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SUBCLASS OF *p*-VALENT ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In the present paper, we introduce subclass $M_p(\gamma, A, B, v)$ of p-valent analytic functions in the open unit disk U. we study some interesting properties, like, coefficient inequalities and closure theorems. Also we obtain integral representation, convolution properties, integral mean connected with fractional integral and weighted mean.

1. Introduction and Preliminaries

Let W_p be the class of all functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \ (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let S_p denote the subclass of W_p containing of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \ (a_n \geq 0, \ p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.2}$$

which are analytic and multivalent in the open unit disk U.

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Definition 1.1. For the functions $f \in S_p$ given by (1.2) and $g \in S_p$ defined by

$$g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \ (b_n \geq 0, \ p \in \mathbb{N} = \{1, 2, \ldots\}),$$

we define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z^p - \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

Definition 1.2. Let $f \in S_p$ given by (1.2). Then the class $M_p(\gamma, A, B, v)$ is defined by

$$M_p(\gamma, A, B, v) = \left\{ f \in S_p : \left| \frac{f''(z) - (p-1)\frac{f'(z)}{z}}{\gamma f''(z) + (A + B + \gamma)\frac{f'(z)}{z}} \right| < v, \right.$$

$$\left. 0 \leq \gamma < 1, \ 0 < A \leq 1, \ 0 \leq B < 1 \text{ and } 0 < v < 1 \right\}. \quad (1.3)$$

Some of the following properties studied for other classes in [1, 2, 3, 5, 8, 9].

2. Coefficient Inequalities

Theorem 2.1. *Let* $f \in S_p$. *Then* $f \in M_p(\gamma, A, B, v)$ *if and only if*

$$\sum_{n=p+1}^{\infty} n[v(A - B) + n(1 + \gamma v) - p]a_n \leq pv(A - B + \gamma p), \quad (2.1)$$

where $0 \leq \gamma < 1, \ 0 < A \leq 1, \ 0 \leq B < 1$ *and* $p \in \mathbb{N}$.

The result is sharp for the function

$$f(z) = z^p - \frac{pv(A - B + \gamma p)}{n[v(A - B) + n(1 + \gamma v) - p]} z^n, \ n = p + 1, \ p + 2, \ \ldots.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then, we have

$$\left| f''(z) - (p-1)\frac{f'(z)}{z} \right| - v\left| f''(z) + (A+B+\gamma)\frac{f'(z)}{z} \right|$$

$$= \left| -\sum_{n=p+1}^{\infty} n(n-p)a_n z^{n-2} \right|$$

$$- \left| pv(A-B+\gamma p)z^{p-2} - \sum_{n=p+1}^{\infty} nv(A-B+\gamma n)a_n z^{n-2} \right|$$

$$\leq \sum_{n=p+1}^{\infty} n(n-p)a_n |z|^{n-2} - pv(A-B+\gamma p)|z|^{p-2}$$

$$+ \sum_{n=p+1}^{\infty} nv(A-B+\gamma p)a_n |z|^{n-2}$$

$$= \sum_{n=p+1}^{\infty} n[v(A-B) + n(1+\gamma v) - p]a_n - pv(A-B+\gamma p) \leq 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in M_p(\gamma, A, B, v)$.

Conversely, suppose that $f \in M_p(\gamma, A, B, v)$. Then from (1.3), we have

$$\left| \frac{f''(z) - (p-1)\frac{f'(z)}{z}}{\gamma f''(z) - (A+B+\gamma)\frac{f'(z)}{z}} \right|$$

$$= \left| \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n z^{n-2}}{p(A-B+\gamma p)z^{p-2} - \sum_{n=p+1}^{\infty} n(A-B+\gamma n)a_n z^{n-2}} \right| < v.$$

Since $\operatorname{Re}(z) \leq |z|$ for all z $(z \in U)$, we get

$$\operatorname{Re}\left\{ \frac{\sum_{n=p+1}^{\infty} n(n-p)a_n z^{n-2}}{p(A-B+\gamma p)z^{p-2} - \sum_{n=p+1}^{\infty} n(A-B+\gamma n)a_n z^{n-2}} \right\} < v. \tag{2.2}$$

We choose the value of z on the real axis so that $\frac{zf''(z)}{f'(z)}$ is real. Letting $z \to 1^-$, through real values, we obtain the inequality (2.1).

Finally, sharpness follows if we take

$$f(z) = z^p - \frac{pv(A - B + \gamma p)}{n[v(A - B) + n(1 + \gamma v) - p]} z^n, \ n = p + 1, \ p + 2, \ \ldots.$$

Corollary 2.1. *Let* $f \in M_p(\gamma, A, B, v)$. *Then*

$$a_n \leq \frac{pv(A - B + \gamma p)}{n[v(A - B) + n(1 + \gamma v) - p]}, \ \ n = p + 1, \ p + 2, \ \ldots. \tag{2.3}$$

3. Closure Theorems

Theorem 3.1. *Let the functions* f_i *defined by*

$$f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, \ \ (a_{n,i} \geq 0, \ p \in \mathbb{N} = \{1, 2, \ldots\}, \ i = 1, 2, \ldots, \ell) \tag{3.1}$$

be in the class $M_p(\gamma, A, B, v)$ *for every* $i = 1, 2, \ldots, \ell$. *Then the function* h_1 *defined by*

$$h_1(z) = z^p - \sum_{n=p+1}^{\infty} e_n z^n, \ \ (e_n \geq 0, \ p \in \mathbb{N} = \{1, 2, \ldots\}),$$

also belongs to class $M_p(\gamma, A, B, v)$, *where*

$$e_n = \frac{1}{\ell} \sum_{i=1}^{\ell} a_{n,i}, \ \ n = p + 1, \ p + 2, \ \ldots.$$

Proof. Since $f_i \in M_p(\gamma, A, B, v)$, we note that

$$\sum_{n=p+1}^{\infty} n[v(A - B) + n(1 + \gamma v) - p] a_{n,i} \leq pv(A - B + \gamma p),$$

for every $i = 1, 2, \ldots, \ell$. Hence

$$\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]e_n$$

$$= \sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]\left(\frac{1}{\ell}\sum_{i=1}^{\ell} a_{n,i}\right)$$

$$= \frac{1}{\ell}\sum_{i=1}^{\ell}\left(\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]a_{n,i}\right)$$

$$\leq pv(A-B+\gamma p).$$

Therefore by Theorem 2.1, we have $h_1 \in M_p(\gamma, A, B, v)$.

This completes the proof of the theorem.

Theorem 3.2. *Let the functions* f_i *defined by* (3.1) *be in the class* $M_p(\gamma, A, B, v)$ *for every* $i = 1, 2, \ldots, \ell$. *Then the function* h_2 *defined by*

$$h_2(z) = \sum_{i=1}^{\ell} c_i f_i(z)$$

is also in the class $M_p(\gamma, A, B, v)$ *where*

$$\sum_{i=1}^{\ell} c_i = 1, \ (c_i \geq 0).$$

Proof. By Theorem 2.1 for every $i = 1, 2, \ldots, \ell$, we have

$$\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]a_{n,i} \leq pv(A-B+\gamma p).$$

But

$$h_2(z) = \sum_{i=1}^{\ell} c_i f_i(z) = \sum_{i=1}^{\ell} c_i\left(z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n\right) = z^p - \sum_{n=p+1}^{\infty}\left(\sum_{i=1}^{\ell} c_i a_{n,i}\right) z^n.$$

Therefore

$$\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]\left(\sum_{i=1}^{\ell} c_i a_{n,i}\right)$$

$$= \sum_{n=p+1}^{\infty} c_i\left(\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]a_{n,i}\right)$$

$$\leq \sum_{i=1}^{\ell} c_i(pv(A-B+\gamma p))$$

$$= pv(A-B+\gamma p),$$

and the proof is complete.

4. Integral Representation

In the following theorem, we obtain integral representation for the unction $f(z)$.

Theorem 4.1. *Let* $f \in M_p(\gamma, A, B, v)$. *Then*

$$f(z) = \int_0^z e^{\int_0^z \frac{v(A-B+\gamma)\phi(t)+p-1}{t(1-\gamma v\phi(t))}dt} dt.$$

Proof. By putting $\frac{zf''(z)}{f'(z)} = N(z)$ in (1.3), we have

$$\left|\frac{N(z)-(p-1)}{\gamma N(z)+(A-B+\gamma)}\right| < v,$$

or equivalently

$$\frac{N(z)-(p-1)}{\gamma N(z)+(A-B+\gamma)} = v\phi(z),$$

where $|\phi(z)| < 1$, $z \in U$.

So

$$\frac{f''(z)}{f'(z)} = \frac{v(A-B+\gamma)\phi(z)+p-1}{z(1-\gamma v\phi(z))},$$

after integration, we obtain

$$\log(f'(z)) = \int_0^z \frac{v(A-B+\gamma)\phi(t)+p-1}{t(1-\gamma v\phi(t))}\,dt.$$

Thus

$$f'(z) = e^{\int_0^z \frac{v(A-B+\gamma)\phi(t)+p-1}{t(1-\gamma v\phi(t))}dt}.$$

After integration, we have

$$f(z) = \int_0^z e^{\int_0^z \frac{v(A-B+\gamma)\phi(t)+p-1}{t(1-\gamma v\phi(t))}dt}\,dt$$

and this gives the result.

5. Convolution Properties

Theorem 5.1. *Let the functions* $f_j\ (j = 1, 2)$ *defined by*

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0,\ p \in \mathbb{N} = \{1, 2, \ldots\},\ j = 1, 2) \tag{5.1}$$

be in the class $M_p(\gamma, A, B, v)$. *Then* $f_1 * f_2 \in M_p(\gamma, A, \sigma, v)$, *where*

$$\sigma \leq \frac{n(A+\gamma p)[v(A-B)+n(1+\gamma v)-p]^2 - pv(A-B+\gamma p)^2[vA+n(1+\gamma v)-p)}{[v(A-B)+n(1+\gamma v)-p]^2 - pv^2(A-B+\gamma p)^2}.$$

Proof. We must find the largest σ such that

$$\sum_{n=p+1}^{\infty} \frac{n[v(A-\sigma)+n(1+\gamma v)-p]}{pv(A-\sigma+\gamma p)} a_{n,1}a_{n,2} \leq 1.$$

Since $f_j \in M_p(\gamma, A, B, v)\ (j = 1, 2)$, then

$$\sum_{n=p+1}^{\infty} \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} a_{nj} \leq 1, \quad (j = 1, 2). \tag{5.2}$$

By Cauchy-Schwarz inequality, we get

$$\sum_{n=p+1}^{\infty} \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{5.3}$$

We want only to show that

$$\frac{n[v(A-\sigma)+n(1+\gamma v)-p]}{pv(A-\sigma+\gamma p)}a_{n,1}a_{n,2} \leq \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\sqrt{a_{n,1}a_{n,2}}.$$

This equivalently to

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(A-\sigma+\gamma p)[v(A-B)+n(1+\gamma v)-p]}{(A-B+\gamma p)[v(A-\sigma)+n(1+\gamma v)-p]}.$$

From (5.3), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{pv(A-B+\gamma p)}{n[v(A-B)+n(1+\gamma v)-p]}.$$

Thus, it is sufficient to show that

$$\frac{pv(A-B+\gamma p)}{n[v(A-B)+n(1+\gamma v)-p]} \leq \frac{(A-\sigma+\gamma p)[v(A-B)+n(1+\gamma v)-p]}{(A-B+\gamma p)[v(A-\sigma)+n(1+\gamma v)-p]},$$

which implies to

$$\sigma \leq \frac{n(A+\gamma p)[v(A-B)+n(1+\gamma v)-p]^2 - pv(A-B+\gamma p)^2[vA+n(1+\gamma v)-p]}{n[v(A-B)+n(1+\gamma v)-p]^2 - pv^2(A-B+\gamma p)^2}.$$

Theorem 5.2. *Let the functions* f_j $(j=1, 2)$ *defined by* (5.1) *be in the class* $M_p(\gamma, A, B, v)$. *Then the function k defined by*

$$k(z) = z^p - \sum_{n=p+1}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n \tag{5.4}$$

belong to the class $M_p(\gamma, A, \varepsilon, v)$, *where*

$$\varepsilon \leq \frac{n(A+\gamma p)[v(A-B)+n(1+\gamma v)-p]^2 - 2pv(A-B+\gamma p)^2[vA+n(1+\gamma v)-p]}{n[v(A-B)+n(1+\gamma v)-p]^2 - 2pv^2(A-B+\gamma p)^2}.$$

Proof. We must find the largest ε such that

$$\sum_{n=p+1}^{\infty}\frac{n[v(A-\varepsilon)+n(1+\gamma v)-p]}{pv(A-\varepsilon+\gamma p)}(a_{n,1}^2 + a_{n,2}^2) \leq 1.$$

Since $f_j \in M_p(\gamma, A, B, v)$ $(j=1, 2)$, we get

$$\sum_{n=p+1}^{\infty}\left(\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2 a_{n,1}^2$$

$$\leq\left(\sum_{n=p+1}^{\infty}\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}a_{n,1}\right)^2\leq 1 \tag{5.5}$$

and

$$\sum_{n=p+1}^{\infty}\left(\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2 a_{n,2}^2$$

$$\leq\left(\sum_{n=p+1}^{\infty}\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}a_{n,2}\right)^2\leq 1. \tag{5.6}$$

Combining the inequalities (5.5) and (5.6), gives

$$\sum_{n=p+1}^{\infty}\frac{1}{2}\left(\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2(a_{n,1}^2+a_{n,2}^2)\leq 1. \tag{5.7}$$

But, $k\in M_p(\gamma, A, \varepsilon, v)$ if and only if

$$\sum_{n=p+1}^{\infty}\frac{n[v(A-\varepsilon)+n(1+\gamma v)-p]}{pv(A-\varepsilon+\gamma p)}(a_{n,1}^2+a_{n,2}^2)\leq 1. \tag{5.8}$$

The inequality (5.8) will be satisfied if

$$\frac{n[v(A-\varepsilon)+n(1+\gamma v)-p]}{pv(A-\varepsilon+\gamma p)}\leq\frac{n^2[v(A-B)+n(1+\gamma v)-p]^2}{2p^2v^2(A-B+\gamma p)^2},\ (n=p+1, p+2, \ldots),$$

so that

$$\varepsilon\leq\frac{n(A+\gamma p)[v(A-B)+n(1+\gamma v)-p]^2-2pv(A-B+\gamma p)^2[vA+n(1+\gamma v)-p]}{n[v(A-B)+n(1+\gamma v)-p]^2-2pv^2(A-B+\gamma p)^2}.$$

Theorem 5.3. *Let the functions* f_j $(j=1, 2)$ *defined by* (5.1) *be in the class* $M_p(\gamma, A, B, v)$ *and*

$$(p+1)[v(A-B+\gamma(p+1))+1]-2pv(A-B+\gamma p)\geq 0. \tag{5.9}$$

Then the function k defined by (5.4) *belongs to the class* $M_p(\gamma, A, B, v)$.

Proof. Since $f_1 \in M_p(\gamma, A, B, v)$, we have

$$\sum_{n=p+1}^{\infty} \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} a_{n,1} \leq 1$$

and so

$$\sum_{n=p+1}^{\infty} \left(\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2 a_{n,1}^2$$

$$\leq \left(\sum_{n=p+1}^{\infty} \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} a_{n,1}\right)^2 \leq 1.$$

Similarly, since $f_2 \in M_p(\gamma, A, B, v)$, we have

$$\sum_{n=p+1}^{\infty} \left(\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2 a_{n,2}^2 \leq 1.$$

Hence

$$\sum_{n=p+1}^{\infty} \frac{1}{2}\left(\frac{n[v(A-\varepsilon)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)}\right)^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1.$$

In view of Theorem 2.1, it is sufficient to show that

$$\sum_{n=p+1}^{\infty} \frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{5.10}$$

Thus the inequality (5.10) will be satisfied if

$$\frac{n[v(A-B)+n(1+\gamma v)-p]}{pv(A-B+\gamma p)} \leq \frac{n^2[v(A-B)+n(1+\gamma v)-p]^2}{2p^2v^2(A-B+\gamma p)^2}, \ (n = p+1, p+2, \ldots),$$

or if

$$n[v(A-B)+n(1+\gamma v)-p] - 2pv(A-B+\gamma p) \geq 0, \ (n = p+1, p+2, \ldots). \tag{5.11}$$

The left hand side of (5.11) is increasing function of n, hence (5.11) is satisfied for all n if

$$(p+1)[v(A-B+\gamma(p+1))+1]-2pv(A-B+\gamma p)\geq 0.$$

This completes the proof.

6. Integral Mean Inequalities for the Fractional Integral

Definition 6.1 [4]**.** The fractional integral of order λ $(\lambda > 0)$ is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(z)}{(z-t)^{1-\lambda}}\, dt, \tag{6.1}$$

where the function f is an analytic in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

Definition 6.2 [7]**.** Let f, g be analytic in U. Then f is said to be subordinate to g, written $f \prec g$, if there exists a schwarz function $w(z)$, which is analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$, $(z \in U)$. In particular, if the function g is univalent in U, we have the following:

$$f(z) \prec g(z)\ (z \in U) \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

In 1925, Littlewood [6] proved the following subordination theorem:

Theorem 6.1 [6]**.** *If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} d\theta.$$

Theorem 6.2. *Let $f \in M_p(\gamma, A, B, v)$ and suppose that f_n is defined by*

$$f_n(z) = z^p - \frac{pv(A-B+\gamma p)}{n[v(A-B)+n(1+\gamma v)-p]} z^n, \quad (n \geq p+1). \tag{6.2}$$

Also let

$$\sum_{m=p+1}^{\infty} (m-\eta)_{n+1} a_m \leq \frac{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+2)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)\Gamma(p-\eta+1)}, \tag{6.3}$$

for $0 \leq \eta \leq m$, $\lambda > 0$, *where* $(m-\eta)_{\eta+1}$ *denote the Pochhammer symbol defined by*

$$(m-\eta)_{\eta+1} = (m-\eta)(m-\eta+1)\ldots m. \tag{6.4}$$

If there exists an analytic function w defined by

$$(w(z))^{n-p} = \frac{n[v(A-B)n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)}{pv(A-B+\gamma p)\Gamma(n+1)} \times \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} H(m) a_m z^{m-p}, \tag{6.5}$$

where $m \geq \eta$ *and*

$$H(m) = \frac{\Gamma(m-\eta)}{\Gamma(m+\lambda+\eta+1)}, \ (\lambda > 0, \ m \geq p+1) \tag{6.6}$$

then, for $z = re^{i\theta}$ *and* $0 < r < 1$

$$\int_0^{2\pi} |D_z^{-\lambda-\eta} f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |D_z^{-\lambda-\eta} f_n(z)|^{\mu} d\theta \ \ (\lambda > 0, \ \mu > 0). \tag{6.7}$$

Proof. Let

$$f(z) = z^p - \sum_{m=p+1}^{\infty} a_m z^m.$$

For $\eta \geq 0$ and Definition 6.1, we get

$$D_z^{-\lambda-\eta} f(z)$$

$$= \frac{\Gamma(p+1)z^{p+\lambda+\eta}}{\Gamma(p+\lambda+\eta+1)}\left(1 - \sum_{m=p+1}^{\infty} \frac{\Gamma(m+1)\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)\Gamma(m+\lambda+\eta+1)} a_m z^{m-p}\right)$$

$$= \frac{\Gamma(p+1)z^{p+\lambda+\eta}}{\Gamma(p+\lambda+\eta+1)}\left(1 - \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} H(m) a_m z^{m-p}\right),$$

where

$$H(m) = \frac{\Gamma(m-\eta)}{\Gamma(m+\lambda+\eta+1)}, \ (\lambda > 0, \ m \geq p+1).$$

Since H is a decreasing function of m, we have

$$0 < H(m) \leq H(p+1) = \frac{\Gamma(p-\eta+1)}{\Gamma(p+\lambda+\eta+2)}.$$

Similarly, from (6.2) and Definition 6.1, we get

$$D_z^{-\lambda-\eta} f_n(z)$$

$$= \frac{\Gamma(p+1)z^{p+\lambda+\eta}}{\Gamma(p+\lambda+\eta+1)}\left(1 - \frac{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p}\right).$$

For $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 + \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} H(m) a_m z^{m-p} \right|^{\mu} d\theta$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p} \right|^{\mu} d\theta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1 - \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} H(m) a_m z^{m-p}$$

$$\prec 1 - \frac{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} z^{n-p}.$$

By setting

$$1 - \sum_{m=p+1}^{\infty} \frac{\Gamma(p+\lambda+\eta+1)}{\Gamma(p+1)} (m-\eta)_{\eta+1} H(m) a_m z^{m-p}$$

$$= 1 - \frac{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+1)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(p+1)\Gamma(n+\lambda+\eta+1)} (w(z))^{n-p},$$

we find that

$$(w(z))^{n-p} = \frac{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)}{pv(A-B+\gamma p)\Gamma(n+1)}$$

$$\times \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} H(m) a_m z^{m-p},$$

which readily yields $w(0) = 0$. For such a function w, we obtain

$$|w(z)|^{n-p} = \frac{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)}{pv(A-B+\gamma p)\Gamma(n+1)}$$

$$\times \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} H(m) a_m |z|^{m-p},$$

$$\leq \frac{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)}{pv(A-B+\gamma p)\Gamma(n+1)}$$

$$\times H(p+1)|z| \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} a_m$$

$$= |z| \frac{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+\eta+1)\Gamma(p-\eta+1)}{pv(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+\eta+2)}$$

$$\times \sum_{m=p+1}^{\infty} (m-\eta)_{\eta+1} a_m \leq |z| < 1.$$

This completes the proof of the theorem.

By taking $\eta = 0$ in the Theorem 6.2, we have the following corollary:

Corollary 6.1. *Let* $f \in M_p(\gamma, A, B, v)$ *and suppose that* f_n *is defined by* (6.2). *Also let*

$$\sum_{m=p+1}^{\infty} m a_m \leq \frac{v(A-B+\gamma p)\Gamma(n+1)\Gamma(p+\lambda+2)}{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+1)\Gamma(p)} \quad (n \geq p+1).$$

If there exists an analytic function w defined by

$$(w(z))^{n-p} = \frac{n[v(A-B)+n(1+\gamma v)-p]\Gamma(n+\lambda+1)}{pv(A-B+\gamma p)\Gamma(n+1)} \sum_{m=p+1}^{\infty} m H(m) a_m z^{m-p},$$

where

$$H(m) = \frac{\Gamma(m)}{\Gamma(m+\lambda+1)}, \quad (\lambda > 0,\ m \geq p+1)$$

then, for $z = re^{i\theta}$ *and* $0 < r < 1$

$$\int_0^{2\pi} | D_z^{-\lambda} f(z) |^{\mu} d\theta \leq \int_0^{2\pi} | D_z^{-\lambda} f_n(z) |^{\mu} d\theta \;\; (\lambda > 0, \mu > 0).$$

7. Weighted Mean

Definition 7.1. Let f and g be in the class $M_p(\gamma, A, B, v)$. Then the weighted mean E_q of f and g is given by

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)], \;\; 0 < q < 1.$$

Theorem 7.1. *Let f and g be in the class* $M_p(\gamma, A, B, v)$*. Then the weighted mean of f and g is also in the class* $M_p(\gamma, A, B, v)$*.*

Proof. By Definition 7.1, we have

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)]$$

$$= \frac{1}{2}\left[(1-q)\left(z^p - \sum_{n=p+1}^{\infty} a_n z^n\right) + (1+p)\left(z^p - \sum_{n=p+1}^{\infty} b_n z^n\right)\right]$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{1}{2}((1-q)a_n + (1+q)b_n)z^n.$$

Since f and g are in the class $M_p(\gamma, A, B, v)$ so by Theorem 2.1, we get

$$\sum_{n=p+1}^{\infty} n[v(A-B) + n(1+\gamma v) - p]a_n \leq pv(A - B + \gamma p)$$

and

$$\sum_{n=p+1}^{\infty} n[v(A-B) + n(1+\gamma v) - p]b_n \leq pv(A - B + \gamma p).$$

Hence

$$\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]\left(\frac{1}{2}(1-q)a_n+\frac{1}{2}(1+q)b_n\right)$$

$$=\frac{1}{2}(1-q)\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]a_n$$

$$+\frac{1}{2}(1+q)\sum_{n=p+1}^{\infty} n[v(A-B)+n(1+\gamma v)-p]b_n$$

$$\leq \frac{1}{2}(1-q)pv(A-B+\gamma p)+\frac{1}{2}(1+q)pv(A-B+\gamma p)$$

$$= pv(A-B+\gamma p).$$

This shows $E_q \in M_p(\gamma, A, B, v)$.

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