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On sandwich theorems for higher-order derivatives of multivalent analytic functions associated with the generalized Noor integral operator

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In this paper, we obtain some subordination and superordination results for higher-order derivatives of multivalent analytic functions in the open unit disk by generalized Noor integral operator. These results are applied to obtain sandwich results. Our results extend corresponding previously known results.

Keywords: Analytic functions; subordination; superordination; higher-order derivatives; generalized Noor integral operator.

AMS Subject Classification: 30C45

1. Introduction and Preliminaries

Let $R(p,m)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p} \quad (p, m \in N = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $U = \{z \in C : |z| < 1\}$. Upon differentiating both sides of (1.1) j-times with respect to z, we obtain (see [7])

$$f^{(j)}(z) = \delta(p,j) z^{p-j} + \sum_{n=m}^{\infty} \delta(n+p,j) a_{n+p} z^{n+p-j}$$

$$(p, m \in N; j \in N_0 = N \cup \{0\}; p > j),$$

where

$$\delta(p,j)=\frac{p!}{(p-j)!}=\begin{cases}1 & (j=0),\\ p(p-1)\cdots(p-j+1) & (j\neq 0).\end{cases}$$

Several researchers have investigated higher-order derivatives of multivalent functions, see, for example [1, 3, 4, 7, 13–15, 20, 22, 28, 29].

Let $H = H(U)$ be the class of analytic functions in U and let $H[e, p]$ be the subclass of H consisting of functions of the form:

$$f(z)=e+a_p z^p+a_{p+1}z^{p+1}+\cdots \quad (e\in C; p\in N)$$

For the functions $f \in R(p, m)$ given by (1.1) and $g \in R(p, m)$ defined by

$$g(z)=z^p+\sum_{n=m}^{\infty} b_{n+p}z^{n+p} \quad (p,m\in N),$$

we define the Hadamard product (or convolution) $f * g$ of the functions f and g (as usual) by

$$(f*g)(z)=z^p+\sum_{n=m}^{\infty} a_{n+p}b_{n+p}z^{n+p}=(g*f)(z).$$

Let $f, g \in H$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well-known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $k, h \in H$ and $\psi(r, s, t; z) : C^3 \times U \to C$. If k and $\psi(k(z), zk'(z), z^2k''(z); z)$ are univalent functions in U and if k satisfies the second-order differential superordination

$$h(z)\prec \psi(k(z), zk'(z), z^2k''(z); z), \tag{1.2}$$

then k is called a solution of the differential superordination (1.2). (If f is subordinate to g, then g is superordinate to f.) An analytic function q is called a subordinant of (1.2), if $q \prec k$ for all the functions k satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants q of (1.2) is called the best subordinant.

Recently, Miller and Mocanu [18] obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z)\prec \psi(k(z), zk'(z), z^2k''(z); z)\Rightarrow q(z)\prec k(z).$$

Using the results due to Miller and Mocanu [18], Bulboacă [5] considered certain classes of first-order differential superordination as well as superordination-preserving integral operators [6]. Ali *et al.* [2] have used the results of Bulboacă [5] to obtain sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Very recently, Shanmugam *et al.* [25–27] and Goyal *et al.* [3] have obtained sandwich results for certain classes of analytic functions.

For real or complex numbers a, b, c other than $0, -1, -2, \ldots$, the hypergeometric function ${}_2F_1(a, b, c; z)$ is defined by the infinite series

$${}_2F_1(a, b, c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{ab(a+1)(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \tag{1.3}$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1), \quad n \in N.$$

We note that the series (1.3) converges absolutely for all $z \in U$ so that it represents an analytic function in U.

Fu and Liu [10] introduced a function $(z^p {}_2F_1(a, b, c; z))^{-1}$ given by

$$(z^p {}_2F_1(a, b, c; z)) * (z^p {}_2F_1(a, b, c; z))^{-1} = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p),$$

which leads us to the following family of linear operators:

$$I_{p,m}^{\lambda}(a, b, c)f(z) = (z^p {}_2F_1(a, b, c; z))^{-1} * f(z),$$

where $f \in R(p, m), a, b, c \in R\backslash Z_0^- = \{0, -1, -2, \ldots\}, \lambda > -p$.

By some easy calculations, we obtain

$$I_{p,m}^{\lambda}(a, b, c)f(z) = z^p + \sum_{n=m}^{\infty} \frac{(c)_n(\lambda+p)_n}{(a)_n(b)_n} a_{n+p} z^{n+p}. \tag{1.4}$$

It is easily verified from (1.4) that

$$z(I_{p,m}^{\lambda}(a, b, c)f(z))' = (\lambda+p)I_{p,m}^{\lambda+1}(a, b, c)f(z) - \lambda I_{p,m}^{\lambda}(a, b, c)f(z). \tag{1.5}$$

Differentiating (1.5), j-times, we get

$$\begin{aligned} z(I_{p,m}^{\lambda}(a, b, c)f(z))^{(j+1)} &= (\lambda+p)(I_{p,m}^{\lambda+1}(a, b, c)f(z))^{(j)} \\ &\quad - (\lambda+j)(I_{p,m}^{\lambda}(a, b, c)f(z))^{(j)}. \end{aligned} \tag{1.6}$$

Note that the generalized Noor integral operator $I_{p,m}^{\lambda}$ unifies many other operators considered earlier. In particular, for $f \in R(p, 1)$ we have the following:

(i) $I_{p,1}^{1}(k+p, c, c) = I_{k,p} (n \in \mathbf{N})$ the operator introduced by Liu and Noor [16] and Patel and Cho [21].
(ii) $I_{p,1}^{\lambda}(a, 1, c) = I_p^{\lambda}(a, c)$ the operator considered by Cho *et al.* [8].
(iii) $I_{p,1}^{\lambda}(a, \lambda+p, c) = I_p(a, c)$ the operator investigated by Saitoh [24].

For our present investigation, we shall need the following definition and results.

Definition 1.1 ([17]). Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta)\neq$ for $\zeta \in \partial U\backslash E(f)$.

Lemma 1.1 ([17]). *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) *$Q(z)$ is starlike univalent in U,*
(ii) $\mathrm{Re}\{\frac{zh'(z)}{Q(z)}\} > 0$ *for $z \in U$.*

If k is analytic in U, with $k(0) = q(0), k(U) \subset D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.7}$$

then $k \prec q$ and q is the best dominant of (1.7).

Lemma 1.2 ([18]). *Let q be a convex univalent function in U and let $\alpha \in C, \beta \in C\backslash\{0\}$ with*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If k is analytic in U and

$$\alpha k(z) + \beta zk'(z) \prec \alpha q(z) + \beta zq'(z), \tag{1.8}$$

then $k \prec q$ and q is the best dominant of (1.8).

Lemma 1.3 ([18]). *Let q be convex univalent in U and let $\beta \in C$. Further assume that* $\mathrm{Re}(\beta) > 0$. *If $k \in H[q(0), 1] \cap Q$ and $k(z) + \beta zk'(z)$ is univalent in U, then*

$$q(z) + \beta zq'(z) \prec k(z) + \beta zk'(z), \tag{1.9}$$

which implies that $q \prec k$ and q is the best subordinant of (1.9).

Lemma 1.4 ([5]). *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\mathrm{Re}\{\frac{\theta'(q(z))}{\phi(q(z))}\} > 0$ *for $z \in U$,*
(ii) *$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.*

If $k \in H[q(0), 1] \cap Q$, with $k(U) \subset D, \theta(k(z)) + zk'(z)\phi(k(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \tag{1.10}$$

then $q \prec k$ and q is the best subordinant of (1.10).

2. Subordination for Analytic Functions

Theorem 2.1. *Let q be convex univalent in U with $q(0) = 1, \gamma > 0, \eta \in C\backslash\{0\}$ and suppose that q satisfies*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{\gamma(p-j)}{\eta}\right)\right\}. \tag{2.1}$$

If $f \in R(p, m)$ satisfies the subordination

$$\left(1 - \frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}$$
$$+ \frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)$$
$$\prec q(z) + \frac{\eta}{\gamma(p-j)}zq'(z), \tag{2.2}$$

then

$$\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.2).

Proof. Define the function k by

$$k(z) = \left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}, \quad z \in U. \tag{2.3}$$

Then the function k is analytic in U and $k(0) = 1$. Therefore, differentiating (2.3) logarithmically with respect to z, we get

$$\frac{zk'(z)}{k(z)} = \gamma\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}} - (p-j)\right).$$

Now, in view of (1.6), we obtain the following subordination

$$\frac{zk'(z)}{k(z)} = \gamma(\lambda+p)\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}} - 1\right).$$

Thus,

$$\frac{zk'(z)}{\gamma(p-j)} = \frac{\lambda+p}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}} - 1\right).$$

The subordination (2.2) from the hypothesis becomes

$$k(z) + \frac{\eta}{\gamma(p-j)}zk'(z) \prec q(z) + \frac{\eta}{\gamma(p-j)}zq'(z).$$

Hence, an application of Lemma 1.2 with $\alpha = 1$ and $\beta = \frac{\eta}{\gamma(p-j)}$, we obtain the desired result. □

Theorem 2.2. *Let $u, v, \varepsilon, \sigma \in C, \gamma > 0, t \in C\backslash\{0\}$ and q be convex univalent in U with $q(0) = 1, q(z) \neq 0 (z \in U)$ and assume that q satisfies*

$$\mathrm{Re}\left\{1 + \frac{v}{t}q(z) + \frac{2\varepsilon}{t}q^2(z) + \frac{3\sigma}{t}q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{2.4}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f \in R(p, m)$ satisfies

$$\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z) \prec u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)}, \tag{2.5}$$

where

$$\begin{aligned}\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z)&\\ = u + v\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{\gamma}&\\ + \varepsilon\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{2\gamma} + \sigma\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{3\gamma}&\\ + t\gamma\left(1 + \frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+2)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}} - \frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right),&\end{aligned} \tag{2.6}$$

then

$$\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.5).

Proof. Define the function k by

$$k(z) = \left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{\gamma}, \quad z \in U. \tag{2.7}$$

Then the function k is analytic in U and $k(0) = 1$. Therefore, by making use of (1.6) and (2.7), we obtain

$$u + vk(z) + \varepsilon k^2(z) + \sigma k^3(z) + t\frac{zk'(z)}{k(z)} = \varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z), \tag{2.8}$$

where $\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z)$ is given by (2.6).

By using (2.8) in (2.5), we have

$$u + vk(z) + \varepsilon k^2(z) + \sigma k^3(z) + t\frac{zk'(z)}{k(z)} \prec u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)}.$$

By Setting

$$\theta(w) = u + vw + \varepsilon w + \sigma w^3 \quad \text{and} \quad \phi(w) = \frac{t}{w}, \quad w \neq 0,$$

we see that $\theta(w)$ is analytic in C, $\phi(w)$ is analytic in $C\backslash\{0\}$ and that $\phi(w) \neq 0, \quad w \in C\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U and that

$$\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{1 + \frac{v}{t}q(z) + \frac{2\varepsilon}{t}q^2(z) + \frac{2\sigma}{t}q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0.$$

Thus, by applying Lemma 1.1, our proof of Theorem 2.2 is completed. □

If we take $\lambda = j = 0, a = c$ and $b = p$ in Theorem 2.2, then we obtain the next result.

Corollary 2.1. *Let $u, v, \varepsilon, \sigma \in C, \gamma > 0, t \in C\backslash\{0\}$ and q be convex univalent in U with $q(0) = 1, q(z) \neq 0 (z \in U)$ and assume that q satisfies (2.4). Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f \in R(p, m)$ satisfies*

$$\Omega(u, v, \varepsilon, \sigma, \gamma, t, m, p; z) \prec u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)},$$

where

$$\Omega(u, v, \varepsilon, \sigma, \gamma, t, m, p; z) = u + v\left(\frac{zf'(z)}{pf(z)}\right)^{\gamma} + \varepsilon\left(\frac{zf'(z)}{pf(z)}\right)^{2\gamma} + \sigma\left(\frac{zf'(z)}{pf(z)}\right)^{3\gamma} + t\gamma\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right), \tag{2.9}$$

then

$$\left(\frac{zf'(z)}{pf(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant.

3. Superordination for Analytic Functions

Theorem 3.1. *Let q be convex univalent in U with $q(0) = 1, \gamma > 0$ and $\text{Re}\{\eta\} > 0$. Let$f \in R(p, m)$ satisfies*

$$\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a, b, c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} \in H[q(0), 1] \cap Q$$

and

$$\left(1-\frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} + \frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)$$

be univalent in U. *If*

$$q(z)+\frac{\eta}{\gamma(p-j)}zq'(z) \prec \left(1-\frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} + \frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right), \tag{3.1}$$

then

$$q(z) \prec \left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}$$

and q *is the best subordinant of* (3.1).

Proof. Define the function k by

$$k(z)=\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}, \quad z\in U.$$

After some computations and using (1.6), we have

$$\left(1-\frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} + \frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right) = k(z)+\frac{\eta}{\gamma(p-j)}zk'(z). \tag{3.2}$$

By using (3.2) in (3.1), we have

$$q(z)+\frac{\eta}{\gamma(p-j)}zq'(z) \prec k(z)+\frac{\eta}{\gamma(p-j)}zk'(z).$$

Using Lemma 1.3, the proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let $u, v, \varepsilon, \sigma \in C, \gamma > 0, t \in C\backslash\{0\}$ and q be convex univalent in U with $q(0) = 1$ and assume that q satisfies*

$$\operatorname{Re}\left\{\frac{v}{t}q(z) + \frac{2\varepsilon}{t}q^2(z) + \frac{3\sigma}{t}q^3(z)\right\} > 0. \tag{3.3}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f \in R(p, m)$ satisfies

$$\left(\frac{z(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j+1)}}{(p-j)(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}\right)^{\gamma} \in H[q(0), 1] \cap Q$$

and $\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z)$ be univalent in U, where $\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda,$ $a, b, c, m, p, j; z)$ is given by (2.6). If

$$u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)} \prec \varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z), \tag{3.4}$$

then

$$q(z) \prec \left(\frac{z(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j+1)}}{(p-j)(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}\right)^{\gamma}$$

and q is the best subordinant of (3.4).

Proof. Define the function k by

$$k(z) = \left(\frac{z(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j+1)}}{(p-j)(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}\right)^{\gamma}, \quad z \in U. \tag{3.5}$$

Simple computations from (3.5), we get

$$\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z) = u + vk(z) + \varepsilon k^2(z) + \sigma k^3(z) + t\frac{zk'(z)}{k(z)}, \tag{3.6}$$

where $\varphi(u, v, \varepsilon, \sigma, \gamma, t, \lambda, a, b, c, m, p, j; z)$ is given by (2.6).

From (3.4) and (3.6), we obtain

$$u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)} \prec u + vk(z) + \varepsilon k^2(z) + \sigma k^3(z) + t\frac{zk'(z)}{k(z)}.$$

By Setting $\theta(w) = u + vw + \varepsilon w + \sigma w^3$ *and* $\phi(w) = \frac{t}{w}, w \neq 0$, we see that $\theta(w)$ is analytic in C, $\phi(w)$ is analytic in $C\backslash\{0\}$ and that $\phi(w) \neq 0, w \in C\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U and that

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \operatorname{Re}\left\{\frac{v}{t}q(z) + \frac{2\varepsilon}{t}q^2(z) + \frac{3\sigma}{t}q^3(z)\right\} > 0.$$

Now Theorem 3.2 follows by applying Lemma 1.4. □

If we take $\lambda = j = 0, a = c$ and $b = p$ in Theorem 3.2, then we obtain the next result.

Corollary 3.1. *Let* $u, v, \varepsilon, \sigma \in C, \gamma > 0, t \in C\backslash\{0\}$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and assume that* q *satisfies* (3.3). *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let* $f \in R(p, m)$ *satisfies*

$$\left(\frac{zf'(z)}{pf(z)}\right)^{\gamma} \in H[q(0), 1] \cap Q$$

and $\Omega(u, v, \varepsilon, \sigma, \gamma, t, m, p; z)$ *be univalent in* U, *where* $\Omega(u, v, \varepsilon, \sigma, \gamma, t, m, p; z)$ *is given by* (2.9). *If*

$$u + vq(z) + \varepsilon q^2(z) + \sigma q^3(z) + t\frac{zq'(z)}{q(z)} \prec \Omega(u, v, \varepsilon, \sigma, \gamma, t, m, p; z)$$

then

$$q(z) \prec \left(\frac{zf'(z)}{pf(z)}\right)^{\gamma}$$

and q *is the best subordinant.*

4. Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 4.1. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$. *Suppose* q_2 *satisfies* (2.1), $\gamma > 0$ *and* $\mathrm{Re}\{\eta\} > 0$. *Let* $f \in R(p, m)$ *satisfies*

$$\left(\frac{(p-j)!}{p!}\frac{(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} \in H[1, 1] \cap Q$$

and

$$\left(1 - \frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma} + \frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I^{\lambda+1}_{p,m}(a,b,c)f(z))^{(j)}}{(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}\right)$$

be univalent in U. *If*

$$q_1(z) + \frac{\eta}{\gamma(p-j)}zq_1'(z) \prec \left(1 - \frac{\eta(\lambda+p)}{p-j}\right)\left(\frac{(p-j)!}{p!}\frac{(I^{\lambda}_{p,m}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}$$

$$+\frac{\eta(\lambda+p)}{p-j}\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\left(\frac{(I_{p,m}^{\lambda+1}(a,b,c)f(z))^{(j)}}{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)$$

$$\prec q_2(z)+\frac{\eta}{\gamma(p-j)}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{(p-j)!}{p!}\frac{(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}{z^{p-j}}\right)^{\gamma}\prec q_2(z)$$

and q_1 *and* q_2, *are, respectively, the best subordinant and the best dominant.*

Theorem 4.2. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0)=q_2(0)=1$. *Suppose* q_1 *satisfies* (3.3) *and* q_2 *satisfies* (2.4). *Let* $f\in R(p,m)$ *satisfies*

$$\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{\gamma}\in H[1,1]\cap Q$$

and $\varphi(u,v,\varepsilon,\sigma,\gamma,t,\lambda,a,b,c,m,p,j;z)$ *be univalent in* U, *where* $\varphi(u,v,\varepsilon,\sigma,\gamma,t,\lambda,a,b,c,m,p,j;z)$ *is given by* (2.6). *If*

$$u+vq_1(z)+\varepsilon q_1^2(z)+\sigma q_1^3(z)+t\frac{zq_1'(z)}{q_1(z)}\prec\varphi(u,v,\varepsilon,\sigma,\gamma,t,\lambda,a,b,c,m,p,j;z)$$

$$\prec u+vq_2(z)+\varepsilon q_2^2(z)+\sigma q_2^3(z)+t\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{z(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j+1)}}{(p-j)(I_{p,m}^{\lambda}(a,b,c)f(z))^{(j)}}\right)^{\gamma}\prec q_2(z)$$

and q_1 *and* q_2, *are, respectively, the best subordinant and the best dominant.*

If we take $\lambda=j=0, a=c$ and $b=p$ in Theorem 4.2, then we obtain the next result.

Corollary 4.1. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0)=q_2(0)=1$. *Suppose* q_1 *satisfies* (3.3) *and* q_2 *satisfies* (2.4). *Let* $f\in R(p,m)$ *satisfies*

$$(\frac{zf'(z)}{pf(z)})^{\gamma}\in H[1,1]\cap Q$$

and $\Omega(u,v,\varepsilon,\sigma,\gamma,t,m,p;z)$ *be univalent in* U, *where* $\Omega(u,v,\varepsilon,\sigma,\gamma,t,m,p;z)$ *is given by* (2.9). *If*

$$u+vq_1(z)+\varepsilon q_1^2(z)+\sigma q_1^3(z)+t\frac{zq_1'(z)}{q_1(z)}\prec\Omega(u,v,\varepsilon,\sigma,\gamma,t,m,p;z)$$

$$\prec u+vq_2(z)+\varepsilon q_2^2(z)+\sigma q_2^3(z)+t\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{zf'(z)}{pf(z)}\right)^{\gamma} \prec q_2(z)$$

and q_1 *and* q_2, *are, respectively, the best subordinant and the best dominant.*

Remark 4.1. By specifying the parameters $\lambda, \gamma, \eta, a, b, c, u, \varepsilon, \sigma, m, j$ and p, we can derive a number of known results. Some of them are given below.

(i) Taking $\lambda = 0, m = 1, a = c$ and $b = p(p \in \mathbf{N})$ in Theorems 2.1, 3.1 and 4.1, we get the results obtained by El-Ashwah and Aouf [9, Theorems 3.1, 4.1 and 5.1].

(ii) Putting $\lambda = 0, \gamma = \eta = m = 1, a = c$ and $b = p(p \in \mathbf{N})$ in Theorem 2.1, we obtain the results obtained by Ali *et al.* [1, Theorem 2.9].

(iii) Selecting $u = \varepsilon = \sigma = j = 0, \gamma = \lambda = m = p = 1, a = k + 1(k \in \mathbf{N_0})$ and $b = c$ in Theorems 2.2, 3.2 and 4.2, we obtain the results obtained by Ibrahim and Darus [12, Theorems 2.1, 2.3 and 2.5].

(iv) Setting $\lambda = j = 0, m = b = p = 1$ and $a = c$ in Theorem 2.1, we get the results obtained by Murugusundaramoorthy and Magesh [19, Corollary 3.3].

(v) By taking $\lambda = j = 0, m = b = p = 1$ and $a = c$ in Theorems 3.1 and 4.1, we get the results obtained by Răducanu and Nechita [23, Corollaries 3.7 and 3.10].

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