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ON A NEW STRONG DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS OF ANALYTIC FUNCTIONS INVOLVING THE GENERALIZED OPERATOR

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Abstract: We introduce two new classes of analytic functions defined by strong differential subordinations and superordinations involving the generalized operator. Also we study some properties of these classes.

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1. Introduction and Preliminaries

Denote by U the open unit disk of the complex plane $U = \{z \in \mathcal{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathcal{C} : |z| \leq 1\}$ the closed unit disk of the complex plane and $\mathcal{H}\left(U \times \overline{U}\right)$ the class of analytic functions in $U \times \overline{U}$.

For n a positive integer and $a \in \mathcal{C}$, let $\mathcal{H}[a, n, \zeta] = \{f \in \mathcal{H}\left(U \times \overline{U}\right) : f(z, \zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + ..., \ z \in U, \ \zeta \in \overline{U}\}$, where $a_j(\zeta)$ are holomorphic functions in $\overline{U}$ for $j \geq n$.

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Let $\mathcal{A}_\zeta$ the class of functions of the form:

$$f(z,\zeta) = z + \sum_{k=2}^{\infty} a_k(\zeta) z^k, \quad (z \in U,\, \zeta \in \overline{U}), \tag{1}$$

which are analytic in $U \times \overline{U}$ and $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$.

Definition 1. [7] We denote by Q_ζ the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where

$$E(f,\zeta) = \left\{ r \in \partial U : \lim_{z \to r} f(z,\zeta) = \infty \right\},$$

and $f'_z(r,\zeta) \neq 0$ for $r \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q_z with $f(0,\zeta) = a$ is denoted by $Q_\zeta(a)$.

Definition 2. [7] Let $f(z,\zeta)$, $F(z,\zeta)$ be analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $F(z,\zeta)$ if there exists a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z,\zeta) = F(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec F(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Remark 3. [7] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 2 is equivalent to $f(0,\zeta) = F(0,\zeta)$ for all $\zeta \in \overline{U}$ and $f\left(U \times \overline{U}\right) \subset F\left(U \times \overline{U}\right)$.

(ii) If $f(z,\zeta) = f(z)$ and $F(z,\zeta) = F(z)$, the strong subordination becomes the usual notion of subordination.

If $f(z,\zeta)$ is strongly subordinate to $F(z,\zeta)$, then $F(z,\zeta)$ is strongly superordinate to $f(z,\zeta)$.

As a dual notion of strong differential subordination, Oros [7] has introduced and developed the notion of strong differential superordinations.

Lemma 4. *[6] Let $h(z,\zeta)$ be an univalent function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and $\mu \in \mathcal{C} \backslash \{0\}$ with $Re(\mu) \geq 0$. If $p \in \mathcal{H}[a,1,\zeta]$ and*

$$p(z,\zeta) + \frac{1}{\mu} z p'_z(z,\zeta) \prec\prec h(z,\zeta), \quad (z \in U,\, \zeta \in \overline{U}), \tag{2}$$

then

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \quad (z \in U,\; \zeta \in \overline{U}),$$

where $q(z,\zeta) = \mu z^{-\mu} \int_0^z h(t,\zeta) t^{\mu-1} dt$ is convex and it is the best dominant of (2).

Lemma 5. *[7]Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and $\mu\in\mathcal{C}\backslash\{0\}$ with $Re(\mu)\geq 0$. If $p\in\mathcal{H}[a,1,\zeta]\cap Q_\zeta$, $p(z,\zeta)+\frac{1}{\mu}zp'_z(z,\zeta)$ is univalent in $U\times\overline{U}$ and*

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\mu}zp'_z(z,\zeta),\quad (z\in U,\ \zeta\in\overline{U}), \tag{3}$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad (z\in U,\ \zeta\in\overline{U}),$$

where $q(z,\zeta)=\mu z^{-\mu}\int_0^z h(t,\zeta)\,t^{\mu-1}dt$ is convex and it is the best subordinant of (3).

Definition 6. [9]For $f\in\mathcal{A}_\zeta$, $m\in N_0=N\cup\{0\}$, $\beta\geq 0$, $\alpha\in R$ with $\alpha+\beta>0$, the generalized operator $I^m_{\alpha,\beta}:\mathcal{A}_\zeta\to\mathcal{A}_\zeta$ is defined by

$$I^m_{\alpha,\beta}f(z,\zeta)=z+\sum_{k=2}^{\infty}\left(\frac{\alpha+k\beta}{\alpha+\beta}\right)^m a_k(\zeta)\,z^k,\quad (z\in U,\ \zeta\in\overline{U}). \tag{4}$$

It follows from (4) that

$$\beta z\left(I^m_{\alpha,\beta}f(z,\zeta)\right)'_z=(\alpha+\beta)\,I^{m+1}_{\alpha,\beta}f(z,\zeta)-\alpha I^m_{\alpha,\beta}f(z,\zeta). \tag{5}$$

Remark 7. (i) For $\alpha=1+l-\lambda$, $\beta=\lambda$, the operator $I^m_{\alpha,\beta}=I(m,\lambda,l)$ was studied by Alb Lupas [1], [2].

(ii) For $\beta=1$, $\alpha>-1$, the operator $I^m_{\alpha,1}=I^m_\alpha$ was introduced and studied by Cho and Kim [4] and Cho and Srivastava [5].

(iii) For $\alpha=1-\beta$, $\beta\geq 0$, the operator $I^m_{1-\beta,\beta}=D^m_\beta$ was introduced and studied by Al-Oboudi [3].

(iv) For $\alpha=0$, $\beta=1$, the operator $I^m_{0,1}=S^m$ was introduced and studied by Sălăgean [8].

Definition 8. Let $\psi(z,\zeta)$ be an analytic function in $U\times\overline{U}$ with $\psi(0,\zeta)=1$ for every $\zeta\in\overline{U}$ and $\lambda>0$, $\alpha\in R$, $\beta\geq 0$, $m\in N_0$. A function $f\in\mathcal{A}_\zeta$ is said to be in the class $S(\lambda,\alpha,\beta,m;\psi)$ if it satisfies the strong differential subordination

$$\frac{1}{z}\left[\left(1-\frac{\lambda(\alpha+\beta)}{\beta}\right)I^m_{\alpha,\beta}f(z,\zeta)+\frac{\lambda(\alpha+\beta)}{\beta}I^{m+1}_{\alpha,\beta}f(z,\zeta)\right]\prec\prec\psi(z,\zeta).$$

A function $f\in\mathcal{A}_\zeta$ is said to be in the class $T(\lambda,\alpha,\beta,m;\psi)$ if it satisfies the strong differential superordination

$$\psi(z,\zeta)\prec\prec\frac{1}{z}\left[\left(1-\frac{\lambda(\alpha+\beta)}{\beta}\right)I^m_{\alpha,\beta}f(z,\zeta)+\frac{\lambda(\alpha+\beta)}{\beta}I^{m+1}_{\alpha,\beta}f(z,\zeta)\right].$$

2. Main Results

Theorem 9. *Let $\psi(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $\lambda > 0$. If $f \in S(\lambda, \alpha, \beta, m; \psi)$, then there exists a convex function $q(z,\zeta)$ such that $q(z,\zeta) \prec\prec \psi(z,\zeta)$ and $f \in S(0, \alpha, \beta, m; q)$.*

Proof. Suppose that

$$p(z,\zeta) = \frac{I^m_{\alpha,\beta} f(z,\zeta)}{z} = 1 + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha+\beta}\right)^m a_k(\zeta) z^{k-1}. \tag{6}$$

Then $p \in \mathcal{H}[1,1,\zeta]$.

Since $f \in S(\lambda, \alpha, \beta, m; \psi)$, then we have

$$\frac{1}{z}\left[\left(1 - \frac{\lambda(\alpha+\beta)}{\beta}\right) I^m_{\alpha,\beta} f(z,\zeta) + \frac{\lambda(\alpha+\beta)}{\beta} I^{m+1}_{\alpha,\beta} f(z,\zeta)\right] \prec\prec \psi(z,\zeta). \tag{7}$$

From (6) and (7), we get

$$\frac{1}{z}\left[\left(1 - \frac{\lambda(\alpha+\beta)}{\beta}\right) I^m_{\alpha,\beta} f(z,\zeta) + \frac{\lambda(\alpha+\beta)}{\beta} I^{m+1}_{\alpha,\beta} f(z,\zeta)\right]$$
$$= p(z,\zeta) + \lambda z p'_z(z,\zeta) \prec\prec \psi(z,\zeta).$$

An application of Lemma 4 with $\mu = \frac{1}{\lambda}$ yields

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta).$$

By using (6), we obtain

$$\frac{I^m_{\alpha,\beta} f(z,\zeta)}{z} \prec\prec q(z,\zeta) \prec\prec \psi(z,\zeta),$$

where

$$q(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \psi(t,\zeta) t^{\frac{1}{\lambda}-1} dt$$

is convex and it is the best dominant. □

Theorem 10. *Let $\psi(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $\psi(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $\lambda > 0$. If $f \in T(\lambda, \alpha, \beta, m; \psi)$, $\frac{I^m_{\alpha,\beta} f(z,\zeta)}{z} \in \mathcal{H}[1,1,\zeta] \cap Q_\zeta$ and $\frac{1}{z}\left[\left(1 - \frac{\lambda(\alpha+\beta)}{\beta}\right) I^m_{\alpha,\beta} f(z,\zeta) + \frac{\lambda(\alpha+\beta)}{\beta} I^{m+1}_{\alpha,\beta} f(z,\zeta)\right]$ is univalent in $U \times \overline{U}$, then there exists a convex function $q(z,\zeta)$ such that $f \in T(0, \alpha, \beta, m, q)$.*

Proof. Suppose that

$$p(z,\zeta) = \frac{I^m_{\alpha,\beta} f(z,\zeta)}{z} = 1 + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m a_k(\zeta) z^{k-1}. \tag{8}$$

Then $p \in \mathcal{H}[1,1,\zeta] \cap Q_\zeta$.

After a short calculation and considering $f \in T(\lambda, \alpha, \beta, m; \psi)$, we can conclude that

$$\psi(z,\zeta) \prec\prec p(z,\zeta) + \lambda z p'_z(z,\zeta).$$

An application of Lemma 5 with $\mu = \frac{1}{\lambda}$ yields

$$q(z,\zeta) \prec\prec p(z,\zeta).$$

By using (8), we obtain

$$q(z,\zeta) \prec\prec \frac{I^m_{\alpha,\beta} f(z,\zeta)}{z},$$

where

$$q(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \psi(t,\zeta) t^{\frac{1}{\lambda}-1} dt$$

is convex and it is the best subordinant. □

If we combine the results of Theorem 9 and Theorem 10, we obtain the following strong differential "sandwich theorem".

Theorem 11. *Let $\psi_1(z,\zeta)$ and $\psi_2(z,\zeta)$ be convex functions in $U \times \overline{U}$ with $\psi_1(0,\zeta) = \psi_2(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $\lambda > 0$. If $f \in S(\lambda, \alpha, \beta, m; \psi_1) \cap T(\lambda, \alpha, \beta, m; \psi_2)$, $\frac{I^m_{\alpha,\beta} f(z,\zeta)}{z} \in \mathcal{H}[1,1,\zeta] \cap Q_\zeta$ and*

$$\frac{1}{z} \left[\left(1 - \frac{\lambda(\alpha+\beta)}{\beta} \right) I^m_{\alpha,\beta} f(z,\zeta) + \frac{\lambda(\alpha+\beta)}{\beta} I^{m+1}_{\alpha,\beta} f(z,\zeta) \right]$$

is univalent in $U \times \overline{U}$, then

$$f \in S(0, \alpha, \beta, m; q_1) \cap T(0, \alpha, \beta, m, q_2),$$

where $q_1(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \psi_1(t,\zeta) t^{\frac{1}{\lambda}-1} dt$ and $q_2(z,\zeta) = \frac{1}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \psi_2(t,\zeta) t^{\frac{1}{\lambda}-1} dt$.

The functions q_1 and q_2 are convex.

Theorem 12. *Let $\psi(z,\zeta)$ be a convex function in $U\times\overline{U}$ with $\psi(0,\zeta)=1$ for every $\zeta\in\overline{U}$ and*

$$G(z,\zeta)=\frac{\epsilon+2}{z^{\epsilon+1}}\int_0^z t^{\epsilon}f(t,\zeta)\,dt,\quad (z\in U,\,\zeta\in\overline{U},\,Re(\epsilon)>-2). \tag{9}$$

If $f\in S(1,\alpha,\beta,m;\psi)$, then there exists a convex function $q(z,\zeta)$ such that $q(z,\zeta)\prec\prec\psi(z,\zeta)$ and $G\in S(1,\alpha,\beta,m;q)$.

Proof. Suppose that

$$p(z,\zeta)=\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z,\quad (z\in U,\,\zeta\in\overline{U}). \tag{10}$$

Then $p\in\mathcal{H}[1,1,\zeta]$.

From (9) we have

$$z^{\epsilon+1}G(z,\zeta)=(\epsilon+2)\int_0^z t^{\epsilon}f(t,\zeta)\,dt. \tag{11}$$

Differentiating both sides of (11) with respect to z, we get

$$(\epsilon+2)f(z,\zeta)=(\epsilon+1)G(z,\zeta)+zG'_z(z,\zeta)$$

and

$$(\epsilon+2)I^m_{a,\beta}f(z,\zeta)=(\epsilon+1)I^m_{\alpha,\beta}G(z,\zeta)+z\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z.$$

Differentiating the last relation with respect to z, we have

$$\left(I^m_{\alpha,\beta}f(z,\zeta)\right)'_z=\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z+\frac{z}{\epsilon+2}\left(I^m_{\alpha,\beta}G(z,\zeta)\right)''_{z^2}. \tag{12}$$

Since $f\in S(1,\alpha,\beta,m;\psi)$, then we get

$$\frac{1}{\beta z}\left[(\alpha+\beta)I^{m+1}_{\alpha,\beta}f(z,\zeta)-\alpha I^m_{\alpha,\beta}f(z,\zeta)\right]\prec\prec\psi(z,\zeta). \tag{13}$$

Now, from (5), (13) is equivalent to

$$\left(I^m_{\alpha,\beta}f(z,\zeta)\right)'_z\prec\prec\psi(z,\zeta). \tag{14}$$

From (12) and (14), we get

$$\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z+\frac{z}{\epsilon+2}\left(I^m_{\alpha,\beta}G(z,\zeta)\right)''_{z^2}\prec\prec\psi(z,\zeta). \tag{15}$$

Replacing (10) in (15), we obtain

$$p(z,\zeta)+\frac{1}{\epsilon+2}zp'_z(z,\zeta)\prec\prec\psi(z,\zeta).$$

An application of Lemma 4 with $\mu=\epsilon+2$ yields

$$p(z,\zeta)\prec\prec q(z,\zeta)\prec\prec\psi(z,\zeta).$$

By using (10), we obtain

$$\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z\prec\prec q(z,\zeta)\prec\prec\psi(z,\zeta),$$

where

$$q(z,\zeta)=(\epsilon+2)\,z^{-(\epsilon+2)}\int_0^z\psi(t,\zeta)\,t^{\epsilon+1}dt$$

is convex and it is the best dominant. □

Theorem 13. *Let $\psi(z,\zeta)$ be a convex function in $U\times\overline{U}$ with $\psi(0,\zeta)=1$ for every $\zeta\in\overline{U}$ and $G(z,\zeta)$ is given by (9). If $f\in T(1,\alpha,\beta,m;\psi)$, $\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z\in\mathcal{H}[1,1,\zeta]\cap Q_\zeta$ and*

$$\frac{1}{\beta z}\left[(\alpha+\beta)\,I^{m+1}_{\alpha,\beta}f(z,\zeta)-\alpha I^m_{\alpha,\beta}f(z,\zeta)\right]$$

is univalent in $U\times\overline{U}$, then there exists a convex function $q(z,\zeta)$ such that $G\in T(1,\alpha,\beta,m,q)$.

Proof. Suppose that

$$p(z,\zeta)=\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z,\quad(z\in U,\ \zeta\in\overline{U}).\tag{16}$$

Then $p\in\mathcal{H}[1,1,\zeta]\cap Q_\zeta$.

After a short calculation and considering $f\in T(1,\alpha,\beta,m;\psi)$, we can conclude that

$$\psi(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\epsilon+2}zp'_z(z,\zeta).$$

An application of Lemma 5 with $\mu=\epsilon+2$ yields

$$q(z,\zeta)\prec\prec p(z,\zeta).$$

By using (16), we obtain

$$q(z,\zeta)\prec\prec\left(I^m_{\alpha,\beta}G(z,\zeta)\right)'_z,$$

where

$$q(z,\zeta) = (\epsilon+2)\, z^{-(\epsilon+2)} \int_0^z \psi(t,\zeta)\, t^{\epsilon+1} dt$$

is convex and it is the best subordinant. □

If we combine the results of Theorem 12 and Theorem 13, we obtain the following strong differential "sandwich theorem".

Theorem 14. *Let $\psi_1(z,\zeta)$ and $\psi_2(z,\zeta)$ be convex functions in $U \times \overline{U}$ with $\psi_1(0,\zeta) = \psi_2(0,\zeta) = 1$ for every $\zeta \in \overline{U}$ and $G(z,\zeta)$ is given by (9). If $f \in S(1,\alpha,\beta,m;\psi_1) \cap T(1,\alpha,\beta,m;\psi_2)$, $\left(I^m_{\alpha,\beta} G(z,\zeta)\right)'_z \in \mathcal{H}[1,1,\zeta] \cap Q_\zeta$ and $\frac{1}{\beta z}\left[(\alpha+\beta) I^{m+1}_{\alpha,\beta} f(z,\zeta) - \alpha I^m_{\alpha,\beta} f(z,\zeta)\right]$ is univalent in $U \times \overline{U}$, then*

$$f \in S(1,\alpha,\beta,m;q_1) \cap T(1,\alpha,\beta,m,q_2),$$

where

$$q_1(z,\zeta) = (\epsilon+2)\, z^{-(\epsilon+2)} \int_0^z \psi_1(t,\zeta)\, t^{\epsilon+1} dt$$

and

$$q_2(z,\zeta) = (\epsilon+2)\, z^{-(\epsilon+2)} \int_0^z \psi_2(t,\zeta)\, t^{\epsilon+1} dt.$$

The functions q_1 and q_2 are convex.

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