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Some Properties of a certain Class of Multivalent Analytic Functions with a Fixed Point

Abbas kareem Wanas

Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya, Iraq

Abstract
We introduce subclass $\mathcal{AB}_p^w(\lambda,\eta,\mu)$ of multivalent analytic functions with a fixed point w. We obtain coefficient inequalities, extreme points, integral representation, Hadamard product and radii of starlikeness and convexity.
Keywords: *Multivalent function, Extreme points, Integral representation.*

1. Introduction

Let $\mathcal{A}(p,w)$ denote the class of functions f of the form:

$$f(z)=(z-w)^p+\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p}\ \ (p\in\mathbb{N}),\quad (1.1)$$

which are analytic in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and w is a fixed point in U.
Let $\mathcal{B}(\mathrm{p},\mathrm{w})$ denote subclass of $\mathcal{A}(p,w)$ containing of functions of the form:

$$f(z)=(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p}\ \left(a_{n+p}\geq 0, p\in\mathbb{N}\right),\quad (1.2)$$

For the functions $f\in\mathcal{B}(p,w)$ given by (1.2) and $g\in\mathcal{B}(p,w)$ defined by

$$g(z)=(z-w)^p-\sum_{n=1}^{\infty}b_{n+p}(z-w)^{n+p}\ (b_{n+p}\geq 0,p\in\mathbb{N}),$$

we define the convolution (or Hadamard product) of f and g by

$$(f*g)(z)=(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}b_{n+p}(z-w)^{n+p}.$$

Now, we define the class $\mathcal{AB}_{\mathrm{p}}^{\mathrm{w}}(\lambda,\eta,\mu)$ consisting the functions $\mathrm{f}\in\mathcal{B}(\mathrm{p},\mathrm{w})$ such that

$$\left|\frac{(z-w)f''(z)+(1-p)f'(z)}{\lambda(z-w)f''(z)+(\lambda+\eta)f'(z)}\right|<\mu,\quad (1.3)$$

where $0\leq\lambda\leq 1, 0<\eta\leq 1$ and $0<\mu\leq 1$.
Such type of study was carried out by various authors for another classes, like, Atshan and Wanas [1], Ghanim and Darus[2], Najafzadeh and Rahimi [3] and Shenan [4].

2. Coefficient Inequalities

Theorem 2.1. Let $f\in\mathcal{B}(p,w)$. Then $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$ if and only if

$$\sum_{n=1}^{\infty}(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]a_{n+p}\leq\mu p(\lambda p+\eta),\quad (2.1)$$

where $0\leq\lambda\leq 1, 0<\eta\leq 1$ *and* $0<\mu\leq 1$.
The result is sharp for the function f given by

$$f(z)=(z-w)^p-\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\times(z-w)^{n+p}\ \ (n\geq 1).\quad (2.2)$$

Proof. Suppose that the inequality (2.1) holds true and $(z-w)\in\partial U$ where ∂U denotes the boundary of U. Then, we find from (1.3) that

$$\left|(z-w)f''(z)+(1-p)f'(z)\right|$$
$$-\mu\left|\lambda(z-w)f''(z)+(\lambda+\eta)f'(z)\right|$$
$$=\left|-\sum_{n=1}^{\infty}n(n+p)a_{n+p}(z-w)^{n+p-1}\right|$$
$$-\Bigg|\mu p(\lambda p+\eta)(z-w)^{p-1}-\sum_{n=1}^{\infty}\mu(n+p)(\lambda(n+p)+\eta)a_{n+p}(z-w)^{n+p-1}\Bigg|$$
$$\leq\sum_{n=1}^{\infty}n(n+p)a_{n+p}|z-w|^{n+p-1}-\mu p(\lambda p+\eta)|z-w|^{p-1}$$
$$+\sum_{n=1}^{\infty}\mu(n+p)(\lambda(n+p)+\eta)a_{n+p}|z-w|^{n+p-1}$$
$$=\sum_{n=1}^{\infty}(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]a_{n+p}-\mu p(\lambda p+\eta)$$
$$\leq 0.$$

Hence, by maximum modulus theorem, we conclude $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$.
Conversely, suppose that $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$. Then from (1.3), we have

$$\left|\frac{(z-w)f''(z)+(1-p)f'(z)}{\lambda(z-w)f''(z)+(\lambda+\eta)f'(z)}\right|$$
$$=\left|\frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}(z-w)^{n+p-1}}{p(\lambda p+\eta)(z-w)^{p-1}-\sum_{n=1}^{\infty}(n+p)(\lambda(n+p)+\eta)a_{n+p}(z-w)^{n+p-1}}\right|$$
$<\mu.$

So, we obtain

$$Re\left\{\frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}(z-w)^{n+p-1}}{p(\lambda p+\eta)(z-w)^{p-1}-\sum_{n=1}^{\infty}(n+p)(\lambda(n+p)+\eta)a_{n+p}(z-w)^{n+p-1}}\right\}$$
$<\mu.$

By letting $(z-w)\to 1^-$, through real values, we have
$$\sum_{n=1}^{\infty}(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]a_{n+p}\le\mu p(\lambda p+\eta).$$

Corollary 2.1. Let $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$. Then
$$a_{n+p}\le\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\quad(n\ge 1).$$

3. Extreme Points

Theorem 3.1. Let $f_p(z)=(z-w)^p$ and
$$f_{n+p}(z)=(z-w)^p-\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}(z-w)^{n+p}\quad(n\ge 1).$$
Then $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$ if and only if it can be expressed in the form
$$f(z)=\sum_{n=0}^{\infty}\gamma_{n+p}f_{n+p}(z),\tag{3.1}$$
where $\gamma_{n+p}\ge 0,\ \sum_{n=0}^{\infty}\gamma_{n+p}=1.$

Proof. Let the f of the form (3.1). Then

$$f(z)=\gamma_p f_p(z)+\sum_{n=1}^{\infty}\gamma_{n+p}$$
$$\times\left((z-w)^p-\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}(z-w)^{n+p}\right).$$
$$=(z-w)^p-\sum_{n=1}^{\infty}\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\gamma_{n+p}$$
$$\times(z-w)^{n+p}.$$

Now,
$$\sum_{n=1}^{\infty}\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}$$
$$\times\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\gamma_{n+p}$$
$$=\sum_{n=1}^{\infty}\gamma_{n+p}=1-\gamma_p\le 1.$$
Thus $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$.

Conversely, let $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$. It follows from Corollary 2.1 that
$$a_{n+p}\le\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\quad(n\ge 1).$$
Setting
$$\gamma_{n+p}=\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}a_{n+p}\quad(n\ge 1)$$
and $\gamma_p=1-\sum_{n=1}^{\infty}\gamma_{n+p}$, we have

$$f(z)=(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p}$$
$$=(z-w)^p-\sum_{n=1}^{\infty}\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\gamma_{n+p}$$
$$\times(z-w)^{n+p}$$
$$=(z-w)^p-\sum_{n=1}^{\infty}\left((z-w)^p-f_{n+p}(z)\right)\gamma_{n+p}$$
$$=\left(1-\sum_{n=1}^{\infty}\gamma_{n+p}\right)(z-w)^p+\sum_{n=1}^{\infty}\gamma_{n+p}f_{n+p}(z)$$
$$=\gamma_p f_p(z)+\sum_{n=1}^{\infty}\gamma_{n+p}f_{n+p}(z)=\sum_{n=0}^{\infty}\gamma_{n+p}f_{n+p}(z),$$

that is the required representation.

4. Integral Representation

Theorem 4.1. Let $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$. Then
$$f(z)=\int_0^z exp\left[\int_0^z\frac{\mu(\lambda+\eta)\psi(t)+p-1}{(t-w)\left(1-\lambda\mu\psi(t)\right)}dt\right]dt,$$
where $|\psi(z)|<1, z\in U$.

Proof. By putting $\frac{(z-w)f''(z)}{f'(z)}=Q(z)$ in (1.3), we have
$$\left|\frac{Q(z)+1-p}{\lambda Q(z)+\lambda+\eta}\right|<\mu,$$
or equivalently
$$\frac{Q(z)+1-p}{\lambda Q(z)+\lambda+\eta}=\mu\psi(z),\quad(|\psi(z)|<1, z\in U).$$
So

$$\frac{f''(z)}{f'(z)}=\frac{\mu(\lambda+\eta)\psi(z)+p-1}{(z-w)\big(1-\lambda\mu\psi(z)\big)},$$

after integration, we obtain

$$\log(f'(z))=\int_0^z\frac{\mu(\lambda+\eta)\psi(t)+p-1}{(t-w)\big(1-\lambda\mu\psi(t)\big)}dt.$$

Therefore

$$f'(z)=exp\left[\int_0^z\frac{\mu(\lambda+\eta)\psi(t)+p-1}{(t-w)\big(1-\lambda\mu\psi(t)\big)}dt\right].$$

After integration, we have

$$f(z)=\int_0^z exp\left[\int_0^z\frac{\mu(\lambda+\eta)\psi(t)+p-1}{(t-w)\big(1-\lambda\mu\psi(t)\big)}dt\right]dt$$

and this the required result.

5. Hadamard Product

Theorem 5.1. Let $f,g\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$. Then

$$f*g\in\mathcal{AB}_p^w(\lambda,\sigma,\mu),$$

where

$$\sigma\le\frac{\lambda p(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}{\mu^2p(\lambda p+\eta)^2-(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}-\frac{\mu p(\lambda p+\eta)^2(n+\lambda\mu(n+p))}{\mu^2p(\lambda p+\eta)^2-(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}.$$

Proof. We must find the largest σ such that

$$\sum_{n=1}^{\infty}\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\sigma)]}{\mu p(\lambda p+\sigma)}a_{n+p}b_{n+p}\le 1. \tag{5.1}$$

Since $f,g\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$, then

$$\sum_{n=1}^{\infty}\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}a_{n+p}\le 1$$

and

$$\sum_{n=1}^{\infty}\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}b_{n+p}\le 1.$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}\sqrt{a_{n+p}b_{n+p}}\le 1. \tag{5.2}$$

We want only to show that

$$\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\sigma)]}{\mu p(\lambda p+\sigma)}a_{n+p}b_{n+p}\le\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)}\sqrt{a_{n+p}b_{n+p}}.$$

This equivalently to

$$\sqrt{a_{n+p}b_{n+p}}\le\frac{(\lambda p+\sigma)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{(\lambda p+\eta)[n(\lambda\mu+1)+\mu(\lambda p+\sigma)]}.$$

From (5.2), we get

$$\sqrt{a_{n+p}b_{n+p}}\le\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}.$$

Thus, it is enough to show that

$$\frac{\mu p(\lambda p+\eta)}{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}\le\frac{(\lambda p+\sigma)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{(\lambda p+\eta)[n(\lambda\mu+1)+\mu(\lambda p+\sigma)]},$$

which implies that

$$\sigma\le\frac{\lambda p(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}{\mu^2p(\lambda p+\eta)^2-(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}-\frac{\mu p(\lambda p+\eta)^2(n+\lambda\mu(n+p))}{\mu^2p(\lambda p+\eta)^2-(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]^2}.$$

6. Radii of Starlikeness and Convexity

Theorem 6.1. If $f\in\mathcal{AB}_p^w(\lambda,\eta,\mu)$, then f is starlike of order δ $(0\le\delta<p)$ in the disk $|z-w|<r_1$, where

$$r_1=\inf_n\left\{\frac{(n+p)(p-\delta)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)(n+p-\delta)}\right\}^{\frac{1}{n}}.$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{(z-w)f'(z)}{f(z)}-p\right|\le p-\delta\quad\text{for } |z-w|<r_1. \tag{6.1}$$

But

$$\left|\frac{(z-w)f'(z)}{f(z)}-p\right|=\left|\frac{-\sum_{n=1}^{\infty}na_{n+p}(z-w)^{n+p}}{(z-w)^p-\sum_{n=1}^{\infty}a_{n+p}(z-w)^{n+p}}\right|\le\frac{\sum_{n=1}^{\infty}na_{n+p}|z-w|^n}{1-\sum_{n=1}^{\infty}a_{n+p}|z-w|^n}.$$

Thus (6.1) will be satisfied if

$$\frac{\sum_{n=1}^{\infty}na_{n+p}|z-w|^n}{1-\sum_{n=1}^{\infty}a_{n+p}|z-w|^n}\le p-\delta,$$

or if

$$\sum_{n=1}^{\infty}\frac{(n+p-\delta)}{(p-\delta)}a_{n+p}|z-w|^n\le 1, \tag{6.2}$$

with the aid of (2.1), (6.2) is true if

$$\frac{(n+p-\delta)}{(p-\delta)}|z-w|^n\le\frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)},$$

or equivalently

$$|z-w| \le \left\{\frac{(n+p)(p-\delta)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)(n+p-\delta)}\right\}^{\frac{1}{n}},$$

which follows the result.

Theorem 6.2. If $f \in \mathcal{AB}_p^w(\lambda,\eta,\mu)$, then f is convex of order δ $(0 \le \delta < p)$ in the disk $|z-w| < r_2$, where

$$r_2 = \inf_n \left\{\frac{(p-\delta)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu(\lambda p+\eta)(n+p-\delta)}\right\}^{\frac{1}{n}}.$$

The result is sharp for the function f given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{(z-w)f''(z)}{f'(z)}+1-p\right| \le p-\delta \quad \text{for } |z-w| < r_2. \quad (6.3)$$

But

$$\left|\frac{(z-w)f''(z)}{f'(z)}+1-p\right|$$

$$= \left|\frac{-\sum_{n=1}^{\infty} n(n+p)a_{n+p}(z-w)^{n+p-1}}{p(z-w)^{p-1}-\sum_{n=1}^{\infty}(n+p)a_{n+p}(z-w)^{n+p-1}}\right|$$

$$\le \frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z-w|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z-w|^n}.$$

Thus (6.3) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z-w|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z-w|^n} \le p-\delta,$$

or if

$$\sum_{n=1}^{\infty}\frac{(n+p)(n+p-\delta)}{p(p-\delta)}a_{n+p}|z-w|^n \le 1, \qquad (6.4)$$

with the aid of (2.1), (6.4) is true if

$$\frac{(n+p)(n+p-\delta)}{p(p-\delta)}|z-w|^n \le \frac{(n+p)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu p(\lambda p+\eta)},$$

or equivalently

$$|z-w| \le \left\{\frac{(p-\delta)[n(\lambda\mu+1)+\mu(\lambda p+\eta)]}{\mu(\lambda p+\eta)(n+p-\delta)}\right\}^{\frac{1}{n}},$$

which follows the result.

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