Volume 2, No. 3, March 2014
Journal of Global Research in Mathematical Archives
ISSN 2320 – 5822
RESEARCH PAPER
Available online at http://www.jgrma.info

Differential Subordinations for Certain Class of Symmetric Functions

Abbas Kareem Wanas
Department of Mathematics
College of Computer Science and Mathematics
University of Al-Qadisiya
Diwaniya – Iraq
E-Mail: abbas.alshareefi@yahoo.com , abbaskareem.w@gmail.com

Abstract

The object of this paper is to introduce and study a new class $L_m(\eta; h)$ of symmetric analytic functions in the open unit disk. Also we obtain some results for this class.

2000 Mathematics Subject Classification: 30C45, 30C50.

Keywords: Analytic function, Symmetric function, Hadamard product, Convex univalent, Subordination.

.

1. Introduction and Preliminaries

Let A_m denote the class of functions f of the form:

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n \quad (m \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

For two functions f and g analytic in U, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Hadamard product (or convolution) $(f_1 * f_2)(z)$ of two functions

© JGRMA 2012, All Rights Reserved

$$f_j(z) = z + \sum_{n=m+1}^{\infty} a_{n,j} z^n \in A_m \quad (j = 1,2)$$

is given by

$$(f_1 * f_2)(z) = z + \sum_{n=m+1}^{\infty} a_{n,1} a_{n,2} z^n .$$

A function $f \in A_m$ is said to be starlike of order α in U if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1\,; z \in U).$$

Denote the class of all starlike functions of order α in U by $S^*(\alpha)$.

A function $f \in A_m$ is said to be prestarlike of order α in U if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \quad (\alpha < 1).$$

Denote the class of all prestarlike functions of order α in U by $\Re(\alpha)$.

Clearly a function $f \in A_m$ is in the class $\Re(0)$ if and only if f is convex univalent in U and $\Re\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$.

Let H be the class of functions h with $h(0) = 1$, which are analytic and convex univalent in U.

Definition 1.1. A function $f \in A_m$ is said to be in the class $L_m(\eta; h)$ if it satisfies the subordination condition:

$$(1-\eta)\left(\frac{f(z) - f(-z)}{2z}\right) + \eta\left(\frac{f'(z) - f'(-z)}{2}\right) \prec h(z), \qquad (1.2)$$

where $\eta \in \mathbb{C}$ and $h \in H$.

We need the following Lemmas in order to derive our main results for the class $L_m(\eta; h)$.

Lemma 1.2 [3]. Let g be analytic in U and let h be analytic and convex univalent in U with $h(0) = g(0)$. If

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \qquad (1.3)$$

where $Re\,\mu \geq 0$ *and* $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t)\,dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (1.3).

Lemma 1.3 [5]. Let $\alpha < 1, f \in S^*(\alpha)$ and $g \in \Re(\alpha)$. Then, for any analytic function F in U,

© JGRMA 2012, All Rights Reserved

$$\frac{g*(fF)}{g*f}(U) \subset \overline{co}\big(F(U)\big),$$

where $\overline{co}\big(F(U)\big)$ denotes the closed convex hull of $F(U)$.

Such type of study was carried out by various authors for another classes, like, Liu [1,2], Prajapat and Raina [4] and Yang et. al. [6].

2. Main Results

Theorem 2.1. Let $0 \le \eta < \xi$. Then $L_m(\xi; h) \subset L_m(\eta; h)$.

Proof. Let $0 \le \eta < \xi$ and $f \in L_m(\xi; h)$.

Suppose that

$$g(z) = \frac{f(z) - f(-z)}{2z}. \tag{2.1}$$

Then, the function g is analytic in U with $g(0) = 1$.

Since $f \in L_m(\xi; h)$, then we have

$$(1-\xi)\left(\frac{f(z) - f(-z)}{2z}\right) + \xi\left(\frac{f'(z) - f'(-z)}{2}\right) \prec h(z), \tag{2.2}$$

From (2.1) and (2.2), we get

$$(1-\xi)\left(\frac{f(z) - f(-z)}{2z}\right) + \xi\left(\frac{f'(z) - f'(-z)}{2}\right) = g(z) + \xi z g'(z) \prec h(z). \tag{2.3}$$

An application of Lemma 1.2, we obtain

$$g(z) \prec h(z). \tag{2.4}$$

Noting that $0 \le \frac{\eta}{\xi} < 1$ and that h is convex univalent in U, it follows from (2.1), (2.3) and (2.4) that

$$(1-\eta)\left(\frac{f(z) - f(-z)}{2z}\right) + \eta\left(\frac{f'(z) - f'(-z)}{2}\right)$$

$$= \frac{\eta}{\xi}\left((1-\xi)\left(\frac{f(z) - f(-z)}{2z}\right) + \xi\left(\frac{f'(z) - f'(-z)}{2}\right)\right) + \left(1 - \frac{\eta}{\xi}\right) g(z) \prec h(z).$$

Therefore $L_m(\eta; h)$ and we obtain the result.

Theorem 2.2. Let $\eta > 0, \delta > 0$ and $f \in L_m(\eta; \delta h + 1 - \delta)$. If $\delta \le \delta_0$, where

$$\delta_0 = \frac{1}{2}\left(1 - \frac{1}{\eta}\int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u}\,du\right)^{-1}, \tag{2.5}$$

© JGRMA 2012, All Rights Reserved

then $f \in L_m(0;h)$. The bound δ_0 is the sharp when $h(z)=\frac{1}{1-z}$.

Proof. Suppose that

$$g(z)=\frac{f(z)-f(-z)}{2z}. \qquad (2.6)$$

Let $f \in L_m(\eta;\delta h+1-\delta)$ with $\eta>0$ and $\delta>0$. Then we have

$$g(z)+\eta z g'(z)=(1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right)+\eta\left(\frac{f'(z)-f'(-z)}{2}\right)$$

$$\prec \delta h(z)+1-\delta.$$

An application of Lemma 1.2, we obtain

$$g(z)\prec \frac{\delta}{\eta} z^{-\frac{1}{\eta}}\int_0^z t^{\frac{1}{\eta}-1}h(t)dt+1-\delta=(h*\phi)(z), \qquad (2.7)$$

where

$$\phi(z)=\frac{\delta}{\eta} z^{-\frac{1}{\eta}}\int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t}dt+1-\delta. \qquad (2.8)$$

If $0<\delta\le\delta_0$, where $\delta_0>1$ is given by (2.5), then it follows from (2.8) that

$$Re\big(\phi(z)\big)=\frac{\delta}{\eta}\int_0^1 u^{\frac{1}{\eta}-1}Re\left(\frac{1}{1-uz}\right)du+1-\delta>\frac{\delta}{\eta}\int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u}du+1-\delta$$

$$\ge\frac{1}{2}.$$

Now, by using the Herglotz representation for ϕ(z), from (2.6) and (2.7), we get

$$\frac{f(z)-f(-z)}{2z}\prec(h*\phi)(z)\prec h(z).$$

Since h is convex univalent in U, then $f \in L_m(0;h)$.

For $h(z)=\frac{1}{1-z}$ and $f \in A_m$ defined by

$$\frac{f(z)-f(-z)}{2z}=\frac{\delta}{\eta} z^{-\frac{1}{\eta}}\int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t}dt+1-\delta,$$

we have

$$(1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right)+\eta\left(\frac{f'(z)-f'(-z)}{2}\right)=\delta h(z)+1-\delta.$$

Thus, $f \in L_m(\eta;\delta h+1-\delta)$.

© JGRMA 2012, All Rights Reserved

Also, for $\delta > \delta_0$, we have

$$Re\left(\frac{f(z)-f(-z)}{2z}\right) \longrightarrow \frac{\delta}{\eta}\int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u}du + 1 - \delta < \frac{1}{2} \qquad (z \to 1),$$

which implies that $f \notin L_m(0;h)$. Therefore, the bound δ_0 cannot be increased when $h(z) = \frac{1}{1-z}$ and this completes the proof of the theorem.

Theorem 2.3. Let $f \in L_m(\eta;h), g \in A_m$ and

$$Re\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}\,. \tag{2.9}$$

Then

$$f * g \in L_m(\eta;h).$$

Proof. Let $f \in L_m(\eta;h)$ and $g \in A_m$. Then, we have

$$(1-\eta)\left(\frac{(f*g)(z)-(f*g)(-z)}{2z}\right) + \eta\left(\frac{(f*g)'(z)-(f*g)'(-z)}{2}\right)$$

$$= (1-\eta)\left(\frac{g(z)}{z}\right) * \left(\frac{f(z)-f(-z)}{2z}\right) + \eta\left(\frac{g(z)}{z}\right) * \left(\frac{f'(z)-f'(-z)}{2}\right)$$

$$= \left(\frac{g(z)}{z}\right) * \psi(z), \tag{2.10}$$

where

$$\psi(z) = (1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right) + \eta\left(\frac{f'(z)-f'(-z)}{2}\right) \prec h(z). \tag{2.11}$$

From (2.9), note that the function $\frac{g(z)}{z}$ has the Herglotz representation

$$\frac{g(z)}{z} = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.12}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|\mathrm{x}| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, it follows from (2.10) to (2.12) that

$$(1-\eta)\left(\frac{(f*g)(z)-(f*g)(-z)}{2z}\right) + \eta\left(\frac{(f*g)'(z)-(f*g)'(-z)}{2}\right)$$

$$= \int_{|x|=1} \psi(xz)\, d\mu(x) \prec h(z).$$

© JGRMA 2012, All Rights Reserved

Therefore, $f * g \in L_m(\eta; h)$.

Theorem 2.4. Let $f \in L_m(\eta; h)$ and let $g \in A_m$ be prestarlike of order α $(\alpha < 1)$. Then

$$f * g \in L_m(\eta; h).$$

Proof. Let $f \in L_m(\eta; h)$ and $g \in A_m$. Then, we have

$$(1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right)+\eta\left(\frac{f'(z)-f'(-z)}{2}\right) \prec h(z). \qquad (2.13)$$

Hence

$$(1-\eta)\left(\frac{(f*g)(z)-(f*g)(-z)}{2z}\right)+\eta\left(\frac{(f*g)'(z)-(f*g)'(-z)}{2}\right)$$

$$= (1-\eta)\left(\frac{g(z)}{z}\right)*\left(\frac{f(z)-f(-z)}{2z}\right)+\eta\left(\frac{g(z)}{z}\right)*\left(\frac{f'(z)-f'(-z)}{2}\right)$$

$$= \frac{g(z)*(z\psi(z))}{g(z)*z} \quad (z \in U), \qquad (1.14)$$

where $\psi(z)$ is defined as in (2.11).

Since h is convex univalent in U, $\psi(z) \prec h(z)$, $g(z) \in \Re(\alpha)$ and $z \in S^*(\alpha)$, $(\alpha < 1)$, it follows from (2.14) and Lemma 1.3, we obtain the result.

Theorem 2.5. Let $f \in L_m(\eta; h)$ be defined as in (1.1). Then

$$k(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \ \ (Re(c) > -1)$$

is also in the class $L_m(\eta; h)$.

Proof. Let $f \in L_m(\eta; h)$ be defined as in (1.1). Then, we have

$$(1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right)+\eta\left(\frac{f'(z)-f'(-z)}{2}\right) \prec h(z). \qquad (2.15)$$

Note that

$$k(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt = z + \sum_{n=m+1}^{\infty}\frac{c+1}{c+n}a_n z^n. \qquad (2.16)$$

We find from (2.16) that $k \in A_m$ and

$$f(z) = \frac{ck(z)+zk'(z)}{c+1}. \qquad (2.17)$$

Define the function p by

© JGRMA 2012, All Rights Reserved

$$p(z) = (1-\eta)\left(\frac{k(z)-k(-z)}{2z}\right) + \eta\left(\frac{k'(z)-k'(-z)}{2}\right). \qquad (2.18)$$

By using (2.17) and (2.18), we get

$$p(z) + \frac{1}{c+1}zp'(z) = \frac{c}{c+1}p(z) + \frac{1}{c+1}\left(zp'(z)+p(z)\right)$$

$$= (1-\eta)\left(\frac{\left(ck(z)+zk'(z)\right)-\left(ck(-z)+zk'(-z)\right)}{2z(c+1)}\right)$$

$$+\eta\left(\frac{\left(ck(z)+zk'(z)\right)'-\left(ck(-z)+zk'(-z)\right)'}{2(c+1)}\right)$$

$$= (1-\eta)\left(\frac{f(z)-f(-z)}{2z}\right) + \eta\left(\frac{f'(z)-f'(-z)}{2}\right). \qquad (2.19)$$

From (2.15) and (2.19), we arrive at

$$p(z) + \frac{1}{c+1}zp'(z) \prec h(z), \quad (Re(c) > -1).$$

An application of Lemma 1.2, we obtain $p(z) \prec h(z)$. By (2.18), we get

$$(1-\eta)\left(\frac{k(z)-k(-z)}{2z}\right) + \eta\left(\frac{k'(z)-k'(-z)}{2}\right) \prec h(z).$$

Therefore, $k \in L_m(\eta; h)$.

References

[1] J. L. Liu, Certain convolution properties of multivalent analytic functions associated with a linear operator, General Mathematics, 17(2)(2009), 41-52.

[2] J. L. Liu, On a class of multivalent analytic functions associated with an integral operator, Bulletin of the Institute of Mathematics, 5(1)(2010), 95-110.

[3] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28(1981), 157-171.

[4] J. K. Prajapat and R. K. Raina, Some applications of differential subordination for a general class of multivalently analytic functions involving a convolution structure, Math. J. Okayama Univ., 52(2010), 147-158.

[5] S. Ruscheweyh, Convolutions in Geometric Function Theory, Les Presses de 1'Université de Montréal, Montréal, 1982.

[6] Y. Yang, Y. Tao and J. L. Liu, Differential subordinations for certain meromorphically multivalent functions defined by Dziok-Srivastava operator, Abstract and Applied Analysis, Art. ID 726518, 9 pages, 2011.

© JGRMA 2012, All Rights Reserved