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On a differential subordinations of multivalent analytic functions defined by linear operator

Research Article

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Abstract: In this paper, we introduce and study a class of multivalent analytic functions which are defined by means of a linear operator. We obtain some results connected to inclusion relationship, argument estimate, integral representation and subordination property.

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Keywords: Multivalent functions • Subordination • Integral representation • Linear operator

1. Introduction

Let $\mathscr{A}_p$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in N = \{1, 2, \cdots\}, \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in C : |z| < 1\}$ and let $\mathscr{A}_1 = \mathscr{A}$.
Given two functions f and g which are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.
For functions f given by (1) and $g \in \mathscr{A}_p$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Recently, Mahzoon and Latha [3] introduced and investigated the operator $D_p(\mu, c, \lambda) : \mathscr{A}_p \longrightarrow \mathscr{A}_p$ defined by

$$D_p(\mu, c, \lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(1 + \frac{k-p}{p+c} \lambda\right)^{\mu} a_k z^k,$$

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where $\mu, c, \lambda \in R, \mu, c, \lambda \geq 0$.
In [6], Mustafa and Darus defined the linear operator $D_p^{\alpha,\delta}(\mu, c, \lambda) : \mathscr{A}_p \longrightarrow \mathscr{A}_p$ in terms of the Hadamard product by

$$D_p^{\alpha,\delta}(\mu, c, \lambda) = k^\alpha * D_p(\mu, c, \lambda) * \mathscr{R}^\delta, \tag{2}$$

where $\mu, c, \lambda \in R, \mu, c, \lambda \geq 0, \alpha, \delta \in N_0 = N \cup \{0\}$ and $\mathscr{R}^\delta$ denotes the Ruscheweyh derivative operator [8] given by

$$\mathscr{R}^\delta f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\delta + k)}{\Gamma(\delta + 1)\Gamma(k)} a_k z^k, \quad (\delta \in N_0, z \in U).$$

It is easy to obtain from (2) that

$$D_p^{\alpha,\delta}(\mu, c, \lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} k^\alpha \left(1 + \frac{k-p}{p+c}\lambda\right)^\mu \frac{\Gamma(\delta + k)}{\Gamma(\delta + 1)\Gamma(k)} a_k z^k. \tag{3}$$

In view of (3), we obtain the following relation:

$$\lambda z \left(D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)\right)' = (p + c) D_p^{\alpha,\delta}(\mu + 1, c, \lambda) f(z) - \left(c - p(\lambda - 1)\right) D_p^{\alpha,\delta}(\mu, c, \lambda) f(z). \tag{4}$$

Let T be the class of functions h of the form:

$$h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k,$$

which are analytic and convex univalent in U and satisfy the condition:

$$Re\{h(z)\} > 0, \quad (z \in U).$$

Definition 1.1.
A function $f \in \mathscr{A}_p$ is said to be in the class $L(\mu, c, \lambda, \alpha, \delta, p, \gamma; h)$ if it satisfies the following differential subordination condition:

$$\frac{1}{p-\gamma}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)\right)'}{D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)} - \gamma\right) \prec h(z), \tag{5}$$

where $\mu, c, \lambda \in R, \mu, c, \lambda \geq 0, \alpha, \delta \in N_0 = N \cup \{0\}, p \in N, 0 \leq \gamma < p$ and $h \in T$.

We will require the following lemmas in proving our main results.

Lemma 1.1 (Eenigenburg and et al. [2]).
Let $u, v \in C$ and suppose that ψ is convex and univalent in U with $\psi(0) = 1$ and $Re\{u\psi(z) + v\} > 0, (z \in U)$. If q is analytic in U with $q(0) = 1$, then the subordination

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z)$$

implies that $q(z) \prec \psi(z)$.

Lemma 1.2 (Miller and Mocanu [4]).
Let h be convex univalent in U and $\mathscr{T}$ be analytic in U with $Re\{\mathscr{T}(z)\} \geq 0, (z \in U)$. If q is analytic in U and $q(0) = h(0)$, then the subordination

$$q(z) + \mathscr{T}(z) z q'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

Lemma 1.3 (Ebadian and et al. [1]).
Let q be analytic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exists two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2} b_1 = arg\left(q(z_1)\right) < arg\left(q(z)\right) < arg\left(q(z_2)\right) = \frac{\pi}{2} b_2,$$

for some b_1 and b_2 $(b_1 > 0, b_2 > 0)$ and for all z $(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right) m \quad and \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right) m,$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad and \quad \varepsilon = i\,tan\frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

Lemma 1.4 (Robertson [7]).
The function

$$(1-z)^{\eta} \equiv \exp\left(\log(1-z)\right), (\eta \neq 0)$$

is univalent if and only if η *is either in the closed disk* $|\eta - 1| \leq 1$ *or in the closed disk* $|\eta + 1| \leq 1$.

Lemma 1.5 (Miller and Mocanu [5]).
Let q *be univalent in the unit disk* U *and let* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = zq'(z)\phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that*
1- $Q(z)$ *is starlike univalent in* U,
2- $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.
If k *is analytic in* U, *with* $k(0) = q(0), k(U) \subset D$ *and*

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $k \prec q$ *and* q *is the best dominant.*

2. Main Results

Theorem 2.1.
Let $Re\left\{(p-\gamma)h(z) + \gamma + \frac{c-p(\lambda-1)}{\lambda}\right\} > 0$. *Then* $L(\mu+1, c, \lambda, \alpha, \delta, p, \gamma; h) \subset L(\mu, c, \lambda, \alpha, \delta, p, \gamma; h)$.

Proof. Let $f \in L(\mu+1, c, \lambda, \alpha, \delta, p, \gamma; h)$ and put

$$q(z) = \frac{1}{p-\gamma}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)\right)'}{D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)} - \gamma\right). \tag{6}$$

Then q is analytic in U with $q(0) = 1$. According to (6) and using the relation (4), we obtain

$$\frac{p+c}{\lambda}\frac{D_p^{\alpha,\delta}(\mu+1, c, \lambda) f(z)}{D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)} = (p-\gamma)q(z) + \gamma + \frac{c-p(\lambda-1)}{\lambda}. \tag{7}$$

By logarithmically differentiating both sides of (7) with respect to z and multiplying by z, we get

$$q(z) + \frac{zq'(z)}{(p-\gamma)q(z) + \gamma + \frac{c-p(\lambda-1)}{\lambda}} = \frac{1}{p-\gamma}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu+1, c, \lambda) f(z)\right)'}{D_p^{\alpha,\delta}(\mu+1, c, \lambda) f(z)} - \gamma\right) \prec h(z). \tag{8}$$

Since $Re\left\{(p-\gamma)h(z) + \gamma + \frac{c-p(\lambda-1)}{\lambda}\right\} > 0$, then applying Lemma 1.1 to the subordination (8), yields $q(z) \prec h(z)$, which implies $f \in L(\mu, c, \lambda, \alpha, \delta, p, \gamma; h)$. □

Theorem 2.2.
Let $f \in \mathcal{A}_p$, $0 < a_1, a_2 \leq 1$ *and* $0 \leq \gamma < p$. *If*

$$-\frac{\pi}{2}a_1 < arg\left(\frac{z\left(D_p^{\alpha,\delta}(\mu+1, c, \lambda) f(z)\right)'}{D_p^{\alpha,\delta}(\mu+1, c, \lambda) g(z)} - \gamma\right) < \frac{\pi}{2}a_2,$$

for some $g \in L(\mu+1, c, \lambda, \alpha, \delta, p, \gamma; \frac{1+Az}{1+Bz})$, $(-1 \leq B < A \leq 1)$, *then*

$$-\frac{\pi}{2}b_1 < arg\left(\frac{z\left(D_p^{\alpha,\delta}(\mu, c, \lambda) f(z)\right)'}{D_p^{\alpha,\delta}(\mu, c, \lambda) g(z)} - \gamma\right) < \frac{\pi}{2}b_2,$$

where b_1 *and* b_2 $(0 < b_1, b_2 \leq 1)$ *are the solutions of the equations:*

$$a_1 = \begin{cases} b_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B} + \gamma + \frac{c-p(\lambda-1)}{\lambda}\right) + (1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1 \\ b_1, & B = -1. \end{cases} \tag{9}$$

and

$$a_2 = \begin{cases} b_2 + \frac{2}{\pi}\tan^{-1}\left(\dfrac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B \neq -1 \\ b_2 & , B=-1. \end{cases} \tag{10}$$

with

$$\varepsilon = i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right) \quad \textit{and} \quad t = \frac{2}{\pi}\sin^{-1}\left(\frac{(A-B)\left(p-\gamma\right)}{\left(\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)(1-B^2)+\left(p-\gamma\right)(1-AB)}\right). \tag{11}$$

Proof. Define the function G by

$$G(z) = \frac{1}{p-\tau}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu,c,\lambda)g(z)} - \tau\right), \tag{12}$$

where $g \in L(\mu+1,c,\lambda,\alpha,\delta,p,\gamma;\frac{1+Az}{1+Bz})$, $(-1 \le B < A \le 1)$ and $0 \le \tau < p$.
Then G is analytic in U with $G(0) = 1$. Therefore by making use of (4) and (12), we obtain

$$\left(\left(p-\tau\right)G(z)+\tau\right)D_p^{\alpha,\delta}(\mu,c,\lambda)g(z) = \frac{p+c}{\lambda}D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z) - \frac{c-p(\lambda-1)}{\lambda}D_p^{\alpha,\delta}(\mu,c,\lambda)f(z).$$

Differentiating above relation with respect to z and multiplying by z, we get

$$\begin{aligned} &\left(\left(p-\tau\right)G(z)+\tau\right)z\left(D_p^{\alpha,\delta}(\mu,c,\lambda)g(z)\right)' + \left(p-\tau\right)zG'(z)D_p^{\alpha,\delta}(\mu,c,\lambda)g(z) \\ &= \frac{p+c}{\lambda}z\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)\right)' - \frac{c-p(\lambda-1)}{\lambda}z\left(D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)'. \end{aligned} \tag{13}$$

Suppose that

$$H(z) = \frac{1}{p-\gamma}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu,c,\lambda)g(z)\right)'}{D_p^{\alpha,\delta}(\mu,c,\lambda)g(z)} - \gamma\right).$$

Using (4) again, we have

$$\frac{p+c}{\lambda}\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)g(z)} = \left(p-\gamma\right)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}. \tag{14}$$

From (13) and (14), we easily get

$$G(z) + \frac{zG'(z)}{\left(p-\gamma\right)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}} = \frac{1}{p-\tau}\left(\frac{z\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z)} - \tau\right). \tag{15}$$

Notice that from Theorem 2.1, $g \in L(\mu+1,c,\lambda,\alpha,\delta,p,\gamma;\frac{1+Az}{1+Bz})$ implies $g \in L(\mu,c,\lambda,\alpha,\delta,p,\gamma;\frac{1+Az}{1+Bz})$. Thus,

$$H(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1).$$

By using the result of Silverman and Silvia [9], we have

$$\left|H(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (B \neq -1, z \in U) \tag{16}$$

and

$$Re\{H(z)\} > \frac{1-A}{2} \quad (B = -1, z \in U). \tag{17}$$

It follows from (16) and (17) that

$$\left|\left(p-\gamma\right)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}-\frac{\left(\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)\left(1-B^2\right)+\left(p-\gamma\right)(1-AB)}{1-B^2}\right|<\frac{(A-B)\left(p-\gamma\right)}{1-B^2},$$

$$(B\neq -1, z\in U)$$

and

$$Re\left\{\left(p-\gamma\right)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}\right\}>\frac{(1-A)\left(p-\gamma\right)}{2}+\gamma+\frac{c-p(\lambda-1)}{\lambda},\quad (B=-1, z\in U).$$

Putting

$$\left(p-\gamma\right)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}=\rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)\left(p-\gamma\right)}{\left(\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)\left(1-B^2\right)+\left(p-\gamma\right)(1-AB)}<\phi<\frac{(A-B)\left(p-\gamma\right)}{\left(\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)\left(1-B^2\right)+\left(p-\gamma\right)(1-AB)},(B\neq -1)$$

and $-1<\phi<1$, $(B=-1)$.
Then

$$\frac{(1-A)\left(p-\gamma\right)}{1-B}+\gamma+\frac{c-p(\lambda-1)}{\lambda}<\rho<\frac{(1+A)\left(p-\gamma\right)}{1+B}+\gamma+\frac{c-p(\lambda-1)}{\lambda},\quad (B\neq -1)$$

and

$$\frac{(1-A)\left(p-\gamma\right)}{1-B}+\gamma+\frac{c-p(\lambda-1)}{\lambda}<\rho<\infty,\quad (B=-1).$$

An application of Lemma 1.2 with $\mathscr{T}(z)=\frac{1}{(p-\gamma)H(z)+\gamma+\frac{c-p(\lambda-1)}{\lambda}}$, yields $G(z)\prec h(z)$.
If there exist two points $z_1, z_2\in U$ such that

$$-\frac{\pi}{2}b_1=arg\left(G(z_1)\right)<arg\left(G(z)\right)<arg\left(G(z_2)\right)=\frac{\pi}{2}b_2,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)}=-\frac{mi}{2}\left(b_1+b_2\right)\quad and\quad \frac{z_2G'(z_2)}{G(z_2)}=\frac{mi}{2}\left(b_1+b_2\right),$$

where

$$m\geq\frac{1-|\varepsilon|}{1+|\varepsilon|}\quad and\quad \varepsilon=i\,tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B\neq -1$, we obtain

$$\begin{aligned}
&arg\left(\frac{1}{p-\tau}\left(\frac{z_1\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z_1)\right)'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z_1)}-\tau\right)\right)\\
&=arg\left(G(z_1)+\frac{z_1G'(z_1)}{\left(p-\gamma\right)H(z_1)+\gamma+\frac{c-p(\lambda-1)}{\lambda}}\right)\\
&=arg\left(G(z_1)\right)+arg\left(1+\frac{z_1G'(z_1)}{\left[\left(p-\gamma\right)H(z_1)+\gamma+\frac{c-p(\lambda-1)}{\lambda}\right]G(z_1)}\right)\\
&=-\frac{\pi}{2}b_1+arg\left(1-\frac{mi}{2\rho}\left(b_1+b_2\right)e^{-i\frac{\pi}{2}\phi}\right)\\
&=-\frac{\pi}{2}b_1+arg\left(1-\frac{m}{2\rho}\left(b_1+b_2\right)\cos\frac{\pi}{2}(1-\phi)+\frac{mi}{2\rho}\left(b_1+b_2\right)\sin\frac{\pi}{2}(1-\phi)\right)\\
&\leq-\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{m\left(b_1+b_2\right)\sin\frac{\pi}{2}(1-\phi)}{2\rho+m\left(b_1+b_2\right)\cos\frac{\pi}{2}(1-\phi)}\right)\\
&\leq-\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{(1-|\varepsilon|)\left(b_1+b_2\right)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)+(1-|\varepsilon|)\left(b_1+b_2\right)\sin\frac{\pi}{2}t}\right)\\
&=-\frac{\pi}{2}a_1,
\end{aligned}$$

where a_1 and t are given by (9) and (11), respectively.
Also,

$$arg\left(\frac{1}{p-\tau}\left(\frac{z_2\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z_2)\right)'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z_2)}-\tau\right)\right)$$
$$\geq \frac{\pi}{2}b_2+\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\gamma+\frac{c-p(\lambda-1)}{\lambda}\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$
$$=\frac{\pi}{2}a_2,$$

where a_2 and t are given by (10) and (11), respectively.
Similarly, for the case $B=-1$, we have

$$arg\left(\frac{1}{p-\tau}\left(\frac{z_1\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z_1)\right)'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z_1)}-\tau\right)\right)\leq -\frac{\pi}{2}b_1$$

and

$$arg\left(\frac{1}{p-\tau}\left(\frac{z_2\left(D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z_2)\right)'}{D_p^{\alpha,\delta}(\mu+1,c,\lambda)g(z_2)}-\tau\right)\right)\geq \frac{\pi}{2}b_2.$$

The above two cases contradict the assumptions. Consequently, the proof of the theorem is complete. □

In the following theorem, we find integral representation of the class $L(\mu,c,\lambda,\alpha,\delta,p,\gamma;h)$.

Theorem 2.3.
Let $f\in L(\mu,c,\lambda,\alpha,\delta,p,\gamma;h)$. Then

$$D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)=z^p.\exp\left[\left(p-\gamma\right)\int_0^z\frac{h(w(s))-1}{s}ds\right],$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1,(z\in U)$.

Proof. Assume that $f\in L(\mu,c,\lambda,\alpha,\delta,p,\gamma;h)$. It is easy to see that subordination condition (5) can be written as follows

$$\frac{z\left(D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}=\left(p-\gamma\right)h(w(z))+\gamma, \tag{18}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1,(z\in U)$.
From (18), we find that

$$\frac{\left(D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)'}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}-\frac{p}{z}=\left(p-\gamma\right)\frac{h(w(z))-1}{z}, \tag{19}$$

After integrating both sides of (19), we have

$$\log\left(\frac{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}{z^p}\right)=\left(p-\gamma\right)\int_0^z\frac{h(w(s))-1}{s}ds. \tag{20}$$

Therefore, from (20), we obtain the required result. □

Theorem 2.4.
Let $1<\beta<2$ and $\eta\in R\backslash\{0\}$ such that either $\left|\frac{2\eta(\beta-1)(p+c)}{\lambda}+1\right|\leq 1$ or $\left|\frac{2\eta(\beta-1)(p+c)}{\lambda}-1\right|\leq 1$. If $f\in\mathscr{A}_p$ satisfies the condition

$$Re\left\{1+\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)}\right\}>2-\beta+\frac{\lambda(1-p)}{p+c}, \tag{21}$$

then

$$\left(zD_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)^\eta\prec(1-z)^{-\frac{2\eta(\beta-1)(p+c)}{\lambda}}$$

and $(1-z)^{-\frac{2\eta(\beta-1)(p+c)}{\lambda}}$ is the best dominant.

Proof. Define the function k by

$$k(z) = \left(zD_p^{\alpha,\delta}(\mu,c,\lambda)f(z)\right)^{\eta}. \tag{22}$$

Differentiating (22) with respect to z logarithmically and using (4), we obtain

$$\frac{zk'(z)}{k(z)} = \frac{\eta(p+c)}{\lambda}\frac{D_p^{\alpha,\delta}(\mu+1,c,\lambda)f(z)}{D_p^{\alpha,\delta}(\mu,c,\lambda)f(z)} - \frac{\eta\left(c-p(\lambda-1)+\lambda\right)}{\lambda}.$$

Now, in view of the condition (21), we have the following subordination

$$1+\frac{\lambda zk'(z)}{\eta(p+c)k(z)} \prec \frac{1+(2\beta-3)z}{1-z}.$$

Assume that

$$\theta(w)=1, \quad \phi(w)=\frac{\lambda}{\eta(p+c)w} \quad and \quad q(z)=(1-z)^{-\frac{2\eta(\beta-1)(p+c)}{\lambda}},$$

then by making use of Lemma 1.4, we know that q is univalent in U. It now follows that

$$Q(z) = zq'(z)\phi\left(q(z)\right) = \frac{2(\beta-1)z}{1-z}.$$

and

$$h(z) = \theta\left(q(z)\right)+Q(z) = \frac{1+(2\beta-3)z}{1-z}.$$

If we define the domain D by

$$q(U) = \left\{w : \left|w^{\frac{1}{\sigma}}-1\right| < \left|w^{\frac{1}{\sigma}}\right|, \sigma = \frac{2\eta\left(\beta-1\right)\left(p+c\right)}{\lambda}\right\} \subset D,$$

then, it is easy to check that the conditions of Lemma 1.5 hold true. Therefore, we get the desired result. □

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