q_2)$$

where

$$q_1(z,\zeta) = (\epsilon+2)z^{-(\epsilon+2)}\int_0^z \psi_1(t,\zeta)\, t^{\epsilon+1}dt$$

and

$$q_2(z,\zeta) = (\epsilon+2)z^{-(\epsilon+2)}\int_0^z \psi_2(t,\zeta)\, t^{\epsilon+1}dt.$$

The functions q_1 and q_2 are convex.

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