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The Exponentiated Inverted Weibull Distribution

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Abstract: The exponentiated -parent distribution is a generalization of the standard parent distribution. [7] introduced a simple generalization to weibull distribution namely the exponentiated weibull distribution. The new distribution was applied to analyzing bathtub failure rates lifetime data. In this paper, we consider the standard exponentiated inverted weibull distribution (EIW) that generalizes the standard inverted weibull distribution (IW), the new distribution has two shape parameters. The moments, median, survival function, hazard function, maximum likelihood estimators, least-squares estimators, fisher information matrix and asymptotic confidence intervals have been discussed. A real data set is analyzed and it is observed that the (EIW) distribution can provide a better fitting than (IW) distribution.

Keywords: Inverted Weibull Distribution; Hazard Function; Median, Maximum Likelihood Estimators; Least Squares Estimators; Asymptotic Confidence Intervals.

1. Introduction

The inverted weibull distribution is one of the most popular probability distribution to analyze the life time data with some monotone failure rates. [5] explained the flexibility of the three parameters inverted weibull distribution and its interested properties. Exponentiated (generalized) inverted weibull distribution is a generalization to the inverted weibull distribution through adding a new shape parameter $\lambda \in \Re^+$ by exponentiation to distribution function F , the new distribution function F^λ. [1] explained that the cumulative distribution function is flexible to monotone and non-monotone failure rates. [8] introduced the exponentiated weibull distribution as generalization of the statndard weibull distribution, the applied the new distribution as a suitable model to the bus-motor failure time data. [10] reviewed the exponentiated weibull distribution with new measures. [2] studied the exponentiated exponential distribution in details as an alternative distribution to weibull distribution and gamma distribution. [9] discussed in details the moments of the exponentiated weibull distribution. [12] compared exponentiated weibull distribution with two parameters weibull distribution and gamma distribution with respect to failure rate as well as some basic properties with data analysis. [3] introduced generalized exponential distribution with different method of parameters estimation. [6] applied the exponentiated weibull distribution to the flood data with some properties. [4] introduced a graphical analysis as approach to study the parameters characterization of the exponentiated weibull distribution.

The key idea of this paper is to extend the standard inverted weibull distribution to the standard exponentiated inverted weibull distribution by adding another shape parameter; the shape parameter might be address the lack of fit of the inverted weibull distribution for modeling life time data which indicated non-monotone failure rates. The paper is organized as follows: in Section 2 we introduce the exponentiated inverted weibull distribution with some interested properties. The maximum likelihood estimators and the asymptotic confidence intervals as well as the least squares method has been discussed in Section 3. We analyze data set to explain how the a real data can be modeled by exponentiated inverted weibull distribution in Section 4. Finally we draw conclusions in Section 5.

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2. Exponentiated Inverted Weibull distribution

We say that the random variable X has a standard exponentiated inverted weibull distribution (EIW) if its distribution function takes the following form:

$$F_{\theta}(x) = (e^{-x^{-\beta}})^{\theta}; \quad x, \beta, \theta > 0 \tag{1}$$

Which is simply the θ-th power of the distribution function of the standard inverted weibull distribution. Here, β and θ are the shape parameters. Therefore, the probability density function is:

$$f(x) = \theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^{\theta}; \quad x > 0 \tag{2}$$

The corresponding reliability function is:

$$R(x) = 1 - (e^{-x^{-\beta}})^{\theta} \tag{3}$$

and the hazard function is:

$$h(x) = \frac{\theta\beta x^{-(\beta+1)}(e^{-x^{-\beta}})^{\theta}}{1 - (e^{-x^{-\beta}})^{\theta}} \tag{4}$$

For $\theta = 1$, it represents the standard inverted weibull distribution, and for $\beta = 1$ it represents the exponentiated standard inverted exponential distribution. Thus, the exponentiated inverted weibull distribution is a generalization of the exponentiated inverted exponential distribution as well as the inverted weibull distribution. The exponentiated inverted weibull distribution also has a physical interpretation. If there are m- components in a parallel system and the life times of the components are independent and identically distributed (i.i.d) as exponentiated inverted weibull distribution, then the system lifetime variable has also exponentiated inverted weibull distribution.

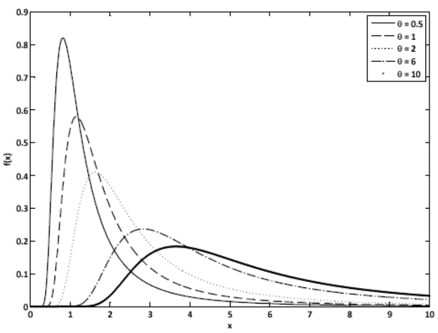

Figure 1 Pdf of the exponentiated inverted weibull distribution for selected values of θ and $\beta = 2$.

We observed that Figure 1 shows that probability density function of the exponentiated inverted weibull distribution is a unimodal.

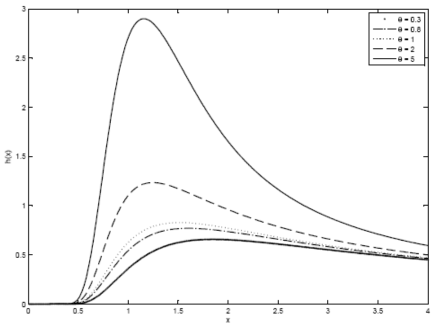

Figure 2 Hazard rate function of the exponentiated inverted weibull distribution selected values of θ and $\beta = 2$.

The k^{th} moments of the exponentiated inverted weibull distribution is given as follows:

$$E(x^k) = \int_0^{\infty} \theta\beta x^k x^{-(\beta+1)}(e^{-x^{-\beta}})^{\theta} dx$$

This can be written as:

$$E(x^k) = \theta^{\frac{k}{\beta}}\Gamma(1 - \frac{k}{\beta}); \quad \beta > k \tag{5}$$

putting $k = 1$ in (5), we obtain the Mean as:

$$E(x) = \theta^{\frac{1}{k}}\Gamma(1 - \frac{1}{k}); \quad \beta > 1 \tag{6}$$

From (5), we can find all the other moments. The quantile function of the exponentiated inverted weibull distribution is given as:

$$x_p = (\frac{-1}{\theta} lnp)^{\frac{-1}{\beta}} \tag{7}$$

The median can be derive from (7) by letting $p = \frac{1}{2}$:

$$x_{\frac{1}{2}} = (\frac{\theta}{ln2})^{\frac{1}{2}} \tag{8}$$

3. Parameters estimation

In this section, we discuss the maximum likelihood estimators of the three-parameter exponentiated inverted weibull distribution and their fisher information matrix as well as asymptotic confidence intervals. Furthermore, we discuss the method of the least square estimators (Regression estimation) of the parameters β and when the third parameter θ is considered to be known.

3.1. Maximum likelihood estimators and fisher information matrix

If $x_1, x_2, ..., x_n$ is a random sample from exponentiated inverted weibull distribution given by (2), then the Log-Likelihood function (LL) becomes:

$$L(\beta,\theta) = log\theta + nlog\beta \quad (9)$$

$$-(\beta+1)\sum_{i=1}^{n} logx_i - \theta\sum_{i=1}^{n} x_i^{-\beta}$$

Therefore, the MLEs of θ and β which maximize (9) must satisfy the nonlinear normal equations given by:

$$\frac{\partial logL}{\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i^{-\beta} = 0 \quad (10)$$

$$\frac{\partial logL}{\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} logx_i + \theta\sum_{i=1}^{n} x_i^{-\beta} logx_i = 0 \quad (11)$$

From (10), we obtain the MLE of θ as a function of β as follows:

$$\widehat{\theta}(\beta) = \frac{n}{\sum_{i=1}^{n} x_i^{-\beta}} \quad (12)$$

Using (12) in (11), we have:

$$\frac{n}{\beta} - \sum_{i=1}^{n} logx_i - \frac{n\sum_{i=1}^{n} x_i^{-\beta} logx_i}{\sum_{i=1}^{n} x_i^{-\beta}} = 0 \quad (13)$$

The MLE of β can be obtained as the fixed point solution of the nonlinear equation of the form $h(\beta) = \beta$, then we have:

$$h(\beta) = \beta - \frac{n}{\beta} + \sum_{i=1}^{n} logx_i + \frac{n\sum_{i=1}^{n} x_i^{-\beta} logx_i}{\sum_{i=1}^{n} x_i^{-\beta}} = 0 (14)$$

Numerical technique method required to find $\widehat{\beta}$ and $\widehat{\theta}$. The asymptotic confidence intervals can be obtained as the results of the asymptotic normality. The asymptotic distribution of the MLEs is given by:

$$\sqrt{n}[(\hat{\beta}-\beta),(\hat{\theta}-\theta)] \xrightarrow{d} N_2(0, I^{-1}(\beta,\theta))$$

Here $I^{-1}(\beta,\theta)$ is the variance-covariance matrix and $I(\beta,\theta)$ is the Fisher information matrix. The elements of $I^{-1}(\beta,\theta)$ can be expressed as follows:

$$\frac{1}{-nE(\frac{\partial^2 logf}{\partial\beta^2})}$$

$$= \frac{\beta^2}{n[1 + \frac{1}{6}\pi^2 - 2\gamma + \gamma^2 - 2ln\theta(1-\gamma) + ln^2\theta}$$

$$= var(\hat{\beta})$$

$$\frac{1}{-nE(\frac{\partial^2 logf}{\partial\theta^2})} = \frac{\theta^2}{n} = var(\hat{\theta})$$

$$\frac{1}{-nE(\frac{\partial^2 logf}{\partial\beta\partial\theta})} = -\frac{\beta\theta}{n(1-\gamma-ln\theta)} = cov(\hat{\beta},\hat{\theta})$$

where $\gamma = 0.5772$ is the Euler's constant, and $\int_0^\infty logze^{-z}dz = -\gamma$.

We can use $I^{-1}(\hat{\beta},\hat{\theta})$ to obtain the asymptotic confidence intervals for θ and β. Therefore, the approximate $100(1-\kappa)\%$ two sided confidence intervals for θ and β are as follows:

$$\hat{\beta} \pm Z_{\frac{\kappa}{2}}\sqrt{I_{11}^{-1}(\hat{\beta})}, \hat{\theta} \pm Z_{\frac{\kappa}{2}}\sqrt{I_{22}^{-1}(\hat{\theta})}$$

The quantity $Z_{\frac{\kappa}{2}}$ can be obtained from the standard normal distribution as the upper $\frac{\kappa}{2}$ percentile.

3.2. Least Squares Method

In this subsection Least Square estimator (LSE) method consider to estimates the parameters θ and β. The LSEs of the parameters θ and β can be estimates by minimizing the following function that assumed the linear relation between two variables, with respect to θ and β. Recall that:

$$F(x) = (e^{-x^{-\beta}})^{\theta}$$

$$lnlnF(x) = -ln\theta - \beta lnx_i$$

let $y_i = lnlnF(x)$, to estimate $F(x_i)$, we can use the following Mean Rank Method:

$$F(x_i) = \frac{i}{n+1}$$

where $x_1, x_2, ..., x_n$ are the rank failure times in ascending order, which can be obtained from the original simple random sample. Therefore, we have:

$$y_i = lnln(\frac{i}{n+1}) \quad (15)$$

So, the straight line equation is given by:

$$y_i = -ln\theta - \beta lnx_i$$

The least squares estimators of $\theta and\beta$ are their value which minimizes the following equation:

$$Q(\theta,\beta) = \sum_{i=1}^{n}[y_i - (-ln\theta - \beta lnx_i)]^2 \quad (16)$$

The first partial derivatives for (16) with respect to θ and β are given by:

$$\frac{\partial Q}{\partial\theta} = \frac{2}{\theta}\sum_{i=1}^{n}[y_i - (-ln\theta - \beta lnx_i)]$$

$$\frac{\partial Q}{\partial\beta} = 2lnx_i\sum_{i=1}^{n}[y_i - (-ln\theta - \beta lnx_i)]$$

Let $\frac{\partial Q}{\partial \theta} = 0$ and $\frac{\partial Q}{\partial \beta} = 0$, then the least squares estimators are as follows:

$$\widehat{\theta} = e^{\left(\frac{\sum_{i=1}^{n} ln^2 x_i \sum_{i=1}^{n} ln y_i + \sum_{i=1}^{n} y_i ln x_i \sum_{i=1}^{n} ln x_i}{(\sum_{i=1}^{n} ln x_i)^2 - n \sum_{i=1}^{n} ln^2 x_i}\right)} \tag{17}$$

$$\widehat{\beta} = \frac{n \sum_{i=1}^{n} y_i ln x_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} ln x_i}{(\sum_{i=1}^{n} ln x_i)^2 - n \sum_{i=1}^{n} ln^2 x_i} \tag{18}$$

where y_i can be obtained from (15).

4. Data Analysis

In this section we provide a data analysis for a simple uncensored data set to see how the new distribution works in practice. The data have been obtained from [11], the data concerning tensile strength of 100 observations of carbon fibers and they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

For the standard exponentiated inverted weibull distribution with shape parameters θ and β, we have:

$\hat{\theta} = 1.6492$, $\hat{\beta} = 0.6175$, with $L = -61.5805$

For the standard inverted weibull distribution:

$\hat{\beta} = 0.80001$, with $L = -78.6322$

We have fitted the exponentiated inverted weibull distribution (EIW) and the inverted weibull distribution (IW) depending on the above data. These two models, with the former having one less parameter, are nested. Our first comparison is based on the likelihood ratio test of $H_0 : \theta = 0$(IW model) against $H_1 : \theta \neq 0$(EIW model). The likelihood ratio test can be used, based on the fact that a log-likelihood (L) ratio statistic is asymptotically chi-square distributed with 1 degree of freedom. The log likelihood functions (L= -61.5807, L= -78.6322) are the log-likelihoods values for (9) and the inverted weibull distribution(IW). However since the value of the test statistics is -2(-78.6322+61.5807) = 34.103, is so large it follows that the exponentiated inverted weibull distrinbution (EIW) provides a significantly better fit.

Our second comparison is based on the probability plot. A probability plot is a graph of the empirical distribution function values (x-axis) against the theoretical distribution function values (y-axis). For the exponentiated inverted weibull distribution we have computed based on (1) for the (y-axis) against the empirical CDF, $(i-0.5)/n$, where $i = 1, 2, ..., n$ and $x_{(i)}$ are the values in the sample of data, in order from smallest to largest. The probability plot corresponding to the two fits, shown in the Figure 3.

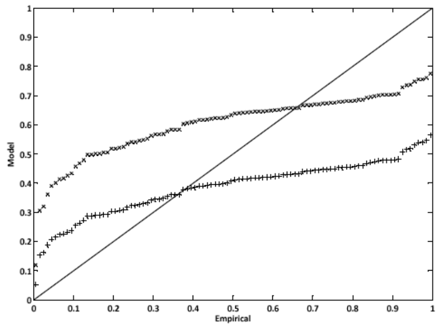

Figure 3 Probability plots for the models based on the exponentiated inverted weibull distribution $(+)$ and the inverted weibull distribution (x).

We have used the sum of the absolute difference between the observed probabilities (Empirical) and the expected probabilities (Theoretical) as a numerical measure of closeness, the values of the measure are 18.7416 for the (EIW) model and 19.0088 for the (IW) model. This supports the conclusions of our results in the first comparison, therefore the EIW will behave better than IW distribution. By using the MLEs of the unknown parameters θ and β, we get the estimation of the Variance-Covariance matrix as follows:

$I^{-1}(\hat{\beta}, \hat{\theta}) = 0\ .01530.13190.13190.0271$

The approximate 95% two sided confidence intervals of the parameters β and θ are given respectively as follows:

$(0.3751, 0.8599)$

and

$(1.3266, 1.9718)$

5. Conclusion

In this paper we have introduced the exponentiated inverted weibull distribution (EIW) as an extension to the inverted weibull distribution (IW). Depending on our data analysis it is observed that the (EIW) model can be serving as alternative to an inverted weibull distribution (IW) and it is expected that in some situations it might work better than the inverted weibull distribution. The (EIW) distribution deserves more works on both aspects, the theoretical (estimation methods) and the applications (analysis further data).

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