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SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS DEFINEND BY CONVOLUTION STRUCTURE WITH GENERALIZAD OPERATOR

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Abstract. The purpose of the present paper is to derive sandwich results involving Hadamard product for certain normalized analytic functions with generalized operator in the open unit disk.

1. Introduction

Let H be the class of analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$ .For n a positive integer and $a \in C$, let $H[a, n]$ be the subclass of H consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a \in C)\,. \tag{1.1}$$

Also, let A be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Let $f, g \in H$. The function f is said to be subordinate to g , or g is said to be superordinate to f , if there exists a schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z) (z \in U)$. If g is univalent in U , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, h \in H$ and $\psi(r, s, t; z) : C^3 \times U \to C$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second -order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). (If f is subordinate to g , then g is superordinate to f). An analytic function q is called a subordinate of (1.3), if $q \prec p$ for all the functions p satisfying (1.3). An univalent subordinat q that satisfies $q \prec q$ for all the subordinants q of (1.3) is called the best subordinant. Recently Miller and Mocanu [10] obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For the functions $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g \in A$ defined by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z)$.

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For $m \in N_0 = N \cup \{0\}, \beta \geq 0, \alpha \in R$ with $\alpha + \beta > 0$ and $f \in A$. The generalized operator $I^m_{\alpha,\beta}$(see [16]) is defined by

$$I^m_{\alpha,\beta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta}{\alpha + \beta} \right)^m a_n z^n. \tag{1.4}$$

It follows from (1.4) that

$$\beta z \left(I^m_{\alpha,\beta} f(z) \right)' = (\alpha + \beta) I^{m+1}_{\alpha,\beta} f(z) - \alpha I^m_{\alpha,\beta} f(z), \beta > 0. \tag{1.5}$$

Note that the genralized operator $I^m_{\alpha,\beta}$ unifies many operators of A. In particular :
(1) $I^m_{\alpha,1} f(z) = I^m_{\alpha} f(z), \alpha > -1$(see Cho and Srivastava [6] and Cho and Kim [7]).
(2) $I^m_{1-\beta,\beta} f(z) = D^m_{\beta} f(z), \beta \geq 0$(see Al-Oboudi [2]).
(3) $I^m_{l+1-\beta,\beta} f(z) = I^m_{l,\beta} f(z), \beta \geq 0$(see Catas [5]).

Using the results of Miller and Mocanu [10], Bulboacã [4] considered certain classes of first order differential super ordinations as well as superordination-preserving integral operators (see [3]). Recently many authors [1,8,11-15] have used the reaults of Bulboacã [4] and obtain certain sufficient conditions applying first order differential subordinations and superordinations.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f in U such that $(f * \Psi)(z) \neq 0$ and f to satisfy $q_1(z) \prec \frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^m_{\alpha,\beta}(f*\Psi)(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in U and $\Phi(z) = z + \sum_{n=2}^{\infty} t_n z^n, \Psi(z) = z + \sum_{n=2}^{\infty} s_n z^n$ are analytic functions in U with $t_n \geq 0, s_n \geq 0$ and $t_n \geq s_n$. Also , we obtain the number of results as their special cases.

2. Preliminaries

To establish our main results , we need the following:

Definition 2.1. [9] Denote by Q the set of all functions f that are analytic and injective on $\bar{U} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 2.1. [9]*Let Q be univalent in the unite disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
(1)$Q(z)$ is starlike univalent in U.
(2)Re $\left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.
If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.2. [4] *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ .Suppose that*
(1)Re $\left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0$ for $z \in U$,
(2)$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent inU.
If $p \in H[q(0), 1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.2}$$

then $q \prec p$ and q is the best subordinat of (2.2).

3. Subordination Results

Theorem 3.1. *Let $\Phi, \Psi \in A$ and q be univalent in U with $q(z) \neq 0, q(0) = 1$ and assume that*

$$Re\left\{1 + \frac{\lambda_2(\gamma - \sigma)}{\lambda_3\sigma} + \frac{\lambda_1\gamma}{\lambda_3\sigma}q(z) + (\frac{\gamma}{\sigma} - 2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0, \tag{3.1}$$

where $\lambda_1, \lambda_2, \gamma \in C, \lambda_3, \sigma \in C\backslash\{0\}$.

Suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ is starlike univalent in U .IF $f \in A$, $\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)} \neq 0, z \in U$, satisfies the differential subordination

$$N_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z) \prec (q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma}, \tag{3.2}$$

where

$$N_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z) = \left(\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\right)^{\gamma} \times$$

$$\left(\lambda_1 + \lambda_2\frac{I^{m}_{\alpha,\beta}(f*\Psi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)} + \frac{\lambda_3(\alpha+\beta)}{\beta}\frac{I^{m}_{\alpha,\beta}(f*\Psi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}\left(\frac{I^{m+2}_{\alpha,\beta}(f*\Phi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)} - \frac{I^{m+1}_{\alpha,\beta}(f*\Psi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\right)\right)^{\sigma}, \tag{3.3}$$

$\beta > 0$, then

$$\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function p by

$$p(z) = \frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}, \qquad z \in U. \tag{3.4}$$

Note that

$$(p(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3\frac{zp'(z)}{(p(z))^2}\right)^{\sigma} = \left(\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\right)^{\gamma} \times$$

$$\left(\lambda_1 + \lambda_2\frac{I^{m}_{\alpha,\beta}(f*\Psi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)} + \frac{\lambda_3(\alpha+\beta)}{\beta}\frac{I^{m}_{\alpha,\beta}(f*\Psi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}\left(\frac{I^{m+2}_{\alpha,\beta}(f*\Phi)(z)}{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)} - \frac{I^{m+1}_{\alpha,\beta}(f*\Psi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\right)\right)^{\sigma} \tag{3.5}$$

From (3.2) and (3.5), we have $(p(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3\frac{zp'(z)}{(p(z))^2}\right)^{\sigma} \prec (q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma}$. This equivalently to $(p(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{p(z)} + \lambda_3\frac{zp'(z)}{(p(z))^2}\right) \prec (q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)$.

By setting $\theta(w) = (\lambda_1 w + \lambda_2)w^{\frac{\gamma}{\sigma}-1}$ and $\phi(w) = \lambda_3 w^{\frac{\gamma}{\sigma}-2}$, we see that $\theta(w)$ and $\phi(w)$ are analytic in $C\backslash\{0\}$ and that $\phi(w) \neq 0, w \in C\backslash\{0\}$.Also , we get $Q(z) = zq'(z)\phi(q(z)) = \lambda_3 z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ and $h(z) = \theta(q(z)) + Q(z) = (q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3\frac{zq'(z)}{(q(z))^2}\right)$.

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{\lambda_2(\gamma - \sigma)}{\lambda_3\sigma} + \frac{\lambda_1\gamma}{\lambda_3\sigma}q(z) + (\frac{\gamma}{\sigma} - 2)\frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\}. \tag{3.6}$$

From (3.1)and (3.6), we have $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$. Therefore by Lemma 2.1, we get $p(z) \prec q(z)$. By using (3.4), we obtain the result . □

By taking $\beta = 1$ and $\alpha > -1$ in Theorem 3.1, we obtain the following Corollary for the operator I_α^m [6].

Corollary 3.2. *Let $\Phi, \Psi \in A$ and q be univalent in U with $q(z) \neq 0, q(0) = 1$ and assume that (3.1) holds true. Suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ is starlike univalent in U .If $f \in A, \frac{I_\alpha^{m+1}(f*\Phi)(z)}{I_\alpha^m(f*\Psi)(z)} \neq 0, z \in U$,satisfies the differential subordination*

$$N_2(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha;z) \prec (q(z))^\gamma \left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^\sigma,$$

where

$$N_2(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha;z) = \left(\frac{I_\alpha^{m+1}(f*\Phi)(z)}{I_\alpha^m(f*\Psi)(z)}\right)^\gamma \times$$

$$\left(\lambda_1 + \lambda_2 \frac{I_\alpha^m(f*\Psi)(z)}{I_\alpha^{m+1}(f*\Phi)(z)} + \lambda_3(\alpha+1)\frac{I_\alpha^m(f*\Psi)(z)}{I_\alpha^{m+1}(f*\Phi)(z)}\left(\frac{I_\alpha^{m+2}(f*\Phi)(z)}{I_\alpha^{m+1}(f*\Phi)(z)} - \frac{I_\alpha^{m+1}(f*\Psi)(z)}{I_\alpha^m(f*\Psi)(z)}\right)\right)^\sigma, \tag{3.7}$$

then

$$\frac{I_\alpha^{m+1}(f*\Phi)(z)}{I_\alpha^m(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By taking $\alpha = 1 - \beta$ and $\beta > 0$ in Theorem 3.1,we obtain the following Corollary for generalized Salagean operator D_β^m [2].

Corollary 3.3. *Let $\Phi, \Psi \in A$ and q be univalent in U with $q(z) \neq 0, q(0) = 1$ and assume that (3.1) holds true. Suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ is starlike univalent in U .If $f \in A, \frac{D_\beta^{m+1}(f*\Phi)(z)}{D_\beta^m(f*\Psi)(z)} \neq 0, z \in U$,satisfies the differential subordination*

$$N_3(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\beta;z) \prec (q(z))^\gamma \left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^\sigma,$$

where

$$N_3(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\beta;z) = \left(\frac{D_\beta^{m+1}(f*\Phi)(z)}{D_\beta^m(f*\Psi)(z)}\right)^\gamma$$

$$\times \left(\lambda_1 + \lambda_2 \frac{D_\beta^m(f*\Psi)(z)}{D_\beta^{m+1}(f*\Phi)(z)} + \frac{\lambda_3}{\beta}\frac{D_\beta^m(f*\Psi)(z)}{D_\beta^{m+1}(f*\Phi)(z)}\left(\frac{D_\beta^{m+2}(f*\Phi)(z)}{D_\beta^{m+1}(f*\Phi)(z)} - \frac{D_\beta^{m+1}(f*\Psi)(z)}{D_\beta^m(f*\Psi)(z)}\right)\right)^\sigma, \tag{3.8}$$

then

$$\frac{D_\beta^{m+1}(f*\Phi)(z)}{D_\beta^m(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 3.1, we obtain the following Corollary:

Corollary 3.4. *Let q be univalent in U with $q(z) \neq 0, q(0) = 1$ and assume that (3.1) holds true. Suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ is starlike univalent in U .If $f \in A, \frac{I_{\alpha,\beta}^{m+1}(f)(z)}{I_{\alpha,\beta}^m(f)(z)} \neq 0, z \in U$,satisfies the differential subordination*

$$N_4(f,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z) \prec (q(z))^\gamma \left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^\sigma,$$

where

$$N_4(f,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z)=\left(\frac{I^{m+1}_{\alpha,\beta}(f)(z)}{I^{m}_{\alpha,\beta}(f)(z)}\right)^{\gamma}$$

$$\times\left(\lambda_1+\lambda_2\frac{I^{m}_{\alpha,\beta}(f)(z)}{I^{m+1}_{\alpha,\beta}(f)(z)}+\frac{\lambda_3(\alpha+\beta)}{\beta}\frac{I^{m}_{\alpha,\beta}(f)(z)}{I^{m+1}_{\alpha,\beta}(f)(z)}\left(\frac{I^{m+2}_{\alpha,\beta}(f)(z)}{I^{m+1}_{\alpha,\beta}(f)(z)}-\frac{I^{m+1}_{\alpha,\beta}(f)(z)}{I^{m}_{\alpha,\beta}(f)(z)}\right)\right)^{\sigma}, \quad (3.9)$$

then

$$\frac{I^{m+1}_{\alpha,\beta}(f)(z)}{I^{m}_{\alpha,\beta}(f)(z)}\prec q(z)$$

and q is the best dominant.

4. Superordination Results

Theorem 4.1. *Let $\Phi,\Psi\in A$ and q be convex univalent in U with $q(z)\neq 0, q(0)=1$ and assume that*

$$Re\left\{\frac{\lambda_2(\gamma-\sigma)}{\lambda_3\sigma}+\frac{\lambda_1\gamma}{\lambda_3\sigma}q(z)\right\}>0, \quad (4.1)$$

suppose that $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ is starlike univalent in U .If $f\in A$, $\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\in H\left[q(0),1\right]\cap Q$ with $\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}\neq 0, z\in U$ and $N_1(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z)$ be univalent in U ,where $N_1(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z)$ is given by (3.3).If

$$(q(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q(z)}+\lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma}\prec N_1(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z), \quad (4.2)$$

then

$$q(z)\prec\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)}$$

and q is the best subordinate.

Proof. Define the function p by

$$p(z)=\frac{I^{m+1}_{\alpha,\beta}(f*\Phi)(z)}{I^{m}_{\alpha,\beta}(f*\Psi)(z)} \qquad z\in U. \quad (4.3)$$

Simple computation from (4.3), we obtain

$$(p(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{p(z)}+\lambda_3\frac{zq'(z)}{(p(z))^2}\right)^{\sigma}=N_1(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z), \quad (4.4)$$

From (4.2) and (4.4), we have $(q(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q(z)}+\lambda_3\frac{zq'(z)}{(q(z))^2}\right)^{\sigma}\prec(p(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{p(z)}+\lambda_3\frac{zp'(z)}{(p(z))^2}\right)^{\sigma}$. This equivalent to $(q(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1+\frac{\lambda_2}{q(z)}+\lambda_3\frac{zq'(z)}{(q(z))^2}\right)\prec(p(z))^{\frac{\gamma}{\sigma}}\left(\lambda_1+\frac{\lambda_2}{p(z)}+\lambda_3\frac{zp'(z)}{(p(z))^2}\right)$.

By setting $\theta(w)=(\lambda_1 w+\lambda_2)w^{\frac{\gamma}{\sigma}-1}$ and $\phi(w)=\lambda_3 w^{\frac{\gamma}{\sigma}-2}$, we see that $\theta(w)$ and $\phi(w)$ are analytic in $C\backslash\{0\}$ and that $\phi(w)\neq 0, w\in C\backslash\{0\}$. Also , we get $Q(z)=zq'(z)\phi(q(z))=\lambda_3 z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$.

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=Re\left\{\frac{\lambda_2(\gamma-\sigma)}{\lambda_3\sigma}+\frac{\lambda_1\gamma}{\lambda_3\sigma}q(z)\right\} \quad (4.5)$$

From (4.1)and (4.5), we have $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}>0$. Therefore by Lemma 2.2, we get $q(z)\prec p(z)$. By using (4.3), we obtain the result . □

By taking $\beta = 1$ and $\alpha > -1$ in Theorem 4.1, we obtain the following Corollary:

Corollary 4.2. *Let* $\Phi, \Psi \in A$ *and* q *be convex univalent in* U *with* $q(z) \neq 0, q(0) = 1$ *and assume that (4.1) holds true.suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in* U *.If* $f \in A$, $\frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha}^{m}(f*\Psi)(z)} \in H[q(0),1] \cap Q$ *with* $\frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha}^{m}(f*\Psi)(z)} \neq 0, z \in U$ *and* $N_2(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha; z)$ *be univalent in* U *,where* $N_2(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha; z)$ *is given by (3.7).If*

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma} \prec N_2(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha; z),$$

then

$$q(z) \prec \frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha}^{m}(f*\Psi)(z)}$$

and q *is the best subordinate.*

By taking $\alpha = 1 - \beta$ and $\beta > 0$ in Theorem 4.1, we obtain the following Corollary:

Corollary 4.3. *Let* $\Phi, \Psi \in A$ *and* q *be convex univalent in* U *with* $q(z) \neq 0, q(0) = 1$ *and assume that (18) holds true.suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in* U *.If* $f \in A$, $\frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)} \in H[q(0),1] \cap Q$ *with* $\frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)} \neq 0, z \in U$ *and* $N_3(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \beta; z)$ *be univalent in* U *,where* $N_3(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \beta; z)$ *is given by (3.8).If*

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma} \prec N_3(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \beta; z),$$

then

$$q(z) \prec \frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)}$$

and q *is the best subordinate.*

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 4.1, we obtain the following Corollary:

Corollary 4.4. *Let* q *be convex univalent in* U *with* $q(z) \neq 0, q(0) = 1$ *and assume that (4.1) holds true.suppose that* $z(q(z))^{\frac{\gamma}{\sigma}-2}q'(z)$ *is starlike univalent in* U *.If* $f \in A$, $\frac{I_{\alpha,\beta}^{m+1}f(z)}{I_{\alpha,\beta}^{m}f(z)} \in H[q(0),1] \cap Q$ *with* $\frac{I_{\alpha,\beta}^{m+1}f(z)}{I_{\alpha,\beta}^{m}f(z)} \neq 0, z \in U$ *and* $N_4(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z)$ *be univalent in* U *,where* $N_4(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z)$ *is given by (3.9).If*

$$(q(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q(z)} + \lambda_3 \frac{zq'(z)}{(q(z))^2}\right)^{\sigma} \prec N_4(f, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z),$$

then

$$q(z) \prec \frac{I_{\alpha,\beta}^{m+1}f(z)}{I_{\alpha,\beta}^{m}f(z)}$$

and q *is the best subordinate.*

5. Sandwich Results

Theorem 5.1. *Let*$\Phi, \Psi \in A$*.Let* q_1 *and* q_2 *be convex univalent in* U *with* $q(z) \neq 0, q_1(0) = q_2(0) = 1$*.suppose* q_2 *satisfies (3.1) and* q_1 *satisfies (4.1). Let* $f \in A$, $\frac{I_{\alpha,\beta}^{m+1}(f*\Phi)(z)}{I_{\alpha,\beta}^{m}(f*\Psi)(z)} \in H[1,1] \cap Q$ *with* $\frac{I_{\alpha,\beta}^{m+1}(f*\Phi)(z)}{I_{\alpha\beta}^{m}(f*\Psi)(z)} \neq 0, z \in U$ *and* $N_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z)$ *be univalent in* U *,where* $N_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z)$ *is given by (3.3). If*

$$(q_1(z))^{\gamma}\left(\lambda_1 + \frac{\lambda_2}{q_1(z)} + \lambda_3 \frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma} \prec N_1(f, \Phi, \Psi, \lambda_1, \lambda_2, \lambda_3, \gamma, \sigma, \alpha, \beta; z),$$

$$\prec (q_2(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_2(z)}+\lambda_3\frac{zq_2'(z)}{(q_2(z))^2}\right)^{\sigma},$$

then

$$q_1(z)\prec\frac{I_{\alpha,\beta}^{m+1}(f*\Phi)(z)}{I_{\alpha,\beta}^{m}(f*\Psi)(z)}\prec q_2(z),$$

and q_1 and q_2 are , respectively , the best subordinate and the best dominant.

By making use of corollaries 3.2 and 4.2, we obtain the following Corollary:

Corollary 5.2. *Let$\Phi,\Psi\in A$.Let q_1 and q_2 be convex univalent in U with $q(z)\neq 0, q_1(0)=q_2(0)=1$. Suppose q_2 satisfies (3.1) and q_1 satisfies (4.1). Let $f\in A$, $\frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha}^{m}(f*\Psi)(z)}\in H[1,1]\cap Q$with$\frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha}^{m}(f*\Psi)(z)}\neq 0, z\in U$ and $N_2(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha;z)$ be univalent in U ,where $N_2(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha;z)$ is given by (3.7).If*

$$(q_1(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_1(z)}+\lambda_3\frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma}\prec N_2(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha;z),$$

$$\prec (q_2(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_2(z)}+\lambda_3\frac{zq_2'(z)}{(q_2(z))^2}\right)^{\sigma},$$

then

$$q_1(z)\prec\frac{I_{\alpha}^{m+1}(f*\Phi)(z)}{I_{\alpha,\beta}^{m}(f*\Psi)(z)}\prec q_2(z),$$

and q_1 and q_2 are , respectively , the best subordinate and the best dominant.

By making use of corollaries 3.4 and 4.4, we obtain the following Corollary:

Corollary 5.3. *Let$\Phi,\Psi\in A$.Let q_1 and q_2 be convex univalent in U with $q(z)\neq 0, q_1(0)=q_2(0)=1$. Suppose q_2 satisfies (3.1) and q_1 satisfies (4.1). Let $f\in A$, $\frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)}\in H[1,1]\cap Q$with$\frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)}\neq 0, z\in U$ and $N_3(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\beta;z)$ be univalent in U ,where $N_3(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\beta;z)$ is given by (3.8).If*

$$(q_1(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_1(z)}+\lambda_3\frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma}\prec N_3(f,\Phi,\Psi,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\beta;z)$$

$$\prec (q_2(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_2(z)}+\lambda_3\frac{zq_2'(z)}{(q_2(z))^2}\right)^{\sigma},$$

then

$$q_1(z)\prec\frac{D_{\beta}^{m+1}(f*\Phi)(z)}{D_{\beta}^{m}(f*\Psi)(z)}\prec q_2(z),$$

and q_1 and q_2 are , respectively , the best subordinate and the best dominant.

By making use of corollaries 3.4 and 4.4, we obtain the following Corollary:

Corollary 5.4. *Let $\Phi,\Psi\in A$.Let q_1 and q_2 be convex univalent in U with $q(z)\neq 0, q_1(0)=q_2(0)=1$. Suppose q_2 satisfies (3.1) and q_1 satisfies (4.1). Let $f\in A$, $\frac{I_{\alpha,\beta}^{m+1}(f)(z)}{I_{\alpha,\beta}^{m}(f)(z)}\in H[1,1]\cap Q$ with $\frac{I_{\alpha,\beta}^{m+1}(f)(z)}{I_{\alpha,\beta}^{m}(f)(z)}\neq 0, z\in U$ and $N_4(f,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z)$ be univalent in U ,where $N_4(f,\lambda_1,\lambda_2,\lambda_3,\gamma,\sigma,\alpha,\beta;z)$ is given by (3.9). If*

$$(q_1(z))^{\gamma}\left(\lambda_1+\frac{\lambda_2}{q_1(z)}+\lambda_3\frac{zq_1'(z)}{(q_1(z))^2}\right)^{\sigma}\prec N_4(f,\lambda_1,\lambda_2,\lambda_3\gamma,\sigma,\alpha,\beta;z),$$

$$\prec (q_2(z))^{\gamma} \left(\lambda_1 + \frac{\lambda_2}{q_2(z)} + \lambda_3 \frac{zq_2'(z)}{(q_2(z))^2} \right)^{\sigma},$$

then

$$q_1(z) \prec \frac{I_{\alpha,\beta}^{m+1} f(z)}{I_{\alpha,\beta}^{m} f(z)} \prec q_2(z)$$

and q_1 and q_2 are , respectively , the best subordinate and the best dominant.

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