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DIFFERENTIAL SUBORDINATIONS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVE

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ABSTRACT. In the present paper, we consider a class $k_p^m(\lambda, \gamma; h)$ which consists of analytic and multivalent functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ associated with Ruscheweyh derivative. Also we obtain some results for this class.

1. Introduction and Preliminaries

Let $R(p, m)$ denote the class of all analytic functions f of the form:

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p} \ (p, m \in \mathbb{N} = \{1, 2, 3, \cdots\}), \ z \in U. \tag{1.1}$$

The Hadamard product (or convolution) $(f_1 * f_2)(z)$ of two functions

$$f_j(z) = z^p + \sum_{n=m}^{\infty} a_{n+p,j} z^{n+p} \in R(p, m) \ (j = 1, 2)$$

is given by

$$(f_1 * f_2) = z^p + \sum_{n=m}^{\infty} a_{n+p,1} a_{n+p,2} z^{n+p}.$$

Given two functions f and g, which are analytic in U, we say that the function g is subordinate to f, written $g \prec f$ or $g(z) \prec f(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $g(z) = f(w(z)), (z \in U)$. In particular, if the function f is univalent in U, then $g \prec f$ if and only if $g(0) = f(0)$ and $g(U) \subset f(U)$.

For $\lambda > -p$ and $f \in R(p, m)$. The Ruscheweyh Derivative of order $\lambda + p - 1$ (see [1]) is denoted by $D^{\lambda+p-1} f$ and defined as

$$D^{\lambda+p-1} f(z) = \frac{z^p}{(1-z)^{p+\lambda}} * f(z) = z^p + \sum_{n=m}^{\infty} \frac{\Gamma(\lambda + n + p)}{\Gamma(\lambda + p) n!} a_{n+p} z^{n+p} \ (\lambda > -p). \tag{1.2}$$

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We note from (1.2) that, we have

$$z(D^{\lambda+p-1}f(z))' = (\lambda+p)(D^{\lambda+p}f(z)) - \lambda D^{\lambda+p-1}f(z). \tag{1.3}$$

Let H be the class of function h with $h(0) = 1$, which are analytic and convex univalent in U.

Definition 1.1. A function $f \in R(p,m)$ is said to be in the class $k_p^m(\lambda,\gamma;h)$ if it satisfies the subordination condition:

$$(1-\gamma)z^{-p}D^{\lambda+p-1}f(z) + \gamma z^{-p}D^{\lambda+p}f(z) \prec h(z),$$

where $\gamma \in \mathbb{C}$ and $h \in H$.

A function $f \in R(1,m)$ is said to be in the class $S^*(\alpha)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in U)$$

for some α $(\alpha < 1)$.

When $0 \le \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order α in U.

A function $f \in R(1,m)$ is said to be prestarlike of order α in U if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \ (\alpha < 1).$$

We note this class by $\Re(\alpha)$.

Clearly a function $f \in R(1,m)$ is in the class $\Re(0)$ if and only if f is convex univalent in U and $\Re\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$.

Lemma 1.1. [4] *Let g be analytic in U and let h be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu}zg'(z) \prec h(z), \tag{1.4}$$

where $Re\ \mu \ge 0$ and $\mu \ne 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\mu-1}h(t)dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (1.4).

Lemma 1.2. [6] *Let $\alpha < 1, f \in S^*(\alpha)$ and $g \in \Re(\alpha)$. Then, for any analytic function F in U*

$$\frac{g*(fF)}{g*f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

Such type of study was carried out by various authors for another classes, like, Dinggong and Liu [2], Liu [3], Prajapat and Raina [5] and Yang et. al. [7].

2. Main Results

Theorem 2.1. *Let* $0 \le \gamma_1 < \gamma_2$. *Then* $k_p^m(\lambda, \gamma_2; h) \subset k_p^m(\lambda, \gamma_1; h)$.

Proof. Let $0 \le \gamma_1 < \gamma_2$ and $f \in k_p^m(\lambda, \gamma_2; h)$.

Suppose that

$$g(z) = z^{-p} D^{\lambda+p-1} f(z). \tag{2.1}$$

Then the function g is analytic in U with $g(0) = 1$.

Since $f \in k_p^m(\lambda, \gamma_2; h)$, then we have

$$(1-\gamma_2) z^{-p} D^{\lambda+p-1} f(z) + \gamma_2 z^{-p} D^{\lambda+p} f(z) \prec h(z). \tag{2.2}$$

From (2.1) and (2.2), we get

$$(1-\gamma_2) z^{-p} D^{\lambda+p-1} f(z) + \gamma_2 z^{-p} D^{\lambda+p} f(z) = g(z) + \frac{\gamma_2}{(\lambda+p)} z g'(z) \prec h(z). \tag{2.3}$$

By using Lemma 1.1, we have

$$g(z) \prec h(z). \tag{2.4}$$

Note that $0 \le \frac{\gamma_1}{\gamma_2} < 1$ and that h is convex univalent in U. Hence

$$\begin{aligned} &(1-\gamma_1) z^{-p} D^{\lambda+p-1} f(z) + \gamma_1 z^{-p} D^{\lambda+p} f(z) \\ &= \frac{\gamma_1}{\gamma_2}((1-\gamma_2) z^{-p} D^{\lambda+p-1} f(z) + \gamma_2 z^{-p} D^{\lambda+p} f(z)) + \left(1 - \frac{\gamma_1}{\gamma_2}\right) g(z) \prec h(z). \end{aligned}$$

Therefore $f \in k_p^m(\lambda, \gamma_1; h)$ and we obtain the result. □

Theorem 2.2. *Let* $f \in k_p^m(\lambda, \gamma; h), g \in R(p, m)$ *and*

$$Re\{z^{-p} g(z)\} > \frac{1}{2}. \tag{2.5}$$

Then

$$(f * g)(z) \in k_p^m(\lambda, \gamma; h).$$

Proof. Let $f \in k_p^m(\lambda, \gamma; h)$ and $g \in R(p, m)$. Then we have

$$\begin{aligned} &(1-\gamma) z^{-p} D^{\lambda+p-1} (f*g)(z) + \gamma z^{-p} D^{\lambda+p} (f*g)(z) \\ &= (1-\gamma)(z^{-p} g(z)) * (z^{-p} D^{\lambda+p-1} f(z)) + \gamma (z^{-p} g(z)) * (z^{-p} D^{\lambda+p} f(z)) \\ &= (z^{-p} g(z)) * \phi(z) \end{aligned} \tag{2.6}$$

where

$$\phi(z) = (1-\gamma) z^{-p} D^{\lambda+p-1} f(z) + \gamma z^{-p} D^{\lambda+p} f(z) \prec h(z). \tag{2.7}$$

From (2.5) note that the function $z^{-p} g(z)$ has the Herglotz representation

$$z^{-p} g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \ (z \in U), \tag{2.8}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, it follows from (2.6) to (2.8) that

$$(1-\gamma)z^{-p}D^{\lambda+p-1}(f*g)(z)+\gamma z^{-p}D^{\lambda+p}(f*g)(z)=\int_{|x|=1}\phi(xz)d\mu(x)\prec h(z).$$

Therefore

$$(f*g)(z)\in k_p^m(\lambda,\gamma;h).$$

□

Corollary 2.3. *Let $f\in k_p^m(\lambda,\gamma;h)$ be defined as in (1.1) and let*

$$Re\left\{1+\sum_{n=m}^{\infty}\frac{c+p}{c+p+n}z^n\right\}>\frac{1}{2}. \tag{2.9}$$

Then

$$r(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt,\ \ (c>-p)$$

is also in the class $k_p^m(\lambda,\gamma;h)$.

Proof. Let $f\in k_p^m(\lambda,\gamma;h)$ be defined as in (1.1). Then

$$r(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt=z^p+\sum_{n=m}^{\infty}\frac{c+p}{c+p+n}a_{n+p}z^{n+p}$$

$$=\left(z^p+\sum_{n=m}^{\infty}a_{n+p}z^{n+p}\right)*\left(z^p+\sum_{n=m}^{\infty}\frac{c+p}{c+p+n}z^{n+p}\right)=(f*F)(z), \tag{2.10}$$

where

$$f(z)=z^p+\sum_{n=m}^{\infty}a_{n+p}z^{n+p}\in k_p^m(\lambda,\gamma;h)$$

and

$$F(z)=z^p+\sum_{n=m}^{\infty}\frac{c+p}{c+p+n}z^{n+p}\in R(p,m).$$

Note that

$$Re\{z^{-p}F(z)\}=Re\left\{1+\sum_{n=m}^{\infty}\frac{c+p}{c+p+n}z^n\right\}>\frac{1}{2}. \tag{2.11}$$

From (2.10) and (2.11) and by using Theorem 2.2, we get $r(z)\in k_p^m(\lambda,\gamma;h)$. □

Theorem 2.4. *Let $f\in k_p^m(\lambda,\gamma;h), g\in R(p,m)$ and $z^{1-p}g(z)\in\Re(\alpha),(\alpha<1)$. Then*

$$(f*g)(z)\in k_p^m(\lambda,\gamma;h).$$

Proof. Let $f \in k_p^m(\lambda,\gamma;h)$ and $g \in R(p,m)$. Then, we have

$$(1-\gamma)z^{-p}D^{\lambda+p-1}f(z)+\gamma z^{-p}D^{\lambda+p}f(z) \prec h(z). \tag{2.12}$$

Now from (1.3), (2.12) is equivalent to

$$\frac{\lambda+p(1-\gamma)}{\lambda+p}z^{-p}D^{\lambda+p-1}f(z)+\frac{\gamma}{\lambda+p}z^{1-p}(D^{\lambda+p-1}f(z))' \prec h(z). \tag{2.13}$$

Hence

$$\begin{aligned}&\frac{\lambda+p(1-\gamma)}{\lambda+p}z^{-p}D^{\lambda+p-1}(f*g)(z)+\frac{\gamma}{\lambda+p}z^{1-p}(D^{\lambda+p-1}(f*g)(z))'\\ &=\frac{\lambda+p(1-\gamma)}{\lambda+p}(z^{-p}g(z))*(z^{-p}D^{\lambda+p-1}f(z))\\ &+\frac{\gamma}{\lambda+p}(z^{-p}g(z))*(z^{1-p}(D^{\lambda+p-1}f(z))')\end{aligned}$$

$$=\frac{(z^{1-p}g(z))*(z\psi(z))}{(z^{1-p}g(z))*z}, \quad (z\in U), \tag{2.14}$$

where

$$\psi(z)=\frac{\lambda+p(1-\gamma)}{\lambda+p}z^{-p}D^{\lambda+p-1}f(z)+\frac{\gamma}{\lambda+p}z^{1-p}(D^{\lambda+p-1}f(z))' \prec h(z). \tag{2.15}$$

Since h is convex univalent in $U, \psi(z) \prec h(z), z^{1-p}g(z) \in \Re(\alpha)$ and $z \in S^*(\alpha), (\alpha<1)$, it follows from (2.14) and Lemma 1.2, we get the result. ☐

Theorem 2.5. *Let $\gamma>0, \sigma>0$ and $f \in k_p^m(\lambda,\gamma;\sigma h+1-\sigma)$. If $\sigma \le \sigma_0$, where*

$$\sigma_0=\frac{1}{2}\left(1-\frac{\lambda+p}{\gamma}\int_0^1\frac{u^{\frac{\lambda+p}{\gamma}-1}}{1+u}du\right)^{-1}, \tag{2.16}$$

then $f \in k_p^m(\lambda,0;h)$. The bound σ_0 is the sharp when $h(z)=\frac{1}{1-z}$.

Proof. Suppose that

$$g(z)=z^{-p}D^{\lambda+p-1}f(z). \tag{2.17}$$

Let $f \in k_p^m(\lambda,\gamma;\sigma h+1-\sigma)$ with $\gamma>0$ and $\sigma>0$. Then, we have

$$g(z)+\frac{\gamma}{(\lambda+p)}zg'(z)=(1-\gamma)z^{-p}D^{\lambda+p-1}f(z)+\gamma z^{-p}D^{\lambda+p}f(z) \prec \sigma h(z)+1-\sigma.$$

By using Lemma 1.1, we have

$$g(z) \prec \frac{\sigma(\lambda+p)}{\gamma}z^{-\frac{(\lambda+p)}{\gamma}}\int_0^z t^{\frac{\lambda+p}{\gamma}-1}h(t)dt+1-\sigma=(h*\varphi)(z), \tag{2.18}$$

where

$$\varphi(z)=\frac{\sigma(\lambda+p)}{\gamma}z^{-\frac{(\lambda+p)}{\gamma}}\int_0^z\frac{t^{\frac{\lambda+p}{\gamma}-1}}{1-t}dt+1-\sigma. \tag{2.19}$$

If $0 < \sigma \leq \sigma_0$, where $\sigma_0 < 1$ is given by (2.16), then it follows from (2.19) that

$$\begin{aligned} Re(\varphi(z)) &= \frac{\sigma(\lambda+p)}{\gamma}\int_0^1 u^{\frac{\lambda+p}{\gamma}-1} Re\left(\frac{1}{1-uz}\right)du + 1 - \sigma \\ &> \frac{\sigma(\lambda+p)}{\gamma}\int_0^1 \frac{u^{\frac{\lambda+p}{\gamma}-1}}{1+u}du + 1 - \sigma \geq \frac{1}{2}. \end{aligned}$$

Now, by using the Herglotz representation for $\varphi(z)$, from (2.17) and (2.18), we arrive at

$$z^{-p}D^{\lambda+p-1}f(z) \prec (h*\varphi)(z) \prec h(z).$$

Since h is convex univalent in U, then $f \in k_p^m(\lambda, 0; h)$.

For $h(z) = \frac{1}{1-z}$ and $f \in R(p,m)$ define by

$$z^{-p}D^{\lambda+p-1}f(z) = \frac{\sigma(\lambda+p)}{\gamma} z^{-\frac{(\lambda+p)}{\gamma}} \int_0^z \frac{t^{\frac{\lambda+p}{\gamma}-1}}{1-t}dt + 1 - \sigma,$$

we have

$$(1-\gamma)z^{-p}D^{\lambda+p-1}f(z) + \gamma z^{-p}D^{\lambda+p}f(z) = \sigma h(z) + 1 - \sigma.$$

Thus $f \in k_p^m(\lambda, \gamma; \sigma h + 1 - \sigma)$.

Also for $\sigma > \sigma_0$, we have

$$Re\{z^{-p}D^{\lambda+p-1}f(z)\} \to \frac{\sigma(\lambda+p)}{\gamma}\int_0^1 \frac{u^{\frac{\lambda+p}{\gamma}-1}}{1+u}du + 1 - \sigma < \frac{1}{2},\ (z \to 1)$$

which implies that $f \notin k_p^m(\lambda, 0; h)$.

Therefore the bound σ_0 cannot be increased when $h(z) = \frac{1}{1-z}$.

This completes the proof of the theorem. □

Theorem 2.6. *Let $f \in k_p^m\left(\lambda+1, \gamma; \frac{1+Az}{1+Bz}\right), \lambda > -p, -1 \leq B < A \leq 1$. Then*

$$z^{-p}D^{\lambda+p}f(z) \prec \tilde{h}(z) = \frac{\lambda+p+1}{\gamma} z^{-\frac{(\lambda+p+1)}{\gamma}} \int_0^z t^{\frac{\lambda+p+1}{g}-1}\left(\frac{1+Az}{1+Bz}\right)dt$$

and $\tilde{h}$ is the best dominant.

Proof. Let $f \in k_p^m\left(\lambda+1, \gamma; \frac{1+Az}{1+Bz}\right)$. Then, we have

$$(1-\gamma)z^{-p}D^{\lambda+p}f(z) + \gamma z^{-p}D^{\lambda+p+1}f(z) \prec \frac{1+Az}{1+Bz}. \tag{2.20}$$

Suppose that

$$g(z) = z^{-p}D^{\lambda+p}f(z). \tag{2.21}$$

Then the function g is analytic in U with $g(0) = 1$.

From (1.3), (2.20) and (2.21), we get

$$(1-\gamma)z^{-p}D^{\lambda+p}f(z) + \gamma z^{-p}D^{\lambda+p+1}f(z) = g(z) + \frac{\gamma}{\lambda+p+1}zg'(z) \prec \frac{1+Az}{1+Bz}. \tag{2.22}$$

By Lemma 1.1, we obtain

$$g(z) \prec \tilde{h}(z) = \frac{\lambda + p + 1}{\gamma} z^{-\frac{(\lambda+p+1)}{\gamma}} \int_0^z t^{\frac{\lambda+p+1}{\gamma}-1} \left(\frac{1 + Az}{1 + Bz} \right) dt$$

and $\tilde{h}$ is the best dominant. Thus we have the result. □

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