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On Second –Order Differential Subordinations for Multivalent Functions Associated with Komatu Operator

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Abstract. In this paper , we obtain some results for second - order differential subordinations $\psi\,(f(z), zf'(z), z^2 f''(z); z) \prec h(z)$, for multivalent functions in the open unit disk associated with the komatu operator .

Keywords: Univalent function , Multivalent function , Differential subordination , Komatu operator.

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1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mu = \mu\,(U)$ denote the class of analytic functions defined in U , for n positive integer and a $\in \mathbb{C}$. Let $\mu\,[a\,, n] = \{\, f \in \mu : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}$, with $\mu_0 = \mu[0,1]$,$\mu = \mu[1,1]$.

Let f and g be members of μ . The function f is said to be subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w(z) analytic in U, with w(0)= 0 and |w(z)|<1 such that $f(z) = g(w(z))$, (z $\in$ U) .
In particular , if the function g is univalent in U , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If f is analytic in U and satisfies the (second –order) differential subordination

$$\psi(f(z), zf'(z), z^2 f''(z); z) \prec h(z)\,, \tag{1.1}$$

then f is called a solution of the differential subordination . The univalent function q is called a dominant of the solutions of the differential subordination , or more simply dominant if $f \prec q$ for all f satisfying (1.1) A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1) .

Let L(p) denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p} \quad (z \in \mathrm{U}\,, p \in \mathbb{N} = \{1,2,3,\dots\})\,, \qquad (1.2)$$

which are analytic and p-valent in U.

For $f \in$ L(p) , let the komatu operator [4] be denote by

$$K_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta) z^{c}} \int_0^z t^{c-1} \left(log\frac{z}{t}\right)^{\delta-1} f(t)\, dt$$

$$= z^p + \sum_{n=1}^{\infty} \left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p}\, z^{n+p} \quad (c > -p\,, \delta > 0)\ . \qquad (1.3)$$

In order to prove the results , we shall use the following definitions and theorem.

Definition 1.1[2] . Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus \ E(q)$, where

$$\mathrm{E}(q) = \left\{\zeta \in \partial U: \lim_{z\to\zeta} q(z) = \infty\right\} \qquad (1.4)$$

and are such that $q^{'}(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. Further let the subclass of Q for which $q(0) =$ a be denoted by $Q(a)\,, Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.2 [2]. Let Ω be a set in $\mathbb{C}$, $q \in$ Q and let n be positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega$, whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq m\ \mathrm{Re}\left\{\frac{\zeta q^{''}(\zeta)}{q^{'}(\zeta)} + 1\right\}, \qquad (1.5)$$

$z \in U, \zeta \in \partial U \setminus E(q), and\ m \geq n$. Let $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

Theorem 1.1[2]. Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $F \in \mu\ [a\,, n]$ satisfies

$$\psi\ (F(z), zF^{'}(z), z^2F^{''}(z); z) \in \Omega\,, \qquad (1.6)$$

then

$$F(z) \prec q(z).$$

2. Main Results

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mu\,[0\,, p]$. The class of admissible functions $\Phi_k[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition :

$$\phi(u, v, w; z) \notin \Omega\,, \tag{2.1}$$

whenever

$$u = q(\zeta)\,, v = \frac{m\zeta q'(\zeta) + cq(\zeta)}{c+p} \quad (p \in \mathbb{N}\,, c > -p\,)\,,$$

and

$$\mathrm{Re}\left\{\frac{(c+p)^2 w - c^2 u}{c+p)v - cu} - 2\mathrm{c}\right\} \geq m\,\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}\,, \tag{2.2}$$

$z \in U, \zeta \in \partial U \backslash E(q)\,, and\ m \geq p$.

Theorem 2.1. Let $\phi \in \Phi_k[\Omega, q]. If\ f \in \mathrm{L}(p)$ satisfies

$$\left\{\phi\left(K_{c,p}^{\delta+2} f(z), K_{c,p}^{\delta+1} f(z), K_{c,p}^{\delta} f(z); z\right): z \in U\,\right\} \subset \Omega\,, \tag{2.3}$$

then

$$K_{c,p}^{\delta+2} f(z) \prec q(z)\,.$$

Proof. We note from (1.3)that, we have

$$z\left(K_{c,p}^{\delta+1} f(z)\right)' = (c+p)K_{c,p}^{\delta} f(z) - c\,K_{c,p}^{\delta+1} f(z)\,, \tag{2.4}$$

is equivalent to

$$K_{c,p}^{\delta} f(z) = \frac{z\left(K_{c,p}^{\delta+1} f(z)\right)' + cK_{c,p}^{\delta+1} f(z)}{(c+p)}\,, \tag{2.5}$$

and

$$K_{c,p}^{\delta+1} f(z) = \frac{z\left(K_{c,p}^{\delta+2} f(z)\right)' + cK_{c,p}^{\delta+2} f(z)}{(c+p)}\,. \tag{2.6}$$

Let the analytic function F in U defined by

$$F(z) = K_{c,p}^{\delta+2} f(z) . \tag{2.7}$$

Then we have

$$K_{c,p}^{\delta+1} f(z) = \frac{zF'(z) + cF(z)}{c+p},$$

$$K_{c,p}^{\delta} f(z) = \frac{z^2F''(z) + (1+2c)zF'(z) + c^2F(z)}{(c+p)^2} . \tag{2.8}$$

Further , let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r \ , \qquad v = \frac{s+cr}{c+p} \ , \qquad w = \frac{t + (1+2c)s + c^2 r}{(c+p)^2} .$$

Let

$$\psi\,(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+cr}{c+p}, \frac{t+(1+2c)s+c^2r}{(c+p)^2}; z\right). \tag{2.9}$$

The proof will make use of Theorem 1.1. Using (2.7) and (2.8) , from (2.9) , we obtain

$$\psi\,(F(z), zF'(z), z^2F''(z); z) = \phi\big(K_{c,p}^{\delta+2} f(z), K_{c,p}^{\delta+1} f(z), K_{c,p}^{\delta} f(z); z\big). \tag{2.10}$$

Therefore (2.3) becomes

$$\psi\,(F(z), zF'(z), z^2F''(z); z) \in \Omega . \tag{2.11}$$

Note that

$$\frac{t}{s} + 1 = \frac{(c+p)^2 w - c^2 u}{c+p)v - cu} - 2c , \tag{2.12}$$

and since the admissibility condition for $\phi \in \Phi_{\mathrm{k}}[\Omega, q]$is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence $\psi \in \Psi_p[\Omega, q]$, and by Theorem 1.1, *F(z)* $\prec q(z)$.

By (2.7), we get

$$K_{c,p}^{\delta+2} f(z) \prec q(z) .$$

In the case $\phi(u,v,w;z) = v$, we have the following example .

Example 2.1. Let the class of admissible functions $\Phi_{kv}[\Omega, q]$ consist of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition :

$$v = \frac{m\zeta q'(\zeta) + cq(\zeta)}{c+p} \notin \Omega \ ,$$

$z \in U, \zeta \in \partial U \backslash E(q)$, $and\ m \geq p. If\ f \in$ L(p) satisfies $K_{c,p}^{\delta+1} f(z) \subset \Omega$, then

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω and the class is written as $\Phi_k[h,q]$. The following result follows immediately from Theorem 2.1.

Theorem 2.2. Let $\phi \in \Phi_k[\Omega, q]. If\ f \in$ L(p) satisfies

$$\phi\left(K_{c,p}^{\delta+2} f(z), K_{c,p}^{\delta+1} f(z), K_{c,p}^{\delta} f(z); z\right) \prec h(z), \tag{2.13}$$

then

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

The next results occurs when the behavior of q on ∂U is not known .

Corollary 2.1. Let $\Omega \subset \mathbb{C}$, q be univalent in U and $q(0) = a$. Let $\phi \in \Phi_k\left[\Omega, q_\rho\right]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. $If\ f \in$ L(p) satisfies

$$\phi\left(K_{c,p}^{\delta+2} f(z), K_{c,p}^{\delta+1} f(z), K_{c,p}^{\delta} f(z); z\right) \in \Omega \, , \tag{2.14}$$

then

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

Proof. From Theorem 2.1,we have $K_{c,p}^{\delta+2} f(z) \prec q_\rho(z)$ and the proof is complete .

Theorem 2.3. Let h and q be univalent in U, with $q(0) = a$ and set $q_\rho(z) = q(\rho z)\ and\ h_\rho(z) = h(\rho z)$. Let $\phi: \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions :

(1) $\phi \in \Phi_k\left[h, q_\rho\right]$, for some $\rho \in (0,1)$, or

(2) there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_k[h_\rho, q_\rho]$, for all $\rho_0 \in (0,1)$.
If $f \in L(p)$ satisfies (2.13) ,then

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

Proof .

case (1): By applying Theorem 2.1 ,we obtain $K_{c,p}^{\delta+2} f(z) \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$ we deduce

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

case (2): If we let $F(z) = K_{c,p}^{\delta+2} f(z)$ and let $F_\rho(z) = F(\rho z)$, then

$$\phi\left(F_\rho(z), zF_\rho{}'(z), z^2 F_\rho{}''(z); \rho z\right) = \phi(F(\rho z), \rho z F'(\rho z), \rho^2 z^2 F''(\rho z); \rho z) \in h_\rho(U) \ .$$

By using Theorem 2.1 and the comment associated with $\phi\ (F(z), zF'(z), z^2 F''(z); w(z)) \in \Omega$, where w is any function mapping U into U , with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0,1)$. By letting $\rho \longrightarrow 1^-$, we get $F(z) \prec q(z)$.
Therefore

$$K_{c,p}^{\delta+2} f(z) \prec q(z) \ .$$

The next result give the best dominant of the differential subordination (2.13).

Theorem 2.4. Let h be univalent in U and let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$. Suppose that the differential equation

$$\phi\ (q(z), zq'(z), z^2 q''(z); z) = h(z) \qquad (2.15)$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions :

(1) $q \in Q_0$ and $\phi \in \Phi_k[h, q]$,

(2) q is univalent in U and $\phi \in \Phi_k[h, q_\rho]$, for some $\rho \in (0,1)$, or

(3) q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_k[h_\rho, q_\rho]$,for all $\rho_0 \in (0,1)$. If $f \in L(p)$ satisfies (2.13) ,then $K_{c,p}^{\delta+2} f(z) \prec q(z)$ and q is the best dominant .

Proof . By applying Theorem 2.2 and Theorem 2.3 , we deduce that q is a dominant of (2.13) . Since q satisfies (2.15) , it is also a solution of (2.13) and therefore q will be dominated by all dominants of (2.13) . Hence q is the best dominant of (2.13) .

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap \mu_0$. The class of admissible functions $\Phi_{k,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition :

$$\phi(u,v,w;z) \notin \Omega ,$$

whenever

$$u = q(\zeta), \quad v = \frac{m\zeta q'(\zeta) + (c+p-1)q(\zeta)}{c+p} \quad (p \in \mathbb{N}, c > -p),$$

and

$$\mathrm{Re}\left\{\frac{(c+p)^2 w - (c+p-1)^2 u}{(c+p)v - (c+p-1)u} - 2(c+p-1)\right\} \geq m \,\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}, \quad (2.16)$$

$z \in U, \zeta \in \partial U \setminus E(q)$, and $m \geq 1$.

Theorem 2.5. Let $\phi \in \Phi_{k,1}[\Omega, q]$. If $f \in \mathrm{L}(p)$ satisfies

$$\left\{\phi\left(\frac{K_{c,p}^{\delta+2} f(z)}{z^{p-1}}, \frac{K_{c,p}^{\delta+1} f(z)}{z^{p-1}}, \frac{K_{c,p}^{\delta} f(z)}{z^{p-1}}; z\right) : z \in U\right\} \subset \Omega, \quad (2.17)$$

then

$$\frac{K_{c,p}^{\delta+2} f(z)}{z^{p-1}} \prec q(z) .$$

Proof . Let the analytic function F in U defined by

$$F(\mathrm{z}) = \frac{K_{c,p}^{\delta+2} f(z)}{z^{p-1}} . \quad (2.18)$$

By using the relations (2.4) and (2.18) , we get

$$\frac{K_{c,p}^{\delta+1} f(z)}{z^{p-1}} = \frac{zF'(z) + (c+p-1)F(z)}{c+p},$$

$$\frac{K_{c,p}^{\delta} f(z)}{z^{p-1}} = \frac{z^2 F''(z) + [2(c+p)-1]zF'(z) + (c+p-1)^2 F(z)}{(c+p)^2}. \quad (2.19)$$

Further ,let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{s + (c+p-1)r}{c+p}, w = \frac{t + [2(c+p)-1]s + (c+p-1)^2 r}{(c+p)^2}.$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$
$$=\phi\left(r,\frac{s+(c+p-1)r}{c+p},\frac{t+[2(c+p)-1]s+(c+p-1)^2r}{(c+p)^2};z\right). \quad (2.20)$$

The proof will make use of Theorem 1.1. Using (2.18) and (2.19) , from (2.20) , we obtain

$$\psi\left(F(z),zF'(z),z^2F''(z);z\right)=\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta+1}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta}f(z)}{z^{p-1}};\ z\right). \quad (2.21)$$

Therefore (2.17) becomes

$$\psi\left(F(z),zF'(z),z^2F''(z);z\right)\in\Omega . \quad (2.22)$$

Note that

$$\frac{t}{s}+1=\frac{(c+p)^2w-(c+p-1)^2u}{(c+p)v-(c+p-1)u}-2(c+p-1)\,, \quad (2.23)$$

and since the admissibility condition for $\phi\in\Phi_{k,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence $\psi\in\Psi[\Omega,q]$, and by Theorem 1.1, $F(z)\prec q(z)$.

By (2.18) , we get

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}}\prec q(z)\,.$$

In case $\phi(u,v,w;z)=v-u$, we have the following example .

Example 2.2. Let the class of admissible functions $\Phi_{kv,1}[\Omega,q]$ consist of those functions $\phi:\mathbb{C}^3\times U\longrightarrow\mathbb{C}$ that satisfy the admissibility condition :

$$v-u=\frac{m\zeta q'(\zeta)-q(\zeta)}{c+p}\notin\Omega\,,$$

$z\in U,\zeta\in\partial U\backslash E(q)$, *and* $m\geq p$. *If* $f\in \mathrm{L}(p)$ satisfies

$$\frac{K_{c,p}^{\delta+1}f(z)}{z^{p-1}}-\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}}\subset\Omega\,,$$

then

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}}\prec q(z)\,.$$

We consider the special situation when $\Omega\neq\mathbb{C}$ is a simply connected domain.

In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω and the class is written as $\Phi_{k,1}[h,q]$. The following result follows immediately from Theorem 2.5.

Theorem 2.6. Let $\phi \in \Phi_{k,1}[\Omega,q]$. $If\ f \in \mathrm{L}(p)$ satisfies

$$\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta+1}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta}f(z)}{z^{p-1}};z\right) \prec h(z), \qquad (2.24)$$

then

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)\,.$$

The next results occurs when the behavior of q on ∂U is not known .

Corollary 2.1. Let $\Omega \subset \mathbb{C}$, q be univalent in U and $q(0) = 0$. Let $\phi \in \Phi_{k,1}[\Omega, q_\rho]$ for some $\rho \in (0,1)$,where $q_\rho(z) = q(\rho z)$. $If\ f \in \mathrm{L}(p)$ satisfies

$$\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta+1}f(z)}{z^{p-1}},\frac{K_{c,p}^{\delta}f(z)}{z^{p-1}};z\right) \in \Omega\,, \qquad (2.25)$$

then

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)\,.$$

Proof. From Theorem 2.5,we have

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q_\rho(z)$$

and the proof is complete .

Theorem 2.7 Let h and q be univalent in U , with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)\ and\ h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions :

(1) $\phi \in \Phi_{k,1}[h, q_\rho]$, for some $\rho \in (0,1)$, or
(2) there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_{k,1}[h_\rho, q_\rho]$, for all $\rho_0 \in (0,1)$.
$If\ f \in \mathrm{L}(p)$ satisfies (2.24) ,then

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)$$

Proof .

case (1): By applying Theorem 2.5, we obtain $\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$ we deduce

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z).$$

case (2): If we let $F(z) = \frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}}$ and let $F_\rho(z) = F(\rho z)$, then

$$\phi\,(F_\rho(z), zF_\rho{}'(z), z^2F_\rho{}''(z); \rho z) = \phi(F(\rho z), \rho zF'(\rho z), \rho^2z^2F''(\rho z); \rho z) \in h_\rho(U).$$

By using Theorem 2.5 and the comment associated with $\phi\,(F(z), zF'(z), z^2F''(z); w(z)) \in \Omega$, where w is any function mapping U into U , with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \longrightarrow 1^-$, we get $F(z) \prec q(z)$.

Therefore

$$\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z).$$

The next result give the best dominant of the differential subordination (2.24).

Theorem 2.8. Let h be univalent in U and let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ .Suppose that the differential equation

$$\phi\,(q(z), zq'(z), z^2q''(z); z) = h(z) \qquad (2.26)$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions :

(1) $q \in Q_0$ and $\phi \in \Phi_{k,1}[h, q]$,

(2) q is univalent in U and $\phi \in \Phi_{k,1}[h, q_\rho]$, for some $\rho \in (0,1)$, or

(3) q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_{k,1}[h_\rho, q_\rho]$,for all $\rho_0 \in (0,1)$. *If* $f \in L(p)$ satisfies (2.24) ,then $\frac{K_{c,p}^{\delta+2}f(z)}{z^{p-1}} \prec q(z)$ and q is the best dominant .

Proof . By applying Theorem 2.6 and Theorem 2.7 , we deduce that q is a dominant of (2.24) . Since q satisfies (2.26) , it is also a solution of (2.24) and therefore q will be dominated by all dominants of (2.24) . Hence q is the best dominant of (2.24) .

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mu$. The class of admissible functions $\Phi_{k,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition :

$$\phi(u, v, w; z) \notin \Omega ,$$

whenever

$$u = q(\zeta) , \quad v = \frac{m\zeta q'(\zeta) + (c+p)\big(q(\zeta)\big)^2}{(c+p)q(\zeta)} \quad (p \in \mathbb{N}, c > -p) ,$$

and

$$Re\left\{\frac{(w-u)(c+p)u}{v-u} - (c+p)(w-3u)\right\} \geq m\, Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} , (2.27)$$

$z \in U, \zeta \in \partial U \backslash E(q) , and\ m \geq 1$.

Theorem 2.9. Let $\phi \in \Phi_{k,2}[\Omega, q]. If\ f \in \mathrm{L}(p)$ satisfies

$$\left\{\phi\left(\frac{K_{c,p}^{\delta+2} f(z)}{K_{c,p}^{\delta+3} f(z)} , \frac{K_{c,p}^{\delta+1} f(z)}{K_{c,p}^{\delta+2} f(z)} , \frac{K_{c,p}^{\delta} f(z)}{K_{c,p}^{\delta+1} f(z)} ; z\right) : z \in \mathrm{U}\right\} \subset \Omega , \qquad (2.28)$$

then

$$\frac{K_{c,p}^{\delta+2} f(z)}{K_{c,p}^{\delta+3} f(z)} \prec q(z) .$$

Proof . Let the analytic function F in U defined by

$$F(z) = \frac{K_{c,p}^{\delta+2} f(z)}{K_{c,p}^{\delta+3} f(z)} . \qquad (2.29)$$

Differentiating (2.29) yields

$$\frac{zF'(z)}{F(z)} = \frac{z\left(K_{c,p}^{\delta+2} f(z)\right)'}{K_{c,p}^{\delta+2} f(z)} + \frac{z\left(K_{c,p}^{\delta+3} f(z)\right)'}{K_{c,p}^{\delta+3} f(z)} . \qquad (2.30)$$

By using the relation (2.4) , we get

$$\frac{z\left(K_{c,p}^{\delta+2} f(z)\right)'}{K_{c,p}^{\delta+2} f(z)} = \frac{zF'(z)}{F(z)} + (c+p)F(z) - c . \qquad (2.31)$$

Therefore

$$\frac{K_{c,p}^{\delta+1} f(z)}{K_{c,p}^{\delta+2} f(z)} = \frac{zF'(z) + (c+p)(F(z))^2}{(c+p)F(z)} . \qquad (2.32)$$

Further computations show that

$$\frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)}=\frac{z^2F''(z)+[1+3(c+p)F(z)]zF'(z)+(c+p)^2(F(z))^3}{(c+p)zF'(z)+(c+p)^2(F(z))^2}\ . \quad (2.33)$$

Further , let us define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r\ ,v\ =\ \frac{s+(c+p)r^2}{(c+p)r}\ ,w\ =\ \frac{t+[1+3(c+p)r]s+(c+p)^2r^3}{(c+p)s+(c+p)^2r^2}\,.$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$
$$=\phi\left(r,\frac{s+(c+p)r^2}{(c+p)r},\frac{t+[1+3(c+p)r]s+(c+p)^2r^3}{(c+p)s+(c+p)^2r^2};z\right).\ (2.34)$$

The proof will make use of Theorem 1.1.Using (2.29), (2.32) and (2.33), from (2.34), we obtain

$$\psi\ (F(z),zF'(z),z^2F''(z);z)=\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)},\frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)},\frac{K_{c,p}^{\delta}f(z)}{K_{c,p}^{\delta+1}f(z)}\ ;\ z\right).\ (2.35)$$

Therefore (2.28) becomes

$$\psi\ (F(z),zF'(z),z^2F''(z);z)\in\Omega\,. \quad (2.36)$$

Note that

$$\frac{t}{s}+1=\ \frac{(w-u)(c+p)u}{v-u}-(c+p)(w-3u)\,, \quad (2.37)$$

and since the admissibility condition for $\phi\in\Phi_{k,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2 , hence $\psi\in\Psi[\Omega,q]$, and by Theorem 1.1, $F(z)\ \prec q(z)$.

By (2.29), we get

$$\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)}\ \prec q(z)\ .$$

We consider the special situation when $\Omega\neq\mathbb{C}$ is a simply connected domain . In this case $\Omega=h(U)$, where h is a conformal mapping of U onto Ω and the class is written as $\Phi_{k,2}[h,q]$. The following result follows immediately from Theorem 2.9.

Theorem 2.10. Let $\phi\in\Phi_{k,1}[\Omega,q]$.$If\ f\in$ L(p) satisfies

$$\phi\left(\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)},\frac{K_{c,p}^{\delta+1}f(z)}{K_{c,p}^{\delta+2}f(z)},\frac{K_{c,p}^{\delta}f(z)}{K_{c,p}^{\delta+1}f(z)}\,;\,\mathrm{z}\right)\prec h(\mathrm{z})\,, \tag{2.38}$$

then

$$\frac{K_{c,p}^{\delta+2}f(z)}{K_{c,p}^{\delta+3}f(z)}\prec q(z)\,.$$

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