SS-Injective Modules and Rings

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Abstract

We introduce and investigate ss-injectivity as a generalization of both soc-injectivity and small injectivity. A module M is said to be ss-N-injective (where N is a module) if every R-homomorphism from a semisimple small submodule of N into M extends to N. A module M is said to be ss-injective (resp. strongly ss-injective), if M is ss-R-injective (resp. ss-N-injective for every right R-module N). Some characterizations and properties of (strongly) ss-injective modules and rings are given. Some results of Amin, Yuosif and Zeyada on soc-injectivity are extended to ss-injectivity. Also, we provide some new characterizations of universally mininjective rings, quasi-Frobenius rings, Artinian rings and semisimple rings.

Key words and phrases: Small injective rings (modules); soc-injective rings (modules); SS-Injective rings (modules); Perfect rings; quasi-Frobenius rings.

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1 Introduction

Throughout this paper, R is an associative ring with identity, and all modules are unitary R-modules. For a right R-module M, we write $\mathrm{soc}(M)$, $J(M)$, $Z(M)$, $Z_2(M)$, $E(M)$ and $\mathrm{End}(M)$ for the socle, the Jacobson radical, the singular submodule, the second singular submodule, the injective hull and the endomorphism ring of M, respectively. Also, we use S_r, S_ℓ, Z_r, Z_ℓ, Z_2^r and J to indicate the right socle, the left socle, the right singular ideal, the left singular ideal, the right second singular ideal, and the Jacobson radical of R, respectively. For a submodule N of M, we write $N \subseteq^{ess} M$, $N \ll M$, $N \subseteq^{\oplus} M$, and $N \subseteq^{max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand, and a maximal submodule of M, respectively. If X is a subset of a right R-module M, the right (resp. left) annihilator of X in R is denoted by $r_R(X)$ (resp. $l_R(X)$). If $M = R$, we write $r_R(X) = r(X)$ and $l_R(X) = l(X)$.

Let M and N be right R-modules, M is called soc-N-injective if every R-homomorphism from the soc(N) into M extends to N. A right R-module M is called soc-injective, if M is soc-R-injective. A right R-module M is called strongly soc-injective, if M is soc-N-injective for all right R-module N [2]

Recall that a right R-module M is called mininjective [14] (resp. small injective [19], principally small injective [20]) if every R-homomorphism from any simple (resp. small, principally small) right ideal to M extend to R. A ring is called right mininjective (resp. small injective, principally small injective) ring, if it is right mininjective (resp. small injective, principally small injective) as right R-module. A ring R is called right Kasch if every simple right R-module embeds in R (see for example [15]. Recall that a ring R is called semilocal if R/J is a semisimple [11]. Also, a ring R is said to be right perfect if every right R-module has a projective cover. Recall that a ring R is said to be quasi-Frobenius (or QF) ring if it is right (or left) artinian and right (or left) self-injective; or equivalently, every injective right R-module is projective.

In this paper, we introduce and investigate the notions of ss-injective and strongly ss-injective modules and rings. Examples are given to show that the (strong) ss-injectivity is distinct from that of mininjectivity, principally small injectivity, small injectivity, simple J-injectivity, and (strong) soc-injectivity. Some characterizations and properties of (strongly) ss-injective modules and rings are given.

W. K. Nicholson and M. F. Yousif in [14] introduced the notion of universally mininjective ring, a ring R is called right universally mininjective if $S_r \cap J = 0$. In Section 2, we show that R is a right universally mininjective ring if and only if every simple right R-module is ss-injective. We also prove that if M is a projective right R-module, then every quotient of an ss-M-injective right R-module is ss-M-injective if and only if every sum of two ss-M-injective submodules of a right R-module is ss-M-injective if and only if $\mathrm{Soc}(M) \cap J(M)$ is projective. Also, some results are given in terms of ss-injectivity modules. For example, every simple singular right R-module is ss-injective implies that S_r projective and $r(a) \subseteq^{\oplus} R_R$ for all $a \in S_r \cap J$, and if M is a finitely generated right R-module, then $\mathrm{Soc}(M) \cap J(M)$ is finitely generated if and only if every direct sum of ss-M-injective right R-modules is ss-M-injective if and only if every direct sum of $\mathbb{N}$ copies of ss-M-injective right R-module is ss-M-injective.

In Section 3, we show that a right R-module M is strongly ss-injective if and only if every small submodule A of a right R-module N, every R-homomorphism $\alpha : A \longrightarrow M$ with $\alpha(A)$ semisimple extends to N. In particular, R is semiprimitive if every simple right R-module is strongly ss-injective, but not conversely. We also prove that if R is a right perfect ring, then a right R-module M is strongly soc-injective if and only if M is strongly ss-injective. A results ([2, Theorem 3.6 and Proposition 3.7]) are extended. We prove that a ring R is right artinian if and only if every direct sum of strongly ss-injective right R-modules is injective, and R is QF ring if and only if every strongly ss-injective right R-module is projective.

In Section 4, we extend the results ([2, Proposition 4.6 and Theorem 4.12]) from a soc-injective ring to an ss-injective ring (see Proposition 4.14 and Corollary 4.15).

In Section 5, we show that a ring R is QF if and only if R is strongly ss-injective and right noetherian with essential right socle if and only if R is strongly ss-injective, $l(J^2)$ is countable generated left ideal, $S_r \subseteq^{ess} R_R$, and the chain $r(x_1) \subseteq r(x_2x_1) \subseteq \ldots \subseteq r(x_nx_{n-1}\ldots x_1) \subseteq \ldots$ terminates for every infinite sequence $x_1, x_2, \ldots$ in R (see Theorem 5.10 and Theorem 5.12). Finally, we prove that a ring R is QF if and only if R is strongly left and right ss-injective, left

Kasch, and J is left t-nilpotent (see Theorem 5.15), extending a result of I. Amin, M. Yousif and N. Zeyada [2, Proposition 5.8] on strongly soc-injective rings.

General background materials can be found in [3], [9] and [10].

2 SS-Injective Modules

Definition 2.1. *Let N be a right R-module. A right R-module M is said to be ss-N-injective, if for any semisimple small submodule K of N, any right R-homomorphism $f : K \longrightarrow M$ extends to N. A module M is said to be ss-quasi-injective if M is ss-M-injective. M is said to be ss-injective if M is ss-R-injective. A ring R is said to be right ss-injective if the right R-module R_R is ss-injective.*

Definition 2.2. *A right R-module M is said to be strongly ss-injective if M is ss-N-injective, for all right R-module N. A ring R is said to be strongly right ss-injective if the right R-module R_R is strongly ss-injective.*

Example 2.3. (1) Every soc-injective module is ss-injective, but not conversely (see Example 5.8).
(2) Every small injective module is ss-injective, but not conversely (see Example 5.6).
(3) Every $\mathbb{Z}$-module is ss-injective. In fact, if M is a $\mathbb{Z}$-module, then M is small injective (by [19, Theorem 2.8]) and hence it is ss-injective.
(4) The two classes of principally small injective rings and ss-injective rings are different (see [15, Example 5.2], Example 4.4 and Example 5.6).
(5) Every strongly soc-injective module is strongly ss-injective, but not conversely (see Example 5.8).
(6) Every strongly ss-injective module is ss-injective, but not conversely (see Example 5.7).

Theorem 2.4. *The following statements hold:*
(1) Let N be a right R-module and let $\{M_i : i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i \in I} M_i$ is ss-N-injective if and only if each M_i is ss-N-injective, for all $i \in I$.
(2) Let M, N and K be right R-modules with $K \subseteq N$. If M is ss-N-injective, then M is ss-K-injective.
(3) Let M, N and K be right R-modules with $M \cong N$. If M is ss-K-injective, then N is ss-K-injective.
(4) Let M, N and K be right R-modules with $K \cong N$. If M is ss-K-injective, then M is ss-N-injective.
(5) Let M, N and K be right R-modules with N is a direct summand of M. If M is ss-K-injective, then N is ss-K-injective.

Proof. Clear. □

Corollary 2.5. *(1) If N is a right R-module, then a finite direct sum of ss-N-injective modules is again ss-N-injective. Moreover, a finite direct sum of ss-injective (resp. strongly ss-injective) modules is again ss-injective (resp. strongly ss-injective).*
(2) A direct summand of an ss-quasi-injective (resp. ss-injective, strongly ss-injective) module is again ss-quasi-injective (resp. ss-injective, strongly ss-injective).

Proof. (1) By taking the index I to be a finite set and applying Theorem 2.4(1).

(2) This follows from Theorem 2.4(5). □

Lemma 2.6. *Every ss-injective right R-module is right mininjective.*

Proof. Let I be a simple right ideal of R. By [16, Lemma 3.8] we have that either I is nilpotent or a direct summand of R. If I is a nilpotent, then $I \subseteq J$ by [6, Corollary 6.2.8] and hence I is a semisimple small right ideal of R. Thus every ss-injective right R-module is right mininjective. □

It easy to prove the following proposition.

Proposition 2.7. *Let N be a right R-module. If $J(N)$ is a small submodule of N, then a right R-module M is ss-N-injective if and only if any R-homomorphism $f : soc(N) \cap J(N) \longrightarrow M$ extends to N.*

Proposition 2.8. *Let N be a right R-module and $\{A_i : i = 1,2,...,n\}$ be a family of finitely generated right R-modules. Then N is ss-$\bigoplus_{i=1}^{n} A_i$-injective if and only if N is ss-A_i-injective, for all $i = 1,2,...,n$.*

Proof. ($\Rightarrow$) This follows from Theorem 2.4((2),(4)).

($\Leftarrow$) By [5, Proposition (I.4.1) and Proposition (I.1.2)] we have $\mathrm{soc}(\bigoplus_{i=1}^{n} A_i) \cap J(\bigoplus_{i=1}^{n} A_i) = (\mathrm{soc} \cap J)(\bigoplus_{i=1}^{n} A_i)$ $= \bigoplus_{i=1}^{n} (\mathrm{soc} \cap J)(A_i) = \bigoplus_{i=1}^{n} (\mathrm{soc}(A_i) \cap J(A_i))$. For $j = 1,2,...,n$, consider the following diagram:

$$\begin{tikzcd}
K_j = soc(A_j) \cap J(A_j) \arrow[r, hook, "i_2"] \arrow[d, "i_{K_j}"'] & A_j \arrow[d, "i_{A_j}"] \\
\bigoplus_{i=1}^{n} (soc(A_i) \cap J(A_i)) \arrow[r, hook, "i_1"] \arrow[d, "f"'] & \bigoplus_{i=1}^{n} A_i \\
N &
\end{tikzcd}$$

where i_1, i_2 are inclusion maps and i_{K_j}, i_{A_j} are injection maps. By hypothesis, there exists an R-homomorphism $h_j : A_j \longrightarrow N$ such that $h_j \circ i_2 = f \circ i_{K_j}$, also there exists exactly one homomorphism $h : \bigoplus_{i=1}^{n} A_i \longrightarrow N$ satisfying $h_j = h \circ i_{A_j}$ by [9, Theorem 4.1.6(2)]. Thus $f \circ i_{K_j} = h_j \circ i_2 = h \circ i_{A_j} \circ i_2 = h \circ i_1 \circ i_{K_j}$ for all $j = 1,2,...,n$. Let $(a_1, a_2, ..., a_n) \in \bigoplus_{i=1}^{n} (\mathrm{soc}(A_i) \cap J(A_i))$, thus $a_j \in \mathrm{soc}(A_j) \cap J(A_j)$, for all $i = 1,2,...,n$ and, $f(a_1, a_2, ..., a_n) = f(i_{K_1}(a_1)) + f(i_{K_2}(a_2)) + ... + f(i_{K_n}(a_n)) = (h \circ i_1)(a_1, a_2, ..., a_n)$. Thus $f = h \circ i_1$ and the proof is complete. □

Corollary 2.9. *Let M be a right R-module and $1 = e_1 + e_2 + ... + e_n$ in R such that e_i are orthogonal idempotent. Then M is ss-injective if and only if M is ss-e_iR-injective for every $i = 1,2,...,n$.*
(2) For idempotents e and f of R. If $eR \cong fR$ and M is ss-eR-injective, then M is ss-fR-injective.

Proof. (1) From [3, Corollary 7.3], we have $R = \bigoplus_{i=1}^{n} e_iR$, thus it follows from Proposition 2.8 that M is ss-injective if and only if M is ss-e_iR-injective for all $1 \leq i \leq n$.

(2) This follows from Theorem 2.4(4). □

Proposition 2.10. *A right R-module M is ss-injective if and only if M is ss-P-injective, for every finitely generated projective right R-module P.*

Proof. ($\Rightarrow$) Let M be an ss-injective R-module, thus it follows from Proposition 2.8 that M is ss-R^n-injective for any $n \in \mathbb{Z}^+$. Let P be a finitely generated projective R-module, thus by [1, Corollary 5.5], we have that P is a direct summand of a module isomorphic to R^m for some $m \in \mathbb{Z}^+$. Since M is ss-R^m-injective, thus M is ss-P-injective by Theorem 2.4((2),(4)).

($\Leftarrow$) By the fact that R is projective. □

Proposition 2.11. *The following statements are equivalent for a right R-module M.*
(1) Every right R-module is ss-M-injective.
(2) Every simple submodule of M is ss-M-injective.
(3) $\operatorname{soc}(M) \cap J(M) = 0$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.

(2) $\Rightarrow$ (3) Assume that $\operatorname{soc}(M) \cap J(M) \neq 0$, thus $\operatorname{soc}(M) \cap J(M) = \bigoplus_{i \in I} x_i R$ where $x_i R$ is a simple small submodule of M, for each $i \in I$. Therefore, $x_i R$ is ss-M-injective for each $i \in I$ by hypothesis. For any $i \in I$, the inclusion map from $x_i R$ to M is split, so we have that $x_i R$ is a direct summand of M. Since $x_i R$ is small submodule of M, thus $x_i R = 0$ and hence $x_i = 0$ for all $i \in I$ and this a contradiction.

□

Lemma 2.12. *Let M be an ss-quasi-injective right R-module and $S = \operatorname{End}(M_R)$, then the following statements hold:*
(1) $l_M r_R(m) = Sm$ for all $m \in \operatorname{soc}(M) \cap J(M)$.
(2) $r_R(m) \subseteq r_R(n)$, where $m \in \operatorname{soc}(M) \cap J(M)$, $n \in M$ implies $Sn \subseteq Sm$.
(3) $l_S(mR \cap r_M(\alpha)) = l_S(m) + S\alpha$, where $m \in \operatorname{soc}(M) \cap J(M)$, $\alpha \in S$.
(4) If kR is a simple submodule of M, then Sk is a simple left S-module, for all $k \in J(M)$. Moreover, $\operatorname{soc}(M) \cap J(M) \subseteq \operatorname{soc}(_S M)$.
(5) $\operatorname{soc}(M) \cap J(M) \subseteq r_M(J(_S S))$.
(6) $l_S(A \cap B) = l_S(A) + l_S(B)$, for every semisimple small right submodules A and B of M.

Proof. (1) Let $n \in l_M r_R(m)$, thus $r_R(m) \subseteq r_R(n)$. Now, let $\gamma : mR \longrightarrow M$ is given by $\gamma(mr) = nr$, thus γ is a well define R-homomorphism. By hypothesis, there exists an endomorphism β of M such that $\beta_{|mR} = \gamma$. Therefore, $n = \gamma(m) = \beta(m) \in Sm$, that is $l_M r_R(m) \subseteq Sm$. The inverse inclusion is clear.

(2) Let $n \in M$ and $m \in \operatorname{soc}(M) \cap J(M)$. Since $r_R(m) \subseteq r_R(n)$, then $n \in l_M r_R(m)$. By (1), we have $n \in Sm$ as desired.

(3) If $f \in l_S(m) + S\alpha$, then $f = f_1 + f_2$ such that $f_1(m) = 0$ and $f_2 = g\alpha$, for some $g \in S$. For all $n \in mR \cap r_M(\alpha)$, we have $n = mr$ and $\alpha(n) = 0$ for some $r \in R$. Since $f_1(n) = f_1(mr) = f_1(m)r = 0$ and $f_2(n) = g(\alpha(n)) = g(0) = 0$, thus $f \in l_S(mR \cap r_M(\alpha))$ and this implies that $l_S(m) + S\alpha \subseteq l_S(mR \cap r_M(\alpha))$. Now, we will prove that the other inclusion. Let $g \in l_S(mR \cap r_M(\alpha))$. If $r \in r_R(\alpha(m))$, then $\alpha(mr) = 0$, so $mr \in mR \cap r_M(\alpha)$ which yields $r_R(\alpha(m)) \subseteq r_R(g(m))$. Since $m \in \operatorname{soc}(M) \cap J(M)$, thus $\alpha(m) \in \operatorname{soc}(M) \cap J(M)$. By (2), we have that $g(m) = \gamma\alpha(m)$ for some $\gamma \in S$. Therefore, $g - \gamma\alpha \in l_S(m)$ which leads to $g \in l_S(m) + S\alpha$. Thus $l_S(mR \cap r_M(\alpha)) = l_S(m) + S\alpha$.

(4) To prove Sk is simple left S-module, we need only show that Sk is cyclic for any nonzero element in it. If $0 \neq \alpha(k) \in Sk$, then $\alpha : kR \longrightarrow \alpha(kR)$ is an R-isomorphism. Since $\alpha \in S$, then $\alpha(kR) \ll M$. Since M is ss-quasi-injective, thus $\alpha^{-1} : \alpha(kR) \longrightarrow kR$ has an extension $\beta \in S$ and hence $\beta(\alpha(k)) = \alpha^{-1}(\alpha(k)) = k$, so $k \in S\alpha k$ which leads to $Sk = S\alpha k$. Therefore Sk is a simple left S-module and this leads to $\operatorname{soc}(M) \cap J(M) \subseteq \operatorname{soc}(_S M)$.

(5) If mR is simple and small submodule of M, then $m \neq 0$. We claim that $\alpha(m) = 0$ for all $\alpha \in J(S)$, thus $mR \subseteq r_M(J(S))$. Otherwise, $\alpha(m) \neq 0$ for some $\alpha \in J(S)$. Thus $\alpha : mR \longrightarrow \alpha(mR)$ is an R-isomorphism. Now, we need prove that $r_R(\alpha(m)) = r_R(m)$. Let $r \in r_R(m)$, so $\alpha(m)r = \alpha(mr) = \alpha(0) = 0$ which leads to $r_R(m) \subseteq r_R(\alpha(m))$. The other inclusion, if $r \in r_R(\alpha(m))$, then $\alpha(mr) = 0$, that is $mr \in \ker(\alpha) = 0$, so $r \in r_R(m)$. Hence $r_R(\alpha(m)) = r_R(m)$. Since $m, \alpha(m) \in \operatorname{soc}(M) \cap J(M)$, thus $S\alpha m = Sm$ (by(2)) and this implies that $m = \beta\alpha(m)$ for some $\beta \in S$, so $(1 - \beta\alpha)(m) = 0$. Since $\alpha \in J(S)$, then the element $\beta\alpha$ is quasi-regular by [3, Theorem 15.3]. Thus $1 - \beta\alpha$ is invertible and hence $m = 0$ which is a contradiction. This shows that $\operatorname{soc}(M) \cap J(M) \subseteq r_M(J(S))$.

(6) Let $\alpha \in l_S(A\cap B)$ and consider $f : A+B \longrightarrow M$ is given by $f(a+b) = \alpha(a)$, for all $a \in A$ and $b \in B$. Since M is ss-quasi-injective, thus there exists $\beta \in S$ such that $f(a+b) = \beta(a+b)$. Thus $\beta(a+b) = \alpha(a)$, so $(\alpha-\beta)(a) = \beta(b)$ which yields $\alpha-\beta \in l_S(A)$. Therefore, $\alpha = \alpha-\beta+\beta \in l_S(A)+l_S(B)$ and this implies that $l_S(A\cap B) \subseteq l_S(A)+l_S(B)$. The other inclusion is trivial and the proof is complete. □

Remark 2.13. Let M be a right R-module, then $D(S) = \{\alpha \in S = \mathrm{End}(M) \mid r_M(\alpha)\cap mR \neq 0$ for each $0 \neq m \in \mathrm{soc}(M)\cap J(M)\}$ is a left ideal in S.

Proof. This is obvious. □

Proposition 2.14. *Let M be an ss-quasi-injective right R-module. Then $r_M(\alpha) \subsetneqq r_M(\alpha-\alpha\gamma\alpha)$, for all $\alpha \notin D(S)$ and for some $\gamma \in S$.*

Proof. For all $\alpha \notin D(S)$. By hypothesis, we can find $0 \neq m \in \mathrm{soc}(M)\cap J(M)$ such that $r_M(\alpha)\cap mR = 0$. Clearly, $r_R(\alpha(m)) = r_R(m)$, so $Sm = S\alpha m$ by Lemma 2.12(2). Thus $m = \gamma\alpha m$ for some $\gamma \in S$ and this implies that $(\alpha-\alpha\gamma\alpha)m = 0$. Therefore, $m \in r_M(\alpha-\alpha\gamma\alpha)$, but $m \notin r_M(\alpha)$ and hence the inclusion is strictly. □

Proposition 2.15. *Let M be an ss-quasi-injective right R-module, then the set $\{\alpha \in S =$ $\mathrm{End}(M) \mid 1-\beta\alpha$ is monomorphism for all $\beta \in S\}$ is contained in $D(S)$. Moreover, $J(_SS) \subseteq$ $D(S)$.*

Proof. Let $\alpha \notin D(S)$, then there exists $0 \neq m \in \mathrm{soc}(M)\cap J(M)$ such that $r_M(\alpha)\cap mR = 0$. If $r \in r_R(\alpha(m))$, then $\alpha(mr) = 0$ and so $mr \in r_M(\alpha)$. Since $r_M(\alpha)\cap mR = 0$. Thus $r \in r_R(m)$ and hence $r_R(\alpha(m)) \subseteq r_R(m)$, so $Sm \subseteq S\alpha m$ by Lemma 2.12(2). Therefore, $m \in \ker(1-\gamma\alpha)$ for some $\gamma \in S$. Since $m \neq 0$, thus $1-\gamma\alpha$ is not monomorphism and hence the inclusion holds. Now, let $\alpha \in J(_SS)$ we have $\beta\alpha$ is a quasi-regular element by [3, Theorem 15.3] and hence $1-\beta\alpha$ is isomorphism for all $\beta \in S$, which completes the proof. □

Theorem 2.16. (ss-Baer's condition) *The following statements are equivalent for a ring R.*
(1) M is an ss-injective right R-module.
(2) If $S_r\cap J = A\oplus B$ and $\alpha : A \longrightarrow M$ is an R-homomorphism, then there exists $m \in M$ such that $\alpha(a) = ma$ for all $a \in A$ and $mB = 0$.
(3) If $S_r\cap J = A\oplus B$, and $\alpha : A \longrightarrow M$ is an R-homomorphism, then there exists $m \in M$ such that $\alpha(a) = ma$, for all $a \in A$ and $mB = 0$.

Proof. (1)⇒(2) Define $\gamma : S_r\cap J \longrightarrow M$ by $\gamma(a+b) = \alpha(a)$ for all $a \in A, b \in B$. By hypothesis, there is a right R-homomorphism $\beta : R \longrightarrow M$ is an extension of γ, so if $m = \beta(1)$, then $\alpha(a) = \gamma(a) = \beta(a) = \beta(1)a = ma$, for all $a \in A$. Moreover, $mb = \beta(b) = \gamma(b) = \alpha(0) = 0$ for all $b \in B$, so $mB = 0$.

(2)⇒(1) Let $\alpha : I \to M$ be any right R-homomorphism, where I is any semisimple small right ideal in R. By (2), there exists $m \in M$ such that $\alpha(a) = ma$ for all $a \in I$. Define $\beta : R_R \longrightarrow M$ by $\beta(r) = mr$ for all $r \in R$, thus β extends α.

(2)⇔(3) Clear. □

A ring R is called right universally mininjective ring if it satisfies the condition $S_r\cap J = 0$ (see for example [14]). In the next results, we give new characterizations of universally mininjective ring in terms of ss-injectivity and soc-injectivity.

Corollary 2.17. *The following are equivalent for a ring R.*
(1) R is right universally mininjective.
(2) R is right mininjective and every quotient of a soc-injective right R-module is soc-injective.
(3) R is right mininjective and every quotient of an injective right R-module is soc-injective.

(4) R is right mininjective and every semisimple submodule of a projective right R-module is projective.
(5) Every right R-module is ss-injective.
(6) Every simple right ideal is ss-injective.

Proof. (1)⇔(2)⇔(3)⇔(4) By [14, Lemma 5.1] and [2, Corollary 2.9].
(1)⇔(5)⇔(6) By Proposition 2.11. □

Theorem 2.18. *If M is a projective right R-module. Then the following statements are equivalent.*
(1) Every quotient of an ss-M-injective right R-module is ss-M-injective.
(2) Every quotient of a soc-*M-injective right R-module is ss-M-injective.*
(3) Every quotient of an injective right R-module is ss-M-injective.
(4) Every sum of two ss-M-injective submodules of a right R-module is ss-M-injective.
(5) Every sum of two soc-*M-injective submodules of a right R-module is ss-M-injective.*
(6) Every sum of two injective submodules of a right R-module is ss-M-injective.
(7) Every semisimple small submodule of M is projective.
(8) Every simple small submodule of M is projective.
(9) $\mathrm{soc}(M)\cap J(M)$ is projective.

Proof. (1)⇒(2)⇒(3), (4)⇒(5)⇒(6) and (9)⇒(7)⇒(8) are obvious.

(8)⇒(9) Since $\mathrm{soc}(M)\cap J(M)$ is a direct sum of simple submodules of M and since every simple in $J(M)$ is small in M, thus $\mathrm{soc}(M)\cap J(M)$ is projective.

(3)⇒(7) Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "i"] \arrow[d, "f"'] & M \\ E \arrow[r, "h"] & N \arrow[r] & 0 \end{tikzcd}$$

where E and N are right R-modules, K is a semisimple small submodule of M, h is a right R-epimorphism and f is a right$R$-homomorphism. We can assume that E is injective (see, e.g. [6, Proposition 5.2.10]). Since N is ss-M-injective, thus f can be extended to an R-homomorphism $g : M \longrightarrow N$. By projectivity of M, thus g can be lifted to an R-homomorphism $\tilde{g} : M \longrightarrow E$ such that $h\circ\tilde{g} = g$. Define $\tilde{f} : K \longrightarrow E$ is the restriction of $\tilde{g}$ over K. Clearly, $h\circ\tilde{f} = f$ and this implies that K is projective.

(7)⇒(1) Let N and L be right R-modules with $h : N \longrightarrow L$ is an R-epimorphism and N is ss-M-injective. Let K be any semisimple small submodule of M and let $f : K \longrightarrow L$ be any left R-homomorphism. By hypothesis K is projective, thus f can be lifted to R-homomorphism $g : K \longrightarrow N$ such that $h\circ g = f$. Since N is ss-M-injective, thus there exists an R-homomorphism $\tilde{g} : M \longrightarrow N$ such that $\tilde{g}\circ i = g$. Put $\beta = h\circ\tilde{g} : M \longrightarrow L$. Thus $\beta\circ i = h\circ\tilde{g}\circ i = h\circ g = f$. Hence L is an ss-M-injective right R-module.

(1)⇒(4) Let N_1 and N_2 be two ss-M-injective submodules of a right R-module N. Thus $N_1 + N_2$ is a homomorphic image of the direct sum $N_1 \oplus N_2$. Since $N_1 \oplus N_2$ is ss-M-injective, thus $N_1 + N_2$ is ss-M-injective by hypothesis.

(6)⇒(3) Let E be an injective right R-module with submodule N. Let $Q = E\oplus E$, $K = \{(n,n) \mid n \in N\}$, $\bar{Q} = Q/K$, $H_1 = \{y+K \in \bar{Q} \mid y \in E\oplus 0\}$, $H_2 = \{y+K \in \bar{Q} \mid y \in 0\oplus E\}$. Then $\bar{Q} = H_1 + H_2$. Since $(E\oplus 0)\cap K = 0$ and $(0\oplus E)\cap K = 0$, thus $E \cong H_i$, $i = 1,2$. Since $H_1\cap H_2 = \{y+K \in \bar{Q} \mid y \in N\oplus 0\} = \{y+K \in \bar{Q} \mid y \in 0\oplus N\}$, thus $H_1\cap H_2 \cong N$ under $y \mapsto y+K$ for all $y \in N\oplus 0$. By hypothesis, $\bar{Q}$ is ss-M-injective. Since H_1 is injective, thus $\bar{Q} = H_1\oplus A$ for some submodule A of $\bar{Q}$, so $A \cong (H_1+H_2)/H_1 \cong H_2/H_1\cap H_2 \cong E/N$. By Theorem 2.4(5), E/N is ss-M-injective. □

Corollary 2.19. *The following statements are equivalent.*
(1) Every quotient of an ss-injective right R-module is ss-injective.
(2) Every quotient of a soc-injective right R-module is ss-injective.
(3) Every quotient of a small injective right R-module is ss-injective.
(4) Every quotient of an injective right R-module is ss-injective.
(5) Every sum of two ss-injective submodules of any right R-module is ss-injective.
(6) Every sum of two soc-injective submodules of any right R-module is ss-injective.
(7) Every sum of two small injective submodules of any right R-module is ss-injective.
(8) Every sum of two injective submodules of any right R-module is ss-injective.
(9) Every semisimple small submodule of any projective right R-module is projective.
(10) Every semisimple small submodule of any finitely generated projective right R-module is projective.
(11) Every semisimple small submodule of R_R is projective.
(12) Every simple small submodule of R_R is projective.
(13) $S_r \cap J$ is projective.
(14) S_r is projective.

Proof. The equivalence of (1), (2), (4), (5), (6), (8), (11), (12) and (13) is from Theorem 2.18.
(1)⇒(3)⇒(4), (5)⇒(7)⇒(8) and (9)⇒(10)⇒(13) are clear.
(14)⇒(9) By [2, Corollary 2.9].
(13)⇒(14) Let $S_r = (S_r \cap J) \oplus A$, where $A = \bigoplus_{i \in I} S_i$ and S_i is a right simple and summand of R_R for all $i \in I$. Thus A is projective, but $S_r \cap J$ is projective, so it follows that S_r is projective. □

Theorem 2.20. *If every simple singular right R-module is ss-injective, then $r(a) \subseteq^{\oplus} R_R$ for every $a \in S_r \cap J$ and S_r is projective.*

Proof. Let $a \in S_r \cap J$ and let $A = RaR + r(a)$. Thus there exists a right ideal B of R such that $A \oplus B \subseteq^{ess} R_R$. Suppose that $A \oplus B \neq R_R$, thus we choose $I \subseteq^{max} R_R$ such that $A \oplus B \subseteq I$ and so $I \subseteq^{ess} R_R$. By hypothesis, R/I is a right ss-injective. Consider the map $\alpha : aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ which is a well-define R-homomorphism. Thus there exists $c \in R$ such that $1 + I = ca + I$ and hence $1 - ca \in I$. But $ca \in RaR \subseteq I$ which leads to $1 \in I$, a contradiction. Thus $A \oplus B = R$ and hence $RaR + (r(a) \oplus B) = R$. Since $RaR \ll R_R$, thus $r(a) \subseteq^{\oplus} R_R$. Put $r(a) = (1-e)R$, for some $e^2 = e \in R$, so it follows that $ax = aex$ for all $x \in R$ and hence $aR = aeR$. Let $\gamma : eR \longrightarrow aeR$ be defined by $\gamma(er) = aer$ for all $r \in R$. Then γ is a well-defined R-epimorphism. Clearly, $\ker(\gamma) = eR \cap r(a)$. Hence γ is an isomorphism and so aR is projective. Since $S_r \cap J$ is a direct sum of simple small right ideals, thus $S_r \cap J$ is projective and it follows from Corollary 2.19 that S_r is projective. □

Corollary 2.21. *The following statements are equivalent for a ring R.*
(1) R is right mininjective and every simple singular right R-module is ss-injective.
(2) R is right universally mininjective.

Proof. By Theorem 2.20 and [14, Lemma 5.1]. □

Recall that a ring R is called zero insertive, if $aRb = 0$ for each $a, b \in R$ with $ab = 0$ (see [19]). Note that if R is zero insertive ring, then $RaR + r(a) \subseteq^{ess} R_R$ for every $a \in R$ (see [19, Lemma 2.11]).

Proposition 2.22. *Let R be a zero insertive ring. If every simple singular right R-module is ss-injective, then R is right universally mininjective.*

Proof. Let $a \in S_r \cap J$. We claim that $RaR + r(a) = R$, thus $r(a) = R$ (since $RaR \ll R$), so $a = 0$ and this means that $S_r \cap J = 0$. Otherwise, if $RaR + r(a) \subsetneqq R$, then there exists a maximal right ideal I of R such that $RaR + r(a) \subseteq I$. Since $I \subseteq^{ess} R_R$, thus R/I is ss-injective by hypothesis. Consider $\alpha : aR \longrightarrow R/I$ is given by $\alpha(ar) = r + I$ for all $r \in R$ which is a well-defined R-homomorphism. Thus $1 + I = ca + I$ for some $c \in R$. Since $ca \in RaR \subseteq I$, thus $1 \in I$ and this a contradicts with a maximality of I, so we must have $RaR + r(a) = R$ and this completes the proof. □

Theorem 2.23. *If M is a finitely generated right R-module, then the following statements are equivalent.*
(1) $\mathrm{soc}(M) \cap J(M)$ *is a Noetherian R-module.*
(2) $\mathrm{soc}(M) \cap J(M)$ *is finitely generated.*
(3) Any direct sum of ss-M-injective right R-modules is ss-M-injective.
(4) Any direct sum of soc-*M-injective right R-modules is ss-M-injective.*
(5) Any direct sum of injective right R-modules is ss-M-injective.
(6) $K^{(S)}$ is ss-M-injective for every injective right R-module K and for any index set S.
(7) $K^{(\mathbb{N})}$ is ss-M-injective for every injective right R-module K.

Proof. (1)⇒(2) and (3)⇒(4)⇒(5)⇒(6)⇒(7) Clear.

(2)⇒(3) Let $E = \bigoplus_{i \in I} M_i$ be a direct sum of ss-M-injective right R-modules and $f : N \longrightarrow E$ be a right R-homomorphism, where N is a semisimple small submodule of M. Since $\mathrm{soc}(M) \cap J(M)$ is finitely generated, thus N is finitely generated and hence $f(N) \subseteq \bigoplus_{j \in I_1} M_j$, for some finite subset I_1 of I. Since a finite direct sums of ss-M-injective right R-modules is ss-M-injective, thus $\bigoplus_{j \in I_1} M_j$ is ss-M-injective and hence f can be extended to an R-homomorphism $g : M \longrightarrow E$. Thus E is ss-M-injective.

(7)⇒(1) Let $N_1 \subseteq N_2 \subseteq \ldots$ be a chain of submodules of $\mathrm{soc}(M) \cap J(M)$. For each $i \geq 1$, let $E_i = E(M/N_i)$, $E = \bigoplus_{i=1}^{\infty} E_i$ and $M_i = \prod_{j=1}^{\infty} E_j = E_i \oplus (\prod_{\substack{j=1 \\ j \neq i}}^{\infty} E_j)$, then M_i is injective. By hypothesis, $\bigoplus_{i=1}^{\infty} M_i = (\bigoplus_{i=1}^{\infty} E_i) \oplus (\bigoplus_{i=1}^{\infty} \prod_{\substack{j=1 \\ j \neq i}}^{\infty} E_j)$ is ss-M-injective, so it follows from Theorem 2.4(5) that E it self is ss-M-injective. Define $f : U = \bigcup_{i=1}^{\infty} N_i \longrightarrow E$ by $f(m) = (m + N_i)_i$. It is clear that f is a well defined R-homomorphism. Since M is finitely generated, thus $\mathrm{soc}(M) \cap J(M)$ is a semisimple small submodule of M and hence $\bigcup_{i=1}^{\infty} N_i$ is a semisimple small submodule of M, so f can be extended to a right R-homomorphism $g : M \longrightarrow E$. Since M is finitely generated, we have $g(M) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$ for some n and hence $f(\bigcup_{i=1}^{\infty} N_i) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$. Since $\pi_i f(x) = \pi_i (x + N_j)_{j \geq 1} = x + N_i$, for all $x \in U$ and $i \geq 1$, where $\pi_i : \bigoplus_{j \geq 1} E(M/N_j) \longrightarrow E(M/N_i)$ be the projection map, thus $\pi_i f(U) = U/N_i$ for all $i \geq 1$. Since $f(U) \subseteq \bigoplus_{i=1}^{n} E(M/N_i)$, thus $U/N_i = \pi_i f(U) = 0$, for all $i \geq n+1$, so $U = N_i$ for all $i \geq n+1$ and hence the chain $N_1 \subseteq N_2 \subseteq \ldots$ terminates at N_{n+1}. Thus $\mathrm{soc}(M) \cap J(M)$ is a Noetherian R-module. □

Corollary 2.24. *If N is a finitely generated right R-module, then the following statements are equivalent.*
(1) $\mathrm{soc}(N) \cap J(N)$ *is finitely generated.*
(2) $M^{(S)}$ is ss-N-injective for every soc-*N-injective right R-module M and for any index set S.*
(3) $M^{(S)}$ is ss-N-injective for every ss-N-injective right R-module M and for any index set S.

(4) $M^{(\mathbb{N})}$ is ss-N-injective for every soc*-N-injective right R-module M.*
(5) $M^{(\mathbb{N})}$ is ss-N-injective for every ss-N-injective right R-module M.

Proof. By Theorem 2.23. $\square$

Corollary 2.25. *The following statements are equivalent.*
(1) $S_r \cap J$ is finitely generated.
(2) Any direct sum of ss-injective right R-modules is ss-injective.
(3) Any direct sum of soc-injective right R-modules is ss-injective.
(4) Any direct sum of small injective right R-modules is ss-injective.
(5) Any direct sum of injective right R-modules is ss-injective.
(6) $M^{(S)}$ is ss-injective for every injective right R-module M and for any index set S.
(7) $M^{(S)}$ is ss-injective for every soc-injective right R-module M and for any index set S.
(8) $M^{(S)}$ is ss-injective for every small injective right R-module M and for any index set S.
(9) $M^{(S)}$ is ss-injective for every ss-injective right R-module M and for any index set S.
(10) $M^{(\mathbb{N})}$ is ss-injective for every injective right R-module M.
(11) $M^{(\mathbb{N})}$ is ss-injective for every soc*-injective right R-module M.*
(12) $M^{(\mathbb{N})}$ is ss-injective for every small injective right R-module M.
(13) $M^{(\mathbb{N})}$ is ss-injective for every ss-injective right R-module M.

Proof. By applying Theorem 2.23 and Corollary 2.24. $\square$

Remark 2.26. Let M be a right R-module. We denote that $r_u(N) = \{a \in S_r \cap J \mid Na = 0\}$ and $l_M(K) = \{m \in M \mid mK = 0\}$ where $N \subseteq M$ and $K \subseteq S_r \cap J$. Clearly, $r_u(N) \subseteq (S_r \cap J)_R$ and $l_M(K) \subseteq {}_sM$, where $S = End(M_R)$ and we have the following:
(1) $N \subseteq l_M r_u(N)$ for all $N \subseteq M$.
(2) $K \subseteq r_u l_M(K)$ for all $K \subseteq S_r \cap J$.
(3) $r_u l_M r_u(N) = r_u(N)$ for all $N \subseteq M$.
(4) $l_M r_u l_M(K) = l_M(K)$ for all $K \subseteq S_r \cap J$.

Proof. This is clear $\square$

Lemma 2.27. *The following statements are equivalent for a right R-module M:*
(1) R satisfies the ACC for right ideals of form $r_u(N)$, where $N \subseteq M$.
(2) R satisfies the DCC for $l_M(K)$, where $K \subseteq S_r \cap J$.
(3) For each semisimple small right ideal I there exists a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$.

Proof. (1)⇔(2) Clear.

(2)⇒(3) Consider $\Omega = \{l_M(A) \mid A$ is finitely generated right ideal and $A \subseteq I\}$ which is non empty set because $M \in \Omega$. Now, let K be a finitely generated right ideal of R and contained in I. such that $l_M(K)$ is minimal in Ω. Put $B = K + xR$, where $x \in I$. Thus B is a finitely generated right ideal contained in I and $l_M(B) \subseteq l_M(K)$. But since $l_M(K)$ is minimal in Ω, thus $l_M(B) = l_M(K)$ which yields $l_M(K)x = 0$ for all $x \in I$. Therefore, $l_M(K)I = 0$ and hence $l_M(K) \subseteq l_M(I)$. But $l_M(I) \subseteq l_M(K)$, so $l_M(I) = l_M(K)$.

(3)⇒(1) Suppose that $r_u(M_1) \subseteq r_u(M_2) \subseteq ... \subseteq r_u(M_n) \subseteq ...$, where $M_i \subseteq M$ for each i. Put $D_i = l_M r_u(M_i)$ for each i, and $I = \bigcup_{i=1}^{\infty} r_u(M_i)$, then $I \subseteq S_r \cap J$. By hypothesis, there exists a finitely generated right ideal K of R and contained in I such that $l_M(I) = l_M(K)$. Since K is a finitely generated, thus there exists $t \in \mathbb{N}$ such that $K \subseteq r_u(M_n)$ for all $n \geq t$, that is $l_M(K) \supseteq l_M r_u(M_n) = D_n$ for all $n \geq t$. Since $l_M(K) = l_M(I) = l_M(\bigcup_{i=1}^{\infty} r_u(M_i)) = \bigcap_{i=1}^{\infty} l_M r_u(M_i) = \bigcap_{i=1}^{\infty} D_i \subseteq D_n$, thus $l_M(K) = D_n$ for all $n \geq t$. Since $D_n = l_M r_u(M_n)$, thus $r_u(M_n) = r_u l_M r_u(M_n) = r_u(D_n) = r_u l_M(K)$ for all $n \geq t$. Thus $r_u(M_n) = r_u(M_t)$ for all $n \geq t$. and hence (3) implies (1), which completes the proof. $\square$

The first part in following proposition is obtained directly by Corollary 2.25, but we will prove it by different way.

Proposition 2.28. *Let E be an ss-injective right R-module. Then $E^{(\mathbb{N})}$ is ss-injective if and only if R satisfies the ACC for right ideals of form $r_u(N)$, where $N \subseteq E$.*

Proof. ($\Rightarrow$) Suppose that $r_u(N_1) \subsetneqq r_u(N_2) \subsetneqq \dots \subsetneqq r_u(N_m) \subsetneqq \dots$ be a strictly chain, where $N_i \subseteq E$. Thus we get, $l_E r_u(N_1) \supsetneqq l_E r_u(N_2) \supsetneqq \dots \supsetneqq l_E r_u(N_m) \supsetneqq \dots$. For each $i \geq 1$, so we can find $t_i \in l_E r_u(N_i)/l_E r_u(N_{i+1})$ and $a_{i+1} \in r_u(N_{i+1})$ such that $t_i a_{i+1} \neq 0$. Let $L = \bigcup\limits_{i=1}^{\infty} r_u(N_i)$, then for all $\ell \in L$ there exists $m_\ell \geq 1$ such that $\ell \in r_u(N_i)$ for all $i \geq m_\ell$ and this implies that $t_i \ell = 0$ for all $i \geq m_\ell$. Put $\bar{t} = (t_i)_i$, we have $\bar{t}\ell \in E^{(\mathbb{N})}$ for every $\ell \in L$. Consider $\alpha_{\bar{t}} : L \longrightarrow E^{(\mathbb{N})}$ is given by $\alpha_{\bar{t}}(\ell) = \bar{t}\ell$, then $\alpha_{\bar{t}}$ is a well-define R-homomorphism. Since L is a semisimple small right ideal, thus $\alpha_{\bar{t}}$ extends to $\gamma : R \longrightarrow E^{(\mathbb{N})}$ (by hypothesis) and hence $\alpha_{\bar{t}}(\ell) = \bar{t}\ell = \gamma(\ell) = \gamma(1)\ell$. Thus there exists $k \geq 1$ such that $t_i \ell = 0$ for all $i \geq k$ and all $\ell \in L$ (since $\gamma(1) \in E^{(\mathbb{N})}$), but this contradicts with $t_k a_{k+1} \neq 0$.

($\Leftarrow$) Let $\alpha : I \longrightarrow E^{(\mathbb{N})}$ be an R-homomorphism, where I is a semisimple small right ideal, thus it follows from Lemma 2.27 that there is a finitely generated right ideal $K \subseteq I$ such that $l_M(I) = l_M(K)$. Since $E^{\mathbb{N}}$ is ss-injective, thus $\alpha = a.$ for some $a \in E^{\mathbb{N}}$. Write $K = \bigoplus\limits_{i=1}^{m} r_i R$, so we have $\alpha(r_i) = a r_i \in E^{(\mathbb{N})}$, $i = 1, 2, \dots, m$. Thus there exists $\tilde{a} \in E^{(\mathbb{N})}$ such that $a_n r_i = \tilde{a}_n r_i$ for all $n \in \mathbb{N}$, $i = 1, 2, \dots, m$, where a_n is the nth-coordinate of a . Since K is generated by $\{r_1, r_2, \dots, r_m\}$, thus $ar = \tilde{a}r$ for all $r \in K$. Therefore, $a_n - \tilde{a}_n \in l_M(K) = l_M(I)$ for all $n \in \mathbb{N}$ which leads to $a_n r = \tilde{a}_n r$ for all $r \in I$ and $n \in \mathbb{N}$, so $ar = \tilde{a}r$ for all $r \in I$. Thus there exists $\tilde{a} \in E^{(\mathbb{N})}$ such that $\alpha(r) = \tilde{a}r$ for all $r \in I$ and this means that $E^{(\mathbb{N})}$ is ss-injective. $\square$

Theorem 2.29. *The following statements are equivalent for a ring R:*
(1) $S_r \cap J$ is finitely generated.
(2) $\bigoplus\limits_{i=1}^{\infty} E(M_i)$ is ss-injective right R-module for every simple right R-modules M_i, $i \geq 1$.

Proof. (1)$\Rightarrow$(2) By Corollary 2.25.

(2)$\Rightarrow$(1) Let $I_1 \subsetneqq I_2 \subsetneqq \dots$ be a properly ascending chain of semisimple small right ideals of R. Clearly, $I = \bigcup\limits_{i=1}^{\infty} I_i \subseteq S_r \cap J$. For every $i \geq 1$, there exists $a_i \in I$, $a_i \notin I_i$ and consider $N_i/I_i \subseteq^{max} (a_i R + I_i)/I_i$, so $K_i = (a_i R + I_i)/N_i$ is a simple right R-module. Define $\alpha_i : (a_i R + I_i)/I_i \longrightarrow (a_i R + I_i)/N_i$ by $\alpha_i(x + I_i) = x + N_i$ which is right R-epimorphism. Let $E(K_i)$ be the injective hull of K_i and $i_i : K_i \to E(K_i)$ be the inclusion map. By injectivity of $E(K_i)$, there there exists $\beta_i : I/I_i \longrightarrow E(K_i)$ such that $\beta_i = i_i \alpha_i$. Since $a_i \notin N_i$, then $\beta_i(a_i + I_i) = i_i(\alpha_i(a_i + I_i)) = a_i + N_i \neq 0$ for each $i \geq 1$. If $b \in I$, then there exists $n_b \geq 1$ such that $b \in I_i$ for all $i \geq n_b$ and hence $\beta_i(b + I_i) = 0$ for all $i \geq n_b$. Thus we can define $\gamma : I \longrightarrow \bigoplus\limits_{i=1}^{\infty} E(K_i)$ by $\gamma(b) = (\beta_i(b + I_i))_i$. Then there exists $\tilde{\gamma} : R \longrightarrow \bigoplus\limits_{i=1}^{\infty} E(K_i)$ such that $\tilde{\gamma}_{|I} = \gamma$ (by hypothesis). Put $\tilde{\gamma}(1) = (c_i)_i$, thus there exists $n \geq 1$ with $c_i = 0$ for all $i \geq n$. Since $(\beta_i(b + I_i))_i = \gamma(b) = \tilde{\gamma}(b) = \tilde{\gamma}(1)b = (c_i b)_i$ for all $b \in I$, thus $\beta_i(b + I_i) = c_i b$ for all $i \geq 1$, so it follows that $\beta_i(b + I_i) = 0$ for all $i \geq n$ and all $b \in I$ and this contradicts with $\beta_n(a_n + I_n) \neq 0$. Hence (2) implies (1). $\square$

3 Strongly SS-Injective Modules

Proposition 3.1. *The following statements are equivalent.*
(1) M is a strongly ss-injective rightR-module.

(2) Every R-homomorphism $\alpha : A \longrightarrow M$ extends to N, for all right R-module N, where $A \ll N$ and $\alpha(A)$ is a semisimple submodule in M.

Proof. (2)⇒(1) Clear.

(1)⇒(2) Let A be a small submodule of N, and $\alpha : A \longrightarrow M$ be an R-homomorphism with $\alpha(A)$ is a semisimple submodule of M. If $B = \ker(\alpha)$, then α induces an embedding $\tilde{\alpha} : A/B \longrightarrow M$ defined by $\tilde{\alpha}(a+B) = \alpha(a)$, for all $a \in A$. Clearly, $\tilde{\alpha}$ is well define because if $a_1 + B = a_2 + B$ we have $a_1 - a_2 \in B$, so $\alpha(a_1) = \alpha(a_2)$, that is $\tilde{\alpha}(a_1 + B) = \tilde{\alpha}(a_2 + B)$. Since M is strongly ss-injective and A/B is semisimple and small in N/B, thus $\tilde{\alpha}$ extends to an R-homomorphism $\gamma : N/B \longrightarrow M$. If $\pi : N \longrightarrow N/B$ is the canonical map, then the R-homomorphism $\beta = \gamma \circ \pi : N \longrightarrow M$ is an extension of α such that if $a \in A$, then $\beta(a) = \gamma \circ \pi(a) = \gamma(a+B) = \tilde{\alpha}(a+B) = \alpha(a)$ as desired. □

Corollary 3.2. *(1) Let M be a semisimple right R-module. If M is a strongly ss-injective, then M is small injective.*
(2) If every simple right R-module is strongly ss-injective, then R is semiprimitive.

Proof. (1) By Proposition 3.1.

(2) By (1) and applying [19, Theorem 2.8]. □

Remark 3.3. The converse of Corollary 3.2 is not true (see Example 3.8).

Theorem 3.4. *If M is a strongly ss-injective (or just ss-$E(M)$-injective) right R-module, then for every semisimple small submodule A of M, there is an injective R-module E_A such that $M = E_A \oplus T_A$ where T_A is a submodule of M with $T_A \cap A = 0$. Moreover, if $A \neq 0$, then E_A can be taken $A \leq^{ess} E_A$.*

Proof. Let A be a semisimple small submodule of M. If $A = 0$, we are done by taking $E_A = 0$ and $T_A = M$. Suppose that $A \neq 0$ and let i_1, i_2 and i_3 be inclusion maps and $D_A = E(A)$ be the injective hull of A in $E(M)$. Since M is strongly ss-injective, thus M is ss-$E(M)$-injective. Since A is a semisimple small submodule of M, it follows from [9, Lemma 5.1.3(a)] that A is a semisimple small submodule in $E(M)$ and hence there exists an R-homomorphism $\alpha : E(M) \longrightarrow M$ such that $\alpha i_2 i_1 = i_3$. Put $\beta = \alpha i_2$, thus $\beta : D_A \longrightarrow M$ is an extension of i_3. Since $A \leq^{ess} D_A$, thus β is a monomorphism. Put $E_A = \beta(D_A)$. Since E_A is an injective submodule of M, thus $M = E_A \oplus T_A$ for some submodule T_A of M. Since $\beta(A) = A$, thus $A \subseteq \beta(D_A) = E_A$ and this means that $T_A \cap A = 0$. Moreover, define $\tilde{\beta} = \beta : D_A \longrightarrow E_A$, thus $\tilde{\beta}$ is an isomorphism. Since $A \leq^{ess} D_A$, thus $\tilde{\beta}(A) \leq^{ess} E_A$. But $\tilde{\beta}(A) = \beta(A) = A$, so $A \leq^{ess} E_A$. □

Corollary 3.5. *If M is a right R-module has a semisimple small submodule A such that $A \leq^{ess} M$, then the following conditions are equivalent.*
(1) M is injective.
(2) M is strongly ss-injective.
(3) M is ss-$E(M)$-injective.

Proof. (1)⇒(2) and (2)⇒(3) are obvious.

(3)⇒(1) By Theorem 3.4, we can write $M = E_A \oplus T_A$ where E_A injective and $T_A \cap A = 0$. Since $A \leq^{ess} M$, thus $T_A = 0$ and hence $M = E_A$. Therefore, M is an injective R-module. □

Example 3.6. $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective. In particular, $\mathbb{Z}_4$ is not ss-$\mathbb{Z}_{2^\infty}$-injective.

Proof. Assume that $\mathbb{Z}_4$ is strongly ss-injective $\mathbb{Z}$-module. Let $A =< 2 >= \{0, 2\}$. It is clear that A is a semisimple small and essential submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module. Thus by Corollary 3.5 we have that $\mathbb{Z}_4$ is injective $\mathbb{Z}$-module and this is a contradiction. Thus $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not strongly ss-injective. Since $E(\mathbb{Z}_{2^2}) = \mathbb{Z}_{2^\infty}$ as $\mathbb{Z}$-module, thus $\mathbb{Z}_4$ is not ss-$\mathbb{Z}_{2^\infty}$-injective, by Corollary 3.5. □

Corollary 3.7. *Let M be a right R-module such that $soc(M) \cap J(M)$ is small submodule in M (in particular, if M is finitely generated). If M is strongly ss-injective, then $M = E \oplus T$, where E is injective and $T \cap soc(M) \cap J(M) = 0$. Moreover, if $soc(M) \cap J(M) \neq 0$, then we can take $soc(M) \cap J(M) \leq^{ess} E$.*

Proof. By taking $A = \mathrm{soc}(M) \cap J(M)$ and applying Theorem 3.4 □

The following example shows that the converse of Theorem 3.4 and Corollary 3.7 is not true.

Example 3.8. Let $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module. Since $J(M) = 0$ and $\mathrm{soc}(M) = M$, thus $\mathrm{soc}(M) \cap J(M) = 0$. So, we can write $M = 0 \oplus M$ with $M \cap (\mathrm{soc}(M) \cap J(M)) = 0$. Let $N = \mathbb{Z}_8$ as $\mathbb{Z}$-module. Since $J(N) =< \bar{2} >$ and $\mathrm{soc}(N) =< \bar{4} >$. Define $\gamma : \mathrm{soc}(N) \cap J(N) \longrightarrow M$ by $\gamma(\bar{4}) = \bar{3}$, thus γ is a $\mathbb{Z}$-homomorphism. Assume that M is strongly ss-injective, thus M is ss-N-injective, so there exists $\mathbb{Z}$-homomorphism $\beta : N \longrightarrow M$ such that $\beta \circ i = \gamma$, where i is the inclusion map from $\mathrm{soc}(N) \cap J(N)$ to N. Since $\beta(J(N)) \subseteq J(M)$, thus $\bar{3} = \gamma(\bar{4}) = \beta(\bar{4}) \in \beta(J(N)) \subseteq J(M) = 0$ and this contradiction, so M is not strongly ss-injective $\mathbb{Z}$-module.

Corollary 3.9. *The following statements are equivalent:*
(1) $\mathrm{soc}(M) \cap J(M) = 0$, *for all right R-module M.*
(2) Every right R-module is strongly ss-injective.
(3) Every simple right R-module is strongly ss-injective.

Proof. By Proposition 2.11. □

Recall that a ring R is called a right V-ring (GV-ring, SI-ring, respectively) if every simple (simple singular, singular, respectively) right R-module is injective. A right R-module M is called strongly s-injective if every R-homomorphism from K to M extends to N for every right R-module N, where $K \subseteq Z(N)$ (see [22]). A submodule K of a right R-module M is called t-essential in M (written $K \subseteq^{tes} M$) if for every submodule L of M, $K \cap L \subseteq Z_2(M)$ implies that $L \subseteq Z_2(M)$, M is said to be t-semisimple if for every submodule A of M there exists a direct summand B of M such that $B \subseteq^{tes} A$ (see [4]) . In the next results, we will give some relations between ss-injectivity and other injectivities and we provide many new equivalences of V-rings, GV-rings, SI rings and QF rings.

Lemma 3.10. *Let M/N be a semisimple right R-module and C any right R-module. Then every homomorphism from a right submodule (resp. a right semisimple submodule) A of M to C can be extended to a homomorphism from M to C if and only if every homomorphism from a right submodule (resp. a right semisimple submodule) B of N to C can be extended to a homomorphism from M to C.*

Proof. ($\Rightarrow$) is obtained directly.

($\Leftarrow$) Let f be a right R-homomorphism from a right submodule A of M to C. Since M/N is semisimple, thus there exists a right submodule L of M such that $A + L = M$ and $A \cap L \leq N$ (see [11, Proposition 2.1]). Thus there exists a right R-homomorphism $g : M \longrightarrow C$ such that $g(x) = f(x)$ for all $x \in A \cap L$. Define $h : M \longrightarrow C$ such that for any $x = a + \ell$, $a \in A$, $\ell \in L$, $h(x) = f(a) + g(\ell)$. Thus h is a well define R-homomorphism, because if $a_1 + \ell_1 = a_2 + \ell_2$, $a_i \in A$, $\ell_i \in L$, $i = 1, 2$, then $a_1 - a_2 = \ell_2 - \ell_1 \in A \cap L$, that is $f(a_1 - a_2) = g(\ell_2 - \ell_1)$ which leads to $h(a_1 + \ell_1) = h(a_2 + \ell_2)$. Thus h is a well define R-homomorphism and extension of f. □

Corollary 3.11. *For right R-modules M and N, then the following hold:*
(1) If M is finitely generated and $M/J(M)$ is semisimple right R-module, then N is right soc-*M-injective if and only if N is right ss-M-injective.*
(2) If $M/\mathrm{soc}(M)$ is a semisimple right R-module, then N is soc-*M-injective if and only if N is M-injective.*
(3) If R/S_r is semisimple right R-module, then N is soc-*injective if and only if N is injective.*
(4) If R/S_r is semisimple right R-module, then N is ss-injective if and only if N is small injective.

Proof. (1). ($\Rightarrow$) Clear.

($\Leftarrow$) Since N is a right ss-M-injective, thus every homomorphism from a semisimple small submodule of M to N extends to M. Since M is finitely generated, thus $J(M) \ll M$ and hence every homomorphism from any semisimple submodule of $J(M)$ to N extends to M. Since $M/J(M)$ is semisimple. Thus every homomorphism from any semisimple submodule of M to N extends to M by Lemma 3.10. Therefore N is a soc-M-injective right R-module.

(2). ($\Rightarrow$) Since N is soc-M-injective. Thus every homomorphism from any submodule of $\mathrm{soc}(M)$ to N extends to M. Since $M/\mathrm{soc}(M)$ is semisimple, thus Lemma 3.10 implies that every homomorphism from any submodule of M to N extends to M. Hence N is M-injective.

($\Leftarrow$) Clear.

(3) By (2).

(4) Since R/S_r is semisimple right R-module, thus $J(R/S_r) = 0$. By [9, Theorem 9.1.4(b)], we have $J \subseteq S_r$ and hence $J = J \cap S_r$. Thus N is ss-injective if and only if N is small injective. □

Corollary 3.12. *Let R be a semilocal ring, then $S_r \cap J$ is finitely generated if and only if S_r is finitely generated.*

Proof. Suppose that $S_r \cap J$ is finitely generated. By Corollary 2.25, every direct sum of soc-injective right R-modules is ss-injective. Thus it follows from Corollary 3.11(1) and [2, Corollary 2.11] that S_r is finitely generated. The converse is clear. □

Theorem 3.13. *If R is a right perfect ring, then a right R-module M is strongly soc-injective if and only if M is strongly ss-injective.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Let R be a right perfect ring and M be a strongly ss-injective right R-module. By [11, Theorem 3.8], R is a semilocal ring and hence by [11, Theorem 3.5], we have every right R-module N is semilocal and hence $N/J(N)$ is semisimple right R-module. Since R is a right perfect ring, thus the Jacobson radical of every right R-module is small by [7, Theorem 4.3 and 4.4, p. 69]. Thus $N/J(N)$ is semisimple and $J(N) \ll N$, for any $N \in$ Mod-R. Since M is strongly ss-injective, thus every homomorphism from a semisimple small submodule of N to M extends to N, for every $N \in$ Mod-R, and this implies that every homomorphism from any semisimple submodule of $J(N)$ to M extends to N, for every $N \in$ Mod-R. Since $N/J(N)$ is semisimple right R-module, for every $N \in$ Mod-R. Thus Lemma 3.10 implies that every homomorphism from any semisimple submodule of N to M extends to N, for every $N \in$ Mod-R and hence M is strongly soc-injective. □

Corollary 3.14. *A ring R is QF ring if and only if every strongly ss-injective right R-module is projective.*

Proof. ($\Rightarrow$) If R is *QF* ring, then R is a right perfect ring, so by Theorem 3.13 and [2, Proposition 3.7] we have every strongly ss-injective right R-module is projective.

($\Leftarrow$) By hypothesis we have every injective right R-module is projective and hence R is *QF* ring (see for instance [6, Proposition 12.5.13]). □

Theorem 3.15. *The following statements are equivalent for a ring R.*
(1) Every direct sum of strongly ss-injective right R-modules is injective.
(2) Every direct sum of strongly soc-injective right R-modules is injective.
(3) R is right artinian.

Proof. (1)⇒(2) Clear.

(2)⇒(3) Since every direct sum of strongly soc-injective right R-modules is injective, thus R is right noetherian and right semiartinian by [2, Theorem 3.3 and Theorem 3.6], so it follows from [18, Proposition 5.2, p.189] that R is right artinian.

(3)⇒(1) By hypothesis, R is right perfect and right noetherian. It follows from Theorem 3.13 and [2, Theorem 3.3] that every direct sum of strongly ss-injective right R-modules is strongly soc-injective. Since R is right semiartinian, so [2, Theorem 3.6] implies that every direct sum of strongly ss-injective right R-modules is injective . □

Theorem 3.16. *If R is a right t-semisimple, then a right R-module M is injective if and only if M is strongly s-injective.*

Proof. (⇒) Obvious.

(⇐) Since M is strongly s-injective, thus $Z_2(M)$ is injective by [22, Proposition 3, p.27]. Thus every homomorphism $f : K \longrightarrow M$, where $K \subseteq Z_2^r$ extends to R by [22, Lemma 1, p.26]. Since R is a right t-semisimple, thus R/Z_2^r is a right semisimple (see [4, Theorem 2.3]). So by applying Lemma 3.10, we conclude that M is injective. □

Corollary 3.17. *The following statements are equivalent for a ring R.*
(1) R is right SI and right t-semisimple.
(2) R is semisimple.

Proof. (1)⇒(2). Since R is a right *SI* ring, thus every right R-module is strongly s-injective by [22, Theorem 1, p.29]. By Theorem 3.16, we have every right R-module is injective and hence R is semisimple ring.

(2)⇒(1). Clear. □

Corollary 3.18. *If R is a right t-semisimple ring, then R is right V-ring if and only if R is right GV-ring.*

Proof. (⇒). Clear.

(⇐). By [22, Proposition 5, p.28] and Theorem 3.16. □

Corollary 3.19. *If R is a right t-semisimple ring, then* R/S_r *is noetherian right R-module if and only if R is right noetherian.*

Proof. If R/S_r is a noetherian right R-module, thus every direct sum of injective right R-modules is strongly s-injective by [22, Proposition 6]. Since R is right t-semisimple, so it follows from Theorem 3.16 that every direct sum of injective right R-modules is injective and hence R is right noetherian. The converse is clear. □

4 SS-Injective Rings

We recall that the dual of a right R-module M is M^d =Hom$_R(M, R_R)$ and clearly that M^d is a left R-module.

Proposition 4.1. *The following statements are equivalent for a ring R.*
(1) R is a right ss-injective ring.
(2) If K is a semisimple right R-module, P and Q are finitely generated projective right R-modules, $\beta : K \longrightarrow P$ is an R-monomorphism with $\beta(K) \ll P$ and $f : K \longrightarrow Q$ is an R-homomorphism, then f can be extended to an R-homomorphism $h : P \longrightarrow Q$.
(3) If M is a right semisimple R-module and f is a nonzero monomorphism from M to R_R with $f(M) \ll R_R$, then $M^d = Rf$.

Proof. (2)$\Rightarrow$(1) Clear.

(1)$\Rightarrow$(2) Since Q is finitely generated, there is an R-epimorphism $\alpha_1 : R^n \longrightarrow Q$ for some $n \in \mathbb{Z}^+$. Since Q is projective, there is an R-homomorphism $\alpha_2 : Q \longrightarrow R^n$ such that $\alpha_1\alpha_2 = I_Q$. Define $\tilde{\beta} : K \longrightarrow \beta(K)$ by $\tilde{\beta}(a) = \beta(a)$ for all $a \in K$. Since R is a right ss-injective ring (by hypothesis), it follows from Proposition 2.8 and Corollary 2.5(1) that R^n is a right ss-P-injective R-module. So there exists an R-homomorphism $h : P \longrightarrow R^n$ such that $hi = \alpha_2 f\ \tilde{\beta}^{-1}$. Put $g = \alpha_1 h : P \longrightarrow Q$. Thus $gi = (\alpha_1 h)i = \alpha_1(\alpha_2 f\ \tilde{\beta}^{-1}) = f\ \tilde{\beta}^{-1}$ and hence $(g\beta)(a) = g(i(\beta(a))) = (f\ \tilde{\beta}^{-1})(\beta(a)) = f\ (a)$ for all $a \in K$. Therefore, there is an R-homomorphism $g : P \longrightarrow Q$ such that $g\beta = f$.

(1)$\Rightarrow$(3) Let $g \in M^d$, we have $gf^{-1} : f(M) \to R_R$. Since $f(M)$ is a semisimple small right ideal of R and R is a right ss-injective ring (by hypothesis), thus $gf^{-1} = a.$ for some $a \in R$. Therefore, $g = af$ and hence $M^d = Rf$.

(3)$\Rightarrow$(1) Let $f : K \longrightarrow R$ be a right R-homomorphism, where K is a semisimple small right ideal of R and let $i : K \longrightarrow R$ be the inclusion map, thus by (2) we have $K^d = Ri$ and hence $f = ci$ in K^d for some $c \in R$. Thus there is $c \in R$ such that $f(a) = ca$ for all $a \in K$ and this implies that R is a right ss-injective ring.

□

Example 4.2. (1) Every universally mininjective ring is ss-injective, but not conversely (see Example 5.7).
(2) The two classes of universally mininjective rings and soc-injective rings are different (see Example 5.7 and Example 5.8).

Corollary 4.3. *Let R be a right ss-injective ring. Then:*
(1) R is a right mininjective ring.
(2) $lr(a) = Ra$, for all $a \in S_r \cap J$.
(3) $r(a) \subseteq r(b)$, $a \in S_r \cap J$, $b \in R$ implies $Rb \subseteq Ra$.
(4) $l(bR \cap r(a)) = l(b) + Ra$, for all $a \in S_r \cap J$, $b \in R$.
(5) $l(K_1 \cap K_2) = l(K_1) + l(K_2)$, for all semisimple small right ideals K_1 and K_2 of R.

Proof. (1) By Lemma 2.6.
(2), (3),(4) and (5) are obtained by Lemma 2.12. □

The following is an example of a right mininjective ring which is not right ss-injective.

Example 4.4. (The Björk Example [15, Example 2.5]). Let F be a field and let $a \mapsto \bar{a}$ be an isomorphism $F \longrightarrow \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. By [15, Example 2.5] we have R is a right mininjective local ring. It is mentioned in [2, Example 4.15], that R is not right soc-injective. Since R is a local ring, thus by Corollary 3.11(1), R is not right ss-injective ring.

Theorem 4.5. *Let R be a right ss-injective ring. Then:*
(1) $S_r \cap J \subseteq Z_r$.
(2) If the ascending chain $r(a_1) \subseteq r(a_2a_1) \subseteq \ldots$ terminates for any sequence $a_1, a_2, \ldots$ in $Z_r \cap S_r$, then $S_r \cap J$ is right t-nilpotent and $S_r \cap J = Z_r \cap S_r$.

Proof. (1) Let $a \in S_r \cap J$ and $bR \cap r(a) = 0$ for any $b \in R$. By Corollary 4.3(4), $l(b) + Ra = l(bR \cap r(a)) = l(0) = R$, so $l(b) = R$ because $a \in J$, implies that $b = 0$. Thus $r(a) \subseteq^{ess} R_R$ and hence $S_r \cap J \subseteq Z_r$.

(2) For any sequence $x_1, x_2, ...$ in $Z_r \cap S_r$, we have $r(x_1) \subseteq r(x_2x_1) \subseteq ...$. By hypothesis, there exists $m \in \mathbb{N}$ such that $r(x_m...x_2x_1) = r(x_{m+1}x_m...x_2x_1)$. If $x_m...x_2x_1 \neq 0$, then $(x_m...x_2x_1)R \cap r(x_{m+1}) \neq 0$ and hence $0 \neq x_m...x_2x_1r \in r(x_{m+1})$ for some $r \in R$. Thus $x_{m+1}x_m...x_2x_1r = 0$ and this implies that $x_m...x_2x_1r = 0$, a contradiction. Thus $Z_r \cap S_r$ is right t-nilpotent, so $Z_r \cap S_r \subseteq J$. Therefore, $S_r \cap J = Z_r \cap S_r$ by (1). □

Proposition 4.6. *Let R be a right ss-injective ring. Then:*
(1) If Ra is a simple left ideal of R, then $soc(aR) \cap J(aR)$ *is zero or simple.*
(2) $rl(S_r \cap J) = S_r \cap J$ *if and only if* $rl(K) = K$ *for all semisimple small right ideals K of R.*

Proof. (1) Suppose that $\mathrm{soc}(aR) \cap J(aR)$ is a nonzero. Let x_1R and x_2R be any simple small right ideals of R with $x_i \in aR$, $i = 1,2$. If $x_1R \cap x_2R = 0$, then by Corollary 4.3(5) $l(x_1) + l(x_2) = R$. Since $x_i \in aR$, thus $x_i = ar_i$ for some $r_i \in R$, $i = 1,2$, that is $l(a) \subseteq l(ar_i) = l(x_i)$, $i = 1,2$. Since Ra is a simple, then $l(a) \subseteq^{max} R$, that is $l(x_1) = l(x_2) = l(a)$. Therefore, $l(a) = R$ and hence $a = 0$ and this contradicts the minimality of Ra. Thus $\mathrm{soc}(aR) \cap J(aR)$ is simple.

(2) Suppose that $rl(S_r \cap J) = S_r \cap J$ and let K be a semisimple small right ideal of R, trivially we have $K \subseteq rl(K)$. If $K \cap xR = 0$ for some $x \in rl(K)$, then by Corollary 4.3(5) $l(K \cap xR) = l(K) + l(xR) = R$, since $x \in rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$. If $y \in l(K)$, then $yx = 0$, that is $y(xr) = 0$ for all $r \in R$ and hence $l(K) \subseteq l(xR)$.Thus $l(xR) = R$, so $x = 0$ and this means that $K \subseteq^{ess} rl(K)$. Since $K \subseteq^{ess} rl(K) \subseteq rl(S_r \cap J) = S_r \cap J$, it follows that $K = rl(K)$. The converse is trivial. □

Lemma 4.7. *The following statements are equivalent.*
(1) $rl(K) = K$*, for all semisimple small right ideals K of R.*
(2) $r(l(K) \cap Ra) = K + r(a)$*, for all semisimple small right ideals K of R and all* $a \in R$*.*

Proof. (1)⇒(2). Clearly, $K + r(a) \subseteq r(l(K) \cap Ra)$ by [3, Proposition 2.16]. Now, let $x \in r(l(K) \cap Ra)$ and $y \in l(aK)$. Then $yaK = 0$ and $y \in l(ax)$. Thus $l(aK) \subseteq l(ax)$, and so $ax \in rl(ax) \subseteq rl(aK) = aK$, since aK is a semisimple small right ideal of R. Hence $ax = ak$ for some $k \in K$, and so $(x - k) \in r(a)$. This leads to $x \in K + r(a)$, that is $r(l(K) \cap Ra) = K + r(a)$.

(2)⇒(1). By taking $a = 1$. □

Recall that a right ideal I of R is said to be lie over a summand of R_R, if there exists a direct decomposition $R_R = A_R \oplus B_R$ with $A \subseteq I$ and $B \cap I \ll R_R$ (see [13]) which leads to $I = A \oplus (B \cap I)$.

Lemma 4.8. *Let K be an m-generated semisimple right ideal lies over summand of* R_R*. If R is right ss-injective, then every homomorphism from K to* R_R *can be extended to an endomorphism of* R_R*.*

Proof. Let $\alpha : K \longrightarrow R$ be a right R-homomorphism. By hypothesis, $K = eR \oplus B$, for some $e^2 = e \in R$, where B is an m-generated semisimple small right ideal of R . Now, we need prove that $K = eR \oplus (1 - e)B$. Clearly, $eR + (1 - e)B$ is a direct sum. Let $x \in K$, then $x = a + b$ for some $a \in eR$, $b \in B$, so we can write $x = a + eb + (1 - e)b$ and this implies that $x \in eR \oplus (1 - e)B$. Conversely, let $x \in eR \oplus (1 - e)B$. Thus $x = a + (1 - e)b$, for some $a \in eR$, $b \in B$. We obtain $x = a + (1 - e)b = (a - eb) + b \in eR \oplus B$. It is obvious that $(1 - e)B$ is an m-generated semisimple small right ideal. Since R is a right ss-injective, then there exists $\gamma \in End(R_R)$ such that $\gamma_{|(1-e)B} = \alpha_{|(1-e)B}$. Define $\beta : R_R \longrightarrow R_R$ by $\beta(x) = \alpha(ex) + \gamma((1 - e)x)$, for all $x \in R$ which is a well defined R-homomorphism. If $x \in K$, then $x = a + b$ where $a \in eR$ and $b \in (1 - e)B$, so $\beta(x) = \alpha(ex) + \gamma((1 - e)x) = \alpha(a) + \gamma(b) = \alpha(a) + \alpha(b) = \alpha(x)$ which yields β is an extension of α. □

Corollary 4.9. *Let R be a semiregular ring (or just every finitely generated semisimple right ideal lies over a summand of R_R). If R is a right ss-injective ring, then every R-homomorphism from a finitely generated semisimple right ideal to R extends to R.*

Proof. By [13, Theorem 2.9] and Lemma 4.8. □

Corollary 4.10. *Let S_r be a finitely generated and lie over a summand of R_R, then R is a right ss-injective ring if and only if R is right soc-injective.*

Recall that a ring R is called right minannihilator if every simple right ideal K of R is an annihilator; equivalently, if $rl(K) = K$ (see [14]).

Lemma 4.11. *A ring R is a right minannihilator if and only if $rl(K) = K$ for any simple small right ideal K of R.*

Lemma 4.12. *A ring R is a left minannihilator if and only if $lr(K) = K$ for any simple small left ideal K of R.*

Corollary 4.13. *Let R be a right ss-injective ring, then the following hold:*
(1) If $rl(S_r \cap J) = S_r \cap J$, then R is right minannihilator.
(2) If $S_\ell \subseteq S_r$, then:
i) $S_\ell = S_r$.
ii) R is a left minannihilator ring.

Proof. (1) Let aR be a simple small right ideal of R, thus $rl(a) = aR$ by Proposition 4.6(2). Therefore, R is a right minannihilator ring.

(2) i) Since R is a right ss-injective ring, thus it is right mininjective and it follows from [14, Proposition 1.14 (4)] that $S_\ell = S_r$.

ii) If Ra is a simple small left ideal of R, then $lr(a) = Ra$ by Corollary 4.3(2) and hence R is a left minannihilator ring. □

Proposition 4.14. *The following statements are equivalent for a right ss-injective ring R.*
(1) $S_\ell \subseteq S_r$.
(2) $S_\ell = S_r$.
(3) R is a left mininjective ring.

Proof. (1)⇒(2) By Corollary 4.13(2) (i).

(2)⇒(3) By Corollary 4.13(2) and [15, Corollary 2.34], we need only show that R is right minannihilator ring. Let aR be a simple small right ideal, then Ra is a simple small left ideal by [14, Theorem 1.14]. Let $0 \neq x \in rl(aR)$, then $l(a) \subseteq l(x)$. Since $l(a) \leq^{max} R$, thus $l(a) = l(x)$ and hence Rx is simple left ideal, that is $x \in S_r$. Now , if $Rx = Re$ for some $e = e^2 \in R$, then $e = rx$ for some $0 \neq r \in R$. Since $(e-1)e = 0$, then $(e-1)rx = 0$, that is $(e-1)ra = 0$ and this implies that $ra \in eR$. Thus $raR \subseteq eR$, but eR is semisimple right ideal, so $raR \subseteq^{\oplus} R$ and hence $ra = 0$. Therefore, $rx = 0$, that is $e = 0$, a contradiction. Thus $x \in J$ and hence $x \in S_r \cap J$. Therefore, $aR \subseteq rl(aR) \subseteq S_r \cap J$. Now, let $aR \cap yR = 0$ for some $y \in rl(aR)$, thus $l(aR) + l(yR) = l(aR \cap yR) = R$. Since $y \in rl(aR)$, thus $l(aR) \subseteq l(yR)$ and hence $l(yR) = R$, that is $y = 0$. Therefore, $aR \subseteq^{ess} rl(aR)$, so $aR = rl(aR)$ as desired.

(3)⇒(1) Follows from [15, Corollary 2.34]. □

Recall that a ring R is said to be right minfull if it is semiperfect, right mininjective and $\mathrm{soc}(eR) \neq 0$ for each local idempotent $e \in R$ (see [15]). A ring R is called right min-*PF*, if it is a semiperfect, right mininjective, $S_r \subseteq^{ess} R_R$, $lr(K) = K$ for every simple left ideal $K \subseteq Re$ for some local idempotent $e \in R$ (see [15]).

Corollary 4.15. *Let R be a right ss-injective ring, semiperfect with $S_r \subseteq^{ess} R_R$. Then R is right minfull ring and the following statements hold:*
(1) Every simple right ideal of R is essential in a summand.
(2) $soc(eR)$ is simple and essential in eR for every local idempotent $e \in R$. Moreover, R is right finitely cogenerated.
(3) For every semisimple right ideal I of R, there exists $e = e^2 \in R$ such that $I \subseteq^{ess} rl(I) \subseteq^{ess} eR$.
(4) $S_r \subseteq S_\ell \subseteq rl(S_r)$.
(5) If I is a semisimple right ideal of R and aR is a simple right ideal of R with $I \cap aR = 0$, then $rl(I \oplus aR) = rl(I) \oplus rl(aR)$.
(6) $rl(\bigoplus_{i=1}^{n} a_iR) = \bigoplus_{i=1}^{n} rl(a_iR)$, where $\bigoplus_{i=1}^{n} a_iR$ is a direct sum of simple right ideals.
(7) The following statements are equivalent.
(a) $S_r = rl(S_r)$.
(b) $K = rl(K)$ for every semisimple right ideals K of R.
(c) $kR = rl(kR)$ for every simple right ideals kR of R.
(d) $S_r = S_\ell$.
(e) $\mathrm{soc}(Re)$ *is simple for all local idempotent $e \in R$.*
(f) $\mathrm{soc}(Re) = S_r e$ *for every local idempotent $e \in R$.*
(g) R is left mininjective.
(h) $L = lr(L)$ for every semisimple left ideals L of R.
(i) R is left minfull ring.
(j) $S_r \cap J = rl(S_r \cap J)$.
(k) $K = rl(K)$ for every semisimple small right ideals K of R.
(l) $L = lr(L)$ for every semisimple small left ideals L of R.
(8) If R satisfies any condition of (7), then $r(S_\ell \cap J) \subseteq^{ess} R_R$.

Proof. (1), (2), (3), (4), (5) and (6) are obtained by Corollary 3.11 and [2, Theorem 4.12].

(7) The equivalence of (a), (b), (c), (d), (e), (f), (g), (h) and (i) follows from Corollary 3.11 and [2, Theorem 4.12].

(b)⇒(j) Clear.

(j)⇔(k) By Proposition 4.6(2).

(k)⇒(c) By Corollary 4.13(1).

(h)⇒(l) Clear.

(l)⇒(d) Let Ra be a simple left ideal of R. By hypothesis, $lr(A) = A$ for any simple small left ideal A of R. By Lemma 4.12, $lr(A) = A$ for any simple left ideal A of R and hence $lr(Ra) = Ra$. Thus R is a right min-PF ring and it follows from [14, Theorem 3.14] that $S_r = S_\ell$.

(8) Let K be a right ideal of R such that $r(S_\ell \cap J) \cap K = 0$. Then $Kr(S_\ell \cap J) = 0$ and we have $K \subseteq lr(S_\ell \cap J) = S_\ell \cap J = S_r \cap J$. Now, $r((S_\ell \cap J) + l(K)) = r(S_\ell \cap J) \cap K = 0$. Since R is left Kasch, then $(S_\ell \cap J) + l(K) = R$ by [10, Corollary 8.28(5)]. Thus $l(K) = R$ and hence $K = 0$, so $r(S_\ell \cap J) \subseteq^{ess} R_R$. □

Recall that a right R-module M is called almost-injective if $M = E \oplus K$, where E is injective and K has zero radical (see [23]). After reflect on [23, Theorem 2.12] we found it is not true always and the reason is due to the homomorphism $h : (L+J)/J \longrightarrow K$ in the part (3)⇒(1) of the proof of Theorem 2.12 in [23] is not well define, in particular see the following example.

Example 4.16. In particular from the proof of the part (3)⇒(1) in [23, Theorem 2.12], we consider $R = \mathbb{Z}_8$ and $M = K =< \bar{4} >= \{\bar{0}, \bar{4}\}$. Thus $M = E \oplus K$, where $E = 0$ is a trivial injective R-module and $J(K) = 0$. Let $f : L \longrightarrow K$ is the identity map, where $L = K$. So, the map $h : (L+J)/J \longrightarrow K$ which is given by $h(\ell+J) = f(\ell)$ is not well define, because $J = \bar{4}+J$ but $h(J) = f(\bar{0}) = \bar{0} \neq \bar{4} = f(\bar{4}) = h(\bar{4}+J)$.

The following example shows that there is a contradiction in [23, Theorem 2.12].

Example 4.17. Assume that R is a right artinian ring but not semisimple (this claim is found because for example $\mathbb{Z}_8$ satisfies this property). Now, let M be a simple right R-module, then M is almost-injective. Clearly, R is semilocal (see [9, Theorem 9.2.2]), thus M is injective by [23, Theorem 2.12]. Therefore, R is V-ring and hence R is a right semisimple ring but this contradiction. In other word, Since $\mathbb{Z}_8$ is semilocal ring and $< \bar{4} >= \{\bar{0}, \bar{4}\}$ is almost injective as $\mathbb{Z}_8$-module, then $< \bar{4} >$ is injective by [23, Theorem 2.12]. Thus $< \bar{4} >\subseteq^{\oplus} \mathbb{Z}_8$ and this contradiction.

Theorem 4.18. *The following statements are equivalent for a ring R.*
(1) R is semiprimitive and every almost-injective right R-module is quasi-continuous.
(2) R is right ss-injective and right minannihilator ring, J is right artinian, and every almost-injective right R-module is quasi-continuous.
(3) R is a semisimple ring.

Proof. (1)⇒(2) and (3)⇒(1) are clear.

(2)⇒(3) Let M be a right R-module with zero radical. If N is an arbitrary nonzero submodule of M, then $N \oplus M$ is quasi-continuous and by [12, Corollary 2.14], N is M-injective. Thus $N \leq^{\oplus} M$ and hence M is semisimple. In particular R/J is semisimple R-module and hence R/J is artinian by [9, Theorem 9.2.2(b)], so R is semilocal ring. Since J is a right artinian, then R is right artinian. So it follows from Corollary 4.15(7) that R is right and left mininjective. Thus [14, Corollary 4.8] implies that R is QF ring. By hypothesis, $R \oplus (R/J)$ is quasi-continuous (since R is self-injective), so again by [12, Corollary 2.14] we have that R/J is injective. Since R is QF ring, then R/J is projective (see [9, Theorem 13.6.1]). Thus the canonical map $\pi : R \longrightarrow R/J$ is splits and hence $J \leq^{\oplus} R$, that is $J = 0$. Therefore R is semisimple. □

5 STRONGLY SS-INJECTIVE RINGS

Proposition 5.1. *A ring R is strongly right ss-injective if and only if every finitely generated projective right R-module is strongly ss-injective.*

Proof. Since a finite direct sum of strongly ss-injective modules is strongly ss-injective, so every finitely generated free right R-module is strongly ss-injective. But a direct summand of strongly ss-injective is strongly ss-injective. Therefore, every finitely generated projective is strongly ss-injective. The converse is clear. □

A ring R is called a right Ikeda-Nakayama ring if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R (see [15, p.148]). In the next proposition, the strongly ss-injectivity gives a new version of Ikeda-Nakayama rings.

Proposition 5.2. *Let R be a strongly right ss-injective ring, then* $l(A \cap B) = l(A) + l(B)$ *for all semisimple small right ideals A and all right ideals B of R.*

Proof. Let $x \in l(A \cap B)$ and define $\alpha : A+B \longrightarrow R_R$ by $\alpha(a+b) = xa$ for all $a \in A$ and $b \in B$. Clearly, α is well define, because if $a_1+b_1 = a_2+b_2$, then $a_1-a_2 = b_2-b_1$, that is $x(a_1-a_2) = 0$, so $\alpha(a_1+b_1) = \alpha(a_2+b_2)$. The map α induces an R-homomorphism $\tilde{\alpha}$: $(A+B)/B \longrightarrow R_R$ which is given by $\tilde{\alpha}(a+B) = xa$ for all $a \in A$. Since $(A+B)/B \subseteq \mathrm{soc}(R/B) \cap J(R/B)$ and R is a strongly right ss-injective, $\tilde{\alpha}$ can be extended to an R-homomorphism γ : $R/B \longrightarrow R_R$. If $\gamma(1+B) = y$, for some $y \in R$, then $y(a+b) = xa$, for all $a \in A$ and $b \in B$. In particular, $ya = xa$ for all $a \in A$ and $yb = 0$ for all $b \in B$. Hence $x = (x-y)+y \in l(A)+l(B)$. Therefore, $l(A \cap B) \subseteq l(A)+l(B)$. Since the converse is always holds, thus the proof is complete. □

Recall that a ring R is said to be right simple J-injective if for any small right ideal I and any R-homomorphism $\alpha : I \longrightarrow R_R$ with simple image, $\alpha = c.$ for some $c \in R$ (see [21]).

Corollary 5.3. *Every strongly right ss-injective ring is right simple J-injective.*

Proof. By Proposition 3.1. □

Remark 5.4. The converse of Corollary 5.3 is not true (see Example 5.7).

Proposition 5.5. *Let R be a right Kasch and strongly right ss-injective ring. Then:*
(1) $rl(K) = K$, for every small right ideal K. Moreover, R is right minannihilator.
(2) If R is left Kasch, then $r(J) \subseteq^{ess} R_R$.

Proof. (1) By Corollary 5.3 and [21, Lemma 2.4].

(2) Let K be a right ideal of R and $r(J) \cap K = 0$. Then $Kr(J) = 0$ and we obtain $K \subseteq lr(J) = J$, because R is left Kasch. By (1), we have $r(J+l(K)) = r(J) \cap K = 0$ and this means that $J+l(K) = R$ (since R is left Kasch). Thus $K = 0$ and hence $r(J) \subseteq^{ess} R_R$. □

The following examples show that the classes of rings: strongly ss-injective rings, soc-injective rings and of small injective rings are different.

Example 5.6. Let $R = \mathbb{Z}_{(p)} = \{\frac{m}{n} \mid p$ does not divide $n\}$, the localization ring of $\mathbb{Z}$ at the prime p. Then R is a commutative local ring and it has zero socle but not principally small injective (see [20, Example 4]). Since $S_r = 0$, thus R is strongly soc-injective ring and hence R is strongly ss-injective ring.

Example 5.7. Let $R = \left\{ \begin{pmatrix} n & x \\ 0 & n \end{pmatrix} \mid \ n \in \mathbb{Z}, x \in \mathbb{Z}_2 \right\}$. Thus R is a commutative ring, $J = S_r = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid \ x \in \mathbb{Z}_2 \right\}$ and R is small injective (see [19, Example(i)]). Let $A = J$ and $B = \left\{ \begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix} \mid \ n \in \mathbb{Z} \right\}$, then $l(A) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid \ n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\}$ and $l(B) = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid \ y \in \mathbb{Z}_2 \right\}$. Thus $l(A)+l(B) = \left\{ \begin{pmatrix} 2n & y \\ 0 & 2n \end{pmatrix} \mid \ n \in \mathbb{Z}, y \in \mathbb{Z}_2 \right\}$. Since $A \cap B = 0$, thus $l(A \cap B) = R$ and this implies that $l(A)+l(B) \neq l(A \cap B)$. Therefore R is not strongly ss-injective and not strongly soc-injective by Proposition 5.2.

Example 5.8. Let $F = \mathbb{Z}_2$ be the field of two elements, $F_i = F$ for $i = 1,2,3,...$, $Q = \prod_{i=1}^{\infty} F_i$, $S = \bigoplus_{i=1}^{\infty} F_i$. If R is the subring of Q generated by 1 and S, then R is a Von Neumann regular ring (see [22, Example (1), p.28]). Since R is commutative, thus every simple R-module is injective by [10, Corollary 3.73]. Thus R is V-ring and hence $J(N) = 0$ for every right R-module N. It follows from Corollary 3.9 that every R-module is strongly ss-injective. In particular, R is strongly ss-injective ring. But R is not soc-injective (see [22, Example (1)]).

Example 5.9. Let $R = \mathbb{Z}_2[x_1, x_2, ...]$ where $\mathbb{Z}_2$ is the field of two elements, $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$ and $x_i^2 = x_j^2 \neq 0$ for all i and j. If $m = x_i^2$, then R is a commutative, semiprimary, local, soc-injective ring with $J =$span$\{$m$, x_1, x_2, ...\ \}$, and R has simple essential socle $J^2 = \mathbb{Z}_2 m$ (see [2, Example 5.7]). It follows from [2, Example 5.7] that the R-homomorphism $\gamma : J \longrightarrow R$ which is given by $\gamma(a) = a^2$ for all $a \in J$ with simple image can be not extended to R, then R is not simple J-injective and not small injective, so it follows from Corollary 5.3 that R is not strongly ss-injective.

Recall that R is said to be right minsymmetric ring if aR is simple right ideal then Ra is simple left ideal (see [14]). Every right mininjective ring is right minsymmetric by [14, Theorem 1.14].

Theorem 5.10. *A ring R is QF if and only if R is a strongly right ss-injective and right noetherian ring with $S_r \subseteq^{ess} R_R$.*

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) By Corollary 4.3(1), R is right minsymmetric. It follows from [19, Lemma 2.2] that R is right perfect. Thus R is strongly right soc-injective, by Theorem 3.13. Since $S_r \subseteq^{ess} R_R$, so it follows from [2, Corollary 3.2] that R is self-injective and hence R is QF. $\square$

Corollary 5.11. *For a ring R the following statements are true.*
(1) R is semisimple if and only if $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective.
(2) R is QF if and only if R is strongly right ss-injective, semiperfect with essential right socle and R/S_r is noetherian as right R-module.

Proof. (1) Suppose that $S_r \subseteq^{ess} R_R$ and every semisimple right R-module is strongly soc-injective, then R is a right noetherian right V-ring by [2, Proposition 3.12], so it follows from Corollary 3.9 that R is strongly right ss-injective. Thus R is QF by Theorem 5.10. But $J = 0$, so R is semisimple. The converse is clear.

(2) By [14, Theorem 2.9], $J = Z_r$. Since R/Z_2^r is a homomorphic image of R/Z_r and R is a semilocal ring, thus R is a right t-semisimple. By Corollary 3.19, R is right noetherian, so it follows from Theorem 5.10 that R is QF. The converse is clear. $\square$

Theorem 5.12. *A ring R is QF if and only if R is a strongly right ss-injective, $l(J^2)$ is a countable generated left ideal, $S_r \subseteq^{ess} R_R$ and the chain $r(x_1) \subseteq r(x_2x_1) \subseteq ... \subseteq r(x_nx_{n-1}...x_1) \subseteq ...$ terminates for every infinite sequence $x_1, x_2, ...$ in R.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) By [19, Lemma 2.2], R is right perfect. Since $S_r \subseteq^{ess} R_R$, thus R is right Kasch (by [14, Theorem 3.7]). Since R is strongly right ss-injective, thus R is right simple J-injective, by Corollary 5.3. Now, by Proposition 5.5(1) we have $rl(S_r \cap J) = S_r \cap J$, so it follows from Corollary 4.15(7) that $S_r = S_\ell$. By [15, Lemma 3.36], $S_2^r = l(J^2)$. The result now follows from [21, Theorem 2.18]. $\square$

Remark 5.13. The condition $S_r \subseteq^{ess} R_R$ in Theorem 5.10 and Theorem 5.12 can be not deleted, for example, $\mathbb{Z}$ is strongly ss-injective noetherian ring but not QF.

The following two results are extension of Proposition 5.8 in [2].

Corollary 5.14. *The following statements are equivalent.*
(1) R is a QF ring.
(2) R is a left perfect, strongly left and right ss-injective ring.

Proof. By Corollary 5.3 and [21, Corollary 2.12]. □

Theorem 5.15. *The following statements are equivalent:*
(1) R is a QF ring.
(2) R is a strongly left and right ss-injective, right Kasch and J is left t-nilpotent.
(3) R is a strongly left and right ss-injective, left Kasch and J is left t-nilpotent.

Proof. (1)⇒(2) and (1)⇒(3) are clear.

(3)⇒(1) Suppose that xR is simple right ideal. Thus either $rl(x) = xR \subseteq^{\oplus} R_R$ or $x \in J$. If $x \in J$, then $rl(x) = xR$ (since R is right minannihilator), so it follows from Theorem 3.4 that $rl(x) \subseteq^{ess} E \subseteq^{\oplus} R_R$. Therefore, $rl(x)$ is essential in a direct summand of R_R for every simple right ideal xR. Let K be a maximal left ideal of R. Since R is left Kasch, thus $r(K) \neq 0$ by [10, Corollary 8.28]. Choose $0 \neq y \in r(K)$, so $K \subseteq l(y)$ and we conclude that $K = l(y)$. Since $Ry \cong R/l(y)$, thus Ry is simple left ideal. But R is left mininjective ring, so yR is right simple ideal by [14, Theorem 1.14] and this implies that $r(K) \subseteq^{ess} eR$ for some $e^2 = e \in R$ (since $r(K) = rl(y)$). Thus R is semiperfect by [15, Lemma 4.1] and hence R is left perfect (since J is left t-nilpotent), so it follows from Corollary 5.14 that R is QF.

(2)⇒(1) It is similar to the proof of (3)⇒(1). □

Theorem 5.16. *The ring R is QF if and only if R is strongly left and right ss-injective, left and right Kasch, and the chain $l(a_1) \subseteq l(a_1a_2) \subseteq l(a_1a_2a_3) \subseteq ...$ terminates for every $a_1, a_2, ... \in Z_\ell$.*

Proof. (⇒) Clear.

(⇐) By Proposition 5.5, $l(J)$ is essential in ${}_RR$. Thus $J \subseteq Z_\ell$. Let $a_1, a_2, ... \in J$, we have $l(a_1) \subseteq l(a_1a_2) \subseteq l(a_1a_2a_3) \subseteq ....$ Thus there exists $k \in \mathbb{N}$ such that $l(a_1...a_k) = l(a_1...a_ka_{k+1})$ (by hypothesis). Suppose that $a_1...a_k \neq 0$, so $R(a_1...a_k) \cap l(a_{k+1}) \neq 0$ (since $l(a_{k+1})$ is essential in ${}_RR$). Thus $ra_1...a_k \neq 0$ and $ra_1...a_ka_{k+1} = 0$ for some $r \in R$, a contradiction. Therefore, $a_1...a_k = 0$ and hence J is left t-nilpotent, so it follows from Theorem 5.15 that R is QF. □

Corollary 5.17. *The ring R is QF if and only if R is strongly left and right ss-injective with essential right socle, and the chain $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq ...$ terminates for every infinite sequence $a_1, a_2, ...$ in R.*

Proof. By [19, Lemma 2.2] and Corollary 5.14. □

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