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Differential Subordination Theorems of Analytic Functions and Some Applications

Waggas Galib Atshan
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq
E-mail: waggashnd@yahoo.com

Abbas kareem Wanas
Department of Mathematics, College of Computer Science and Mathematics
University of Al-Qadisiya, Diwaniya, Iraq
E-mail: k.abbaswmk@yahoo.com

Abstract

In the present paper , we study some classes of analytic functions , we obtain a number of sufficient conditions for normalized analytic functions in the unit disk . Also we give some applications of the first – order differential subordination for generalized Briot-Bouquet differential subordination .

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1. Introduction and Preliminaries

Let $L(\lambda)$ denote the class of all functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^{n-\lambda} \quad (0\le\lambda<1), \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and satisfying $f(0)=f'(0)-1=0$.

Also let $T(\lambda)$ denote the class of all functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^{n-\lambda} \quad (0\le\lambda<1\ ,a_n\ge 0\ ), \tag{1.2}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$ and satisfying $f(0)=f'(0)-1=0$.

A function $f\in L(\lambda)$ is said to be starlike of order β , if f

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta \quad (z\in U\ ,0\le\beta<1). \tag{1.3}$$

Denote this class by $S^*(\beta)$

A function $f\ \varepsilon\ L(\lambda)$ is said to be convex of order β if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta \quad (z\in U\ ,0\le\beta<1)\ . \tag{1.4}$$

Denote this class by C β

For two functions f and F analytic in U, we say f is subordinate to F in U, denote by $f(z)<F(z)$ or $f<F$, if there exists a schwarz function w(z) analytic in U, with w(0)=0 and |w(z)|<1 such that $f(z)=F(w(z)),(z\ \varepsilon$ U) .

In particular , if the function F is univalent in U , then $f<F$ if and only if $f(0)=F(0)$ and f(U) ⊂ F (U) .

Let ψ C^3 U $\to$C and let h be univalent in U. Assume that p, ϕ are analytic and univalent in U if p satisfies the differential subordination

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right)\prec h(z), \tag{1.5}$$

Then p is called a solution of the differential subordination . The univalent function q is called a dominant of the solutions of the differential subordination , or more simply dominant if p < qfor all p satisfying (1.5) A dominant p that satisfies q <q for all dominants q of (1.5) is said to be the best dominant of (1.5) .

Let a, b,c E Cwith . c= 0,- 1, 2,The Gaussian hypergeometric function $_2F_1$ is defined by

$$(z)=(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n\ (b)_n}{(c)_n}\frac{z^n}{n!}\ , \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined , in terms of the Gamma function , by

$$(\lambda)_n=\frac{r(\lambda+n)}{r(\lambda)}=\begin{cases}1 & (n=0)\\ \gamma(\lambda+1)...(\lambda+n-1) & (n\varepsilon\, 0)\end{cases}$$

Definition 1.1. [1] Let $0\le\lambda<1$ and $\mu,\upsilon\ \varepsilon\ R$. Then , In terms of familiar (Gauss 's) hypergeometric function $_2F_1$, the generalized fractional derivative operator $I_{0.z}^{\lambda,\mu,v}$ of a function f is defined by :

$$I_{0,z}^{\lambda,\mu,v}\ f(z)=\begin{cases}\dfrac{1}{r(1-\lambda)}\dfrac{d}{dz}\left\{z^{\lambda-\mu}\int_0^z(z-\varepsilon)\ f(\varepsilon).{}_2F_1(\mu-\lambda,-v;1-1-\lambda;1-\frac{\varepsilon}{z})d\,\varepsilon\right\}, & (0\le\lambda<1)\\ \dfrac{d^n}{dz^n}J_{0,Z}^{\lambda-n\mu,v}\ f(z), & (n\le\lambda<n+1,n\varepsilon N)\end{cases} \tag{1.7}$$

where the function f is analytic in a simply-connected region of the z-plane containing the origin , with the order.

$$f\ (z)=0(|z|^{\varepsilon}),(z\to 0),...$$

for $\varepsilon>\max\{o,\mu-v\}-1$, and the multiplicity of $(z-\varepsilon)^{-\lambda}$ is removed by requiring $\log(z-\varepsilon)$ to be real, when $(z-\varepsilon)>0$.

Definition 1.2. [1] The fractional derivative of order λ ,$(0\le\lambda<1)$ of a function f is defined by

$$D_z^{\lambda}f(z)=\frac{1}{r(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\varepsilon)}{(z-\varepsilon)^{\lambda}}d\varepsilon \tag{1.8}$$

where f it is chosen as in Definition1.1 , and the multiplicity of $(z-\varepsilon)^{-\lambda}$ is removed by requiring $\log(z-\varepsilon)$ to be real , when. $(z-\varepsilon)>0$

By comparing (1.7) with (1.8) , we find

$$I_{0,z}^{\lambda,\lambda,v} f(z) = D_z^{\lambda} f(z), \qquad (0 \le \lambda < 1). \tag{1.9}$$

In terms of gamma function ,we have

$$I_{0,z}^{\lambda,\lambda,v} z^n = \frac{r(n+1)r(n-\mu+v+1)}{r(n-\mu+1)r(n-\lambda+v+1)} z^{n-\mu}, \tag{1.10}$$

$(0 \le \lambda < 1, \mu, v, \varepsilon$ R and n > max $\{0, \mu\text{-}v\} - 1)$

Lemma 1.1. [5] Let q be univalent in the unit disk U and θ and ϕ be analytic in domain D containing q(U) with $\phi(w)=0$ when $w \in$ q(U) . Set $Q(z) = zq'(z)\varphi(q(z)); h(z) = \theta(q(z)) + Q(z)$ Suppose that

1. Q (z) is starlike univalent in U , and
2. Re $\{\frac{zh'(z)}{Q(z)}\} > 0$for z$\varepsilon$ U.

If $\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(q(z)) + zq'(z)\varphi(q(z))$, then p(z) < q(z) and q is the best dominant .**Lemma 1.2. [6]** Let q be convex univalent in the unit disk U and and t E C with Re $\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{t}\right\} > 0$. If p is analytic in U and $\psi p(z) + tzp'(z) < \psi q(z) + tzq'(z)$, then p(z) < q(z) and q is the best dominant .

2. Main Result

Theorem 2.1. Let the function q be univalent in the unit disk U , q(z) =0 and assume that

$$\text{Re } \{1 + \frac{\lambda_2}{t} q(z) + \frac{2\lambda_3}{t} q^2(z) + \frac{3\lambda_4}{t} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} >\}0, \tag{2.1}$$

where $\lambda_i \; \varepsilon \, C, i = 1,2,3,4, t \; \varepsilon \, C \setminus \{0\}$. .

Suppose that $\frac{zq'(z)}{q(z)}$is starlike univalent in U . If $\in$ L (λ) satisfies

$$\lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{\delta} + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{2\delta} + \lambda_4 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{z + f'(z)} - \frac{z + zf'(z)}{z + f(z)}\right]$$

$$< \lambda_1 + \lambda_2 q(z) + \lambda_3 q^2(z) + \lambda_4 q^3(z) + t\frac{zq'(z)}{q(z)}, \tag{2.2}$$

then

$$\left[\frac{z + zf'(z)}{z + f(z)}\right]^{\delta} < q(z), \qquad z \; \varepsilon \text{ U}, f(z) \ne -z$$

and q is the best dominant .

Proof . Define the function p by

$$p(z) = \left[\frac{z + zf'(z)}{z + f(z)}\right]^{\delta}, z \; \varepsilon \text{ U}, f(z) \ne -z \tag{2.3}$$

Note that

$$\lambda_1 + \lambda_2 p(z) + \lambda_3 p^2(z) + \lambda_4 p^3(z) + t\frac{zp'(z)}{p(z)} = \lambda_1 + \lambda_2 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{\delta} + \lambda_3 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{2\delta}$$

$$+\lambda_4 \left[\frac{z + zf'(z)}{z + f(z)}\right]^{3\delta} + t\delta \left[1 + \frac{zf''(z)}{1 + f'(z)} - \frac{z + zf'(z)}{z + f(z)}\right] \tag{2.4}$$

From (2.2) and (2.4) , we have

$$\lambda_1+\lambda_2 p(z)+\lambda_3 p^2(z)+\lambda_4 p^3(z)+t\frac{zp'(z)}{p(z)}<\lambda_1+\lambda_2 q(z)+\lambda_3 q^2(z)+\lambda_4 q^3(z)+t\ \frac{zp'(z)}{q(z)}. \quad (2.5)$$

By setting

$$\theta(w)=\lambda_1+\lambda_2+\lambda_3 w^2+\lambda_4 w^3 \text{ and } \varphi(w)=\frac{t}{w}' w\neq 0,$$

We see that ϕ(w) is analytic in c, ϕ(w) is analytic in C\{0}and that ϕ(w)=0, w E C . Also , we get

$$Q(z)=zq'(z)\varphi(q(z))=t\frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=\lambda_1+\lambda_2 q(z)+\lambda_3 q^2(z)+\lambda_4 q^3(z)+t\frac{zp'(z)}{q(z)}.$$

It is clear that Q(z) is starlike univalent in U ,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}\operatorname{Re}\left\{1+\frac{\lambda_2}{t}q(z)+\frac{2\lambda_3}{t}q^2(z)+\frac{3\lambda_4}{t}q^3(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\} \quad (2.6)$$

From (2.1) and (2.6) , we have

$$Re\ \left\{\frac{zh'(z)}{Q(z)}\right\}>0.$$

Therefore by Lemma 1.1 , we get p(z)<q(z) By using (2.3) , we obtain the result .

By fixing $\lambda_1=\lambda_3=\lambda_4=0$ and $\lambda_2=t=\delta=1$ in the Theorem 2.1,we obtain the following corollary :

Corollary 2.1. Let the function q be univalent in the unit disk U ,q(z)=0 and assume that

$$\operatorname{Re}\left\{1+q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0.$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . If $f\in L(\lambda)$ satisfies

$$1+\frac{zf''(z)}{1+f'(z)}<q(z)+\frac{zq'(z)}{q(z)},$$

then

$$\frac{z+zf'(z)}{z+f(z)}<q(z),$$

and q is the best dominant .

By taking q(z)=(1+*Az*)/(1+*Bz*) (-1≤*B*<*A*≤1) in the Theorem 2.1 , we obtain the following corollary :

Corollary 2.2. Let the function q be convex univalent in the unit disk U and assume that

$$\operatorname{Re}\left\{\frac{\lambda_2}{t}\left[\frac{1+Az}{1+Bz}\right]+\frac{2\lambda_3}{t}\left[\frac{1+Az}{1+Bz}\right]^2+\frac{3\lambda_4}{t}\left[\frac{1+Az}{1+Bz}\right]^3+\frac{1-ABz^2}{(1+Az)(1+Bz)}\right\}>0.$$

If $f\in L(\lambda)$ satisfies

$$\lambda_1+\lambda_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}+\lambda_3\left[\frac{z+zf'(z)}{z+f(z)}\right]^{2\delta}+\lambda_4\left[\frac{z+zf'(z)}{z+f(z)}\right]^{3\delta}+t\delta\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$

$$<\lambda_1+\lambda_2\left[\frac{1+Az}{1+Bz}\right]+\lambda_3\left[\frac{1+Az}{1+Bz}\right]^2+\lambda_4\left[\frac{1+Az}{1+Bz}\right]^3+\frac{t(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} < \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1$$

and $q(z)=\frac{1=Az}{1+Bz}$ is the best dominant .

By taking q(z)e^{yAx}, y=0 in the Theorem 2.1 , we obtain the following corollary :

Corollary 2.3. Let the function **q** be convex univalent in the unit disk U and assume that

$$\text{Re}\left\{1+\frac{\lambda_2}{t}e^{yAz}+\frac{2\lambda_3}{t}e^{2yAz}+\frac{3\lambda_4}{t}e^{3yAz}\right\}>0.$$

If $f \in L(\lambda)$ satisfies

$$\lambda_1+\lambda_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}+\lambda_3\left[\frac{z+zf'(z)}{z+f(z)}\right]^{2\delta}+\lambda_4\left[\frac{z+zf'(z)}{z+f(z)}\right]^{3\delta}+t\delta\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$
$$<\lambda_1+\lambda_2 e^{yAz}+\lambda_3 e^{2yAz}+\lambda_4 e^{3yAz}+tyAz$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} < e^{yAz}, \quad y \ne 0$$

and q(z)=^{yAz} is the best dominant .

By taking $q(z)=\left[\frac{1+z}{1-z}\right]^{y}$ $y \ne 0$ in the Theorem 2.1 , we obtain the following corollary :

Corollary 2.4. Let the function q be convex univalent in the unit disk U and assume that

$$\text{Re}\left\{\frac{\lambda_2}{t}\left[\frac{1+z}{1-z}\right]^{y}+\frac{2\lambda_3}{t}\left[\frac{1+z}{1-z}\right]^{2y}+\frac{3\lambda_4}{t}\left[\frac{1+z}{1-z}\right]^{3y}+\frac{z^2+1}{(1\_z)(1+z)}\right\}>0.$$

If $f \in$ L(λ) satisfies

$$\lambda_1+\lambda_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}+\lambda_3\left[\frac{z+zf'(z)}{z+f(z)}\right]^{2\delta}+\lambda_4\left[\frac{z+zf'(z)}{z+f(z)}\right]^{3\delta}+t\delta\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$
$$\prec \lambda_1+\lambda_2\left[\frac{1+z}{1-z}\right]^{\gamma}+\lambda_3\left[\frac{1+z}{1-z}\right]^{2\gamma}+\lambda_4\left[\frac{1+z}{1-z}\right]^{3\gamma}+\frac{2\gamma tz}{(1-z)(1+z)},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \prec \left[\frac{1+z}{1-z}\right]^{\gamma}, \gamma \ne 0$$

and $q(z)\left[\frac{1+z}{1-z}\right]^{\gamma}$ is the best dominant.

Theorem 2.2. Let the function q be univalent in the unit disk U , $q(z) \ne 0$and assume that

$$Re\left\{1+\frac{\eta_1 k}{t}+\frac{\eta_2(k+1)}{t}q(z)+\frac{zq''(z)}{q'(z)}+(k-1)\frac{zq'(z)}{q(z)}\right\}>0, \tag{2.7}$$

where η_1, η_2, $k \in$ C, $t \in$ C\{0}.

Suppose that $z(q(z))^{k-1}q'(z)$ is starlike univalent in U . If $f \in$ L(λ) satisfies

$$\eta_1\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+\eta_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta(K+1)}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$
$$\prec [\eta_1+\eta_2 q(z)](q(z))^k+tz(q(z))^{k-1}q'(z), \tag{2.8}$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \prec q(z), z\in U, f(z)\neq -z$$

and q is the best dominant .

Proof . Define the function p by

$$p(z)=\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \text{'} z\in U, f(z)\neq -z. \tag{2.9}$$

Note that

$$[\eta_1+\eta_2 p(z)](p(z))^k+tz(p(z))^{k-1}p'(z)=\eta_1\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+\eta_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta(k+1)}$$

$$+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right] \tag{2.10}$$

From (2.8) and (2.10),we have

$$[\eta_1+\eta_2 p(z)](p(z))^k+tz(p(z))^{k-1}p'(z)\prec[\eta_1+\eta_2 q(z)](\mathrm{q}(z))^k+tz(q(z))^{k-1}q'(z) \tag{2.11}$$

By setting

$\theta(w)=(\eta_1+\eta_2 w)w^k and \varphi(w)=tw^{k-1}, w\neq 0,$

We see that $\theta(w)$ is analytic in C, $\phi(w)$ is analytic in C\{0} and that $\phi(w)\neq 0$, $w\in$ C\{0}. Also , we get

$Q(z)=zq'(z)\phi(q(z))=tz(q(z))k\text{-}1q'(z)$

and

$h(z)=\theta(q(z))+Q(z)=[\eta_1+\eta_2 q(z)](q(z))^k+tz(q(z))^{k-1}q'(z).$

It is clear that $Q(z)$ is starlike univalent in U ,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}=Re\left\{1+\frac{\eta_1 k}{t}+\frac{\eta_2(k+1)}{t}q(z)+\frac{zq''(z)}{q'(z)}+(k-1)\frac{zq'(z)}{q(z)}\right\}. \tag{2.12}$$

From (2.7) and (2.12) , we have

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\}>0.$$

Therefore by Lemma 1.1 , we get

$p(z)\prec q(z)$. By using (2.9), we obtain the result .

Remark 2.1 . Taking

$\eta_1=k=0$, $\eta_2=\delta=t=1$ in Theorem 2.2 , we obtain the result in corollary 2.1 .

By taking in the Theorem 2.2 , we obtain the $q(z)=(1+Az)/(1+Bz)(-1\le B<A\le 1)$ following corollary :

Corollary 2.5. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{\frac{\eta_1 k}{t}+\frac{\eta_2(k+1)(1+Az)}{t(1+Bz)}+\frac{1+k(A-B)z-ABz^2)}{(1+Az)(1+Bz)}\right\}>0.$$

If $f\in L(\lambda)$ satisfies

$$\eta_1\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+\eta_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta(k+1)}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$

$$\prec\left[\eta_1+\eta_2\left(\frac{1+Az}{1+Bz}\right)\right]\left(\frac{1+Az}{1+Bz}\right)^k+\frac{t(A-B)(1+Az)^{k-1_z}}{(1+Bz)},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant .

By taking

$q(z)=e^{\gamma Az}, \gamma \neq 0$ in the Theorem 2.2 , we obtain the following corollary :

Corollary 2.6. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{1+k\gamma Az+\frac{\eta_1 k}{t}+\frac{\eta_2(k+1)}{t}e^{\gamma Az}\right\}>0.$$

If $f\in L(\lambda)$ satisfies

$$\eta_1\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+\eta_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta(k+1)}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$

$$\prec[\eta_1+\eta_2 e^{\gamma Az}]e^{k\gamma Az}+t\gamma Aze^{k\gamma Az},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \prec e^{\gamma Az}, \gamma \neq 0$$

and $q(z)=e^{\gamma Az}$ is the best dominant .

By taking $q(z)=\left[\frac{1+z}{1+z}\right]^{\gamma}$ ' $\gamma \neq 0.$

in the Theorem 2.2 , we obtain the following corollary :

Corollary 2.7. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{\frac{\eta_1 k}{t}+\frac{\eta_2(k+1)(1+z)^{\gamma}}{t(1\text{-}z)^{\gamma}}+\frac{z^2+2kyz+1}{(1-z)(1+z)}\right\}>0\,.$$

If $f\in L(\lambda)$ satisfies

$$\eta_1\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+\eta_2\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta(k+1)}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]$$

$$\prec\left[\eta_1+\eta_2\left(\frac{1+z}{1-z}\right)^{\gamma}\right]\left(\frac{1+z}{1-z}\right)^{k\gamma}+\frac{2t\gamma(1+z)^{k\gamma\text{-}1_z}}{(1\text{-}z)^{k\gamma+1}},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta} \prec \left[\frac{1+z}{1-z}\right]^{\gamma}, \gamma \neq 0$$

and

$q(z)=\left[\frac{1+z}{1+z}\right]^{\gamma}$ is the best dominant .

Theorem 2.3. Let the function q be convex univalent in the unit disk U, $q(z) \neq 0$ and assume that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{t}\right\}>0, \tag{2.13}$$

where $t\in C\backslash\{0\}$. If $f\in T(\lambda)$ satisfies

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]\prec q(z)+tzq'(z), \quad (2.14)$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\prec q(z), z\in U, f(z)\neq -z$$

and q is the best dominant .

Proof . Define the function p by

$$p(z)=\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}, z\in U, f(z)\neq -z. \quad (2.15)$$

Note that

$$p(z)+tzp'(z)=\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]. \quad (2.16)$$

From (2.14) and (2.16) , we have

$$p(z)+tzp'(z)\prec q(z)+tzq'(z). \quad (2.17)$$

By setting $\psi = 1$in Lemma 1.2 , we get $p(z) < q(z)$.By using (2.15) , we obtain the result .

By taking $q(z) = (1+Az)/(1+Bz)(-1\le B<A\le 1)$ in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.8. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{\frac{1-Bz}{1+Bz}+\frac{1}{t}\right\}>0.$$

If $f\in T(\lambda)$ satisfies

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta k}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]\prec\frac{1+Az}{1+Bz}+\frac{t(A-B)z}{(1+Bz)^2},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\prec\frac{1+Az}{1+Bz}, -1\le B<A\le 1$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ is the best dominant .

By taking $q(z)=e^{\gamma Az}, \gamma\neq 0$ in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.9. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{1+\gamma Az+\frac{1}{t}\right\}>0.$$

If $f\in T(\lambda)$. Satisfies

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]\prec e\gamma Az+t\gamma Aze^{\gamma Az},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\prec e^{\gamma Az}, \gamma\neq 0$$

and $q(z)=e^{\gamma Az}$ is the best dominant.

By taking

$q(z)=\left[\frac{1+z}{1-z}\right]^{\gamma}, \gamma\neq 0$ in the Theorem 2.3 , we obtain the following corollary :

Corollary 2.10. Let the function q be convex univalent in the unit disk U and assume that

$$Re\left\{\frac{z^2+2\gamma z+1}{(1-z)(1+z)}+\frac{1}{t}\right\}>0.$$

If $f\in T(\lambda)$. satisfies

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}+t\delta\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\left[1+\frac{zf''(z)}{1+f'(z)}-\frac{z+zf'(z)}{z+f(z)}\right]\prec\left[\frac{1+z}{1-z}\right]^{\gamma}+\frac{2t\gamma(1+z)^{\gamma-1}z}{(1-z)^{\gamma+1}},$$

then

$$\left[\frac{z+zf'(z)}{z+f(z)}\right]^{\delta}\prec\left[\frac{1+z}{1-z}\right]^{\gamma}, \gamma\neq 0$$

and

$q(z)=\left[\frac{1+z}{1-z}\right]^{\gamma}$ is the best dominant .

3. Applications

We introduce some applications containing fractional derivative operators . Assume that

$$g(z)=\sum_{n=2}^{\infty}\sigma_n z^n.$$

By Definition 1.2 , we have

$$D_z^{\gamma}g(z)=\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}\sigma_n z^{n-\lambda}=\sum_{n=2}^{\infty}\sigma_n z^{n-\lambda},$$

where

$$a_n=\frac{\Gamma(n+1)}{\Gamma(n+1-\lambda)}\sigma_n, n=2,3,....$$

Thus $z+D_z^{\lambda}g(z)\in L(\lambda)$ and $z-D_z^{\lambda}g(z)\in T(\lambda).(\sigma_n\geq 0)$, then we have the following results:

Theorem 3.1. Let the assumptions of Theorem 2.1 hold .Then

$$\left[\frac{2z+zD_z^{\lambda}g'(z)}{2z+D_z^{\lambda}g(z)}\right]^{\delta}\prec q(z),$$

and q is the best dominant .

Proof . Define the function f by

$$f(z)=z+D_z^{\lambda}g(z)(z\in U, f(z)\neq -z),$$

it can easily observed that $f\in L(\lambda)$. Since $g(0) = 0$, then

$$\left[D_z^{\lambda}g(z)\right]'=D_z^{\lambda}g'(z).$$

Thus by using Theorem 2.1 , we obtain the result .

Theorem 3.2. Let the assumptions of Theorem 2.2 hold .Then

$$\left[\frac{2z+zD_z^{\lambda}g'(z)}{2z+D_z^{\lambda}g(z)}\right]^{\delta}\prec q(z),$$

and q is the best dominant .

Proof . The proof is similar to that of Theorem 3.1 .

Theorem 3.3. Let the assumptions of Theorem 2.3 hold .Then

$$\left[\frac{2z - zD_z^{\lambda} g'(z)}{2z - D_z^{\lambda} g(z)}\right]^{\delta} \prec q(z),$$

and q is the best dominant .

Proof . Define the function *f* by

$$f(z) = z - D_z^{\lambda} g(z)(z \in U, f(z) \neq -z),$$

it can easily observed that $f \in T(\lambda)$. Since $g(0) = 0$, then

$$\left[D_z^{\lambda} g(z)\right]' = D_z^{\lambda} g'(z).$$

Thus by using Theorem 2.3 , we obtain the result .

By using (1.10) , we have

$$J_{o,z}^{\lambda\mu v} g(z) = \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+v+1)}{\Gamma(n-\mu+1)\Gamma(n-\lambda+v+1)} \sigma_n z^{n-\mu} = \sum_{n=2}^{\infty} \sigma_n z^{n-\mu},$$

where

$$a_n = \frac{\Gamma(n+1)\Gamma(n-\mu+v+1)}{\Gamma(n-\mu+1)\Gamma(n-\lambda+v+1)} \sigma_n, n = 2,3,....$$

Let $\mu=\lambda$. Then

$z + J_{o,z}^{\lambda,\mu,v} g(z) \in L(\lambda)$ and $z - J_{o,z}^{\lambda,\mu,v} g(z) \in T(\lambda)(\sigma_n \geq 0)$, thus we have the following results :

Theorem 3.4. Let the assumptions of Theorem 2.1 hold .Then

$$\left[\frac{z(2z + J_{o,z}^{\lambda,\mu,v} g(z))'}{2z + J_{o,z}^{\lambda,\mu,v} g(z)}\right]^{\delta} \prec q(z),$$

and q is the best dominant .

Proof . Define the function *f* by

$$f(z) = z + J_{o,z}^{\lambda,\mu,v} g(z)(z \in U, f(z) \neq -z),$$

it can easily observed that $f \in L(\lambda)$. Thus by using Theorem 2.1 , we obtain the result .

Theorem 3.5. Let the assumptions of Theorem 2.2 hold .Then

$$\left[\frac{z(2z - J_{o,z}^{\lambda,\mu,v} g(z))'}{2z - J_{o,z}^{\lambda,\mu,v} g(z)}\right]^{\delta} \prec q(z),$$

and q is the best dominant.

Proof . The proof is similar to that of Theorem 3.4 .

Theorem 3.6. Let the assumptions of Theorem 2.3 hold .Then

$$\left[\frac{z(2z - J_{o,z}^{\lambda,\mu,v} g(z))'}{2z - J_{o,z}^{\lambda,\mu,v} g(z)}\right]^{\delta} \prec q(z),$$

and q is the best dominant .

Proof . Define the function *f* by

$$f(z) = z - J_{o,z}^{\lambda,\mu,v} g(z)(z \in U, f(z) \neq -z),$$

it can easily observed that $f \in T(\lambda)$. Thus by using Theorem 2.3 , we obtain the result.

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